ROST'S DEGREE FORMULA

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Some parts of algebraic quadratic form theory and theory of simple algebras (with involutions) can be translated into the language of algebraic geometry.

Example 0.1. Let (V, q) be a quadratic form over a field F, Q the associated projective quadric in $\mathbb{P}(V)$. Then q is isotropic iff $Q(F) \neq \emptyset$.

1. Numbers n_X and n_L

Notation: "scheme" = separated scheme of finite type over a field F.
"variety" = integral scheme over F, $F(X)$ function field of X.

Let X be a scheme over F. For a closed point $x \in X$ we define *degree* of x: $\deg(x) = [F(x) : F]$ and the number

$$n_X = \gcd \deg(x),$$

where the gcd is taken over all closed points of X.

Example 1.1. For an anisotropic quadric Q, $n_Q = 2$.

Example 1.2. For a Severi-Brauer variety $S = SB(A)$, $n_S = \operatorname{ind}(A)$.

Proposition 1.3. (i) *If for a scheme X over F, $X(F) \neq \emptyset$, then $n_X = 1$;*
(ii) *Let X and Y be two schemes. If there is a morphism $f : Y \to X$, then n_X divides n_Y.*

Proof. (ii) For a closed point $y \in Y$, the point $x = f(y)$ is closed and $\deg(x)$ divides $\deg(y)$. □

Let L/F be a field extension, v a valuation on L over F with residue field $F(v)$. We define *degree* of v: $\deg(v) = [F(v) : F]$ and the number

$$n_L = \gcd \deg(v),$$

where the gcd is taken over all valuations of finite degree.

Proposition 1.4. (i) *Let X be a complete scheme over F. If $X(L) \neq \emptyset$, then n_X divides n_L.*
(ii) *Let X be a variety, $L = F(X)$. If X is smooth, then n_L divides n_X.*
(iii) *If a variety X is complete and smooth, then $n_X = n_L$ where $L = F(X)$. In particular, n_X is a birational invariant of a complete smooth variety X.*
(iv) *If $f : Y \to X$ is a rational morphism of complete varieties with Y smooth, then n_X divides n_Y.*

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Proof. (i) Let $\mathcal{O} \subset L$ be the valuation ring. Since X is complete, a point in $X(L)$ factors as

$$\operatorname{Spec} L \to \operatorname{Spec} \mathcal{O} \to^{f} X.$$

Let m be the closed point in $\operatorname{Spec} \mathcal{O}$, $x = f(m)$. Then $F(x) \subset \mathcal{O}/m = F(v)$ and therefore, $\deg(x)$ divides $\deg(v)$.

(ii) Let $x \in X$. Since x is smooth, there is a valuation v on L with residue field $F(x)$, so that $\deg(v) = \deg(x)$.

(iii) follows from (i) and (ii).

(iv) Let $L = F(Y)$. Since $X(L) \neq \emptyset$, by (i), n_X divides n_L. Finally, by (ii), n_L divides n_Y. □

2. Vector bundles and Chern classes

A *vector bundle* over a schemes X is a morphism $E \to X$ which is locally on X (in Zariski topology) is isomorphic to the projection $X \times \mathbb{A}^n \to X$ with linear transition functions.

Let $f : V \to W$ be a dominant morphism of integral schemes. We define *degree* of f by

$$\deg(f) = \begin{cases} [F(V) : F(W)] & \text{if } \dim V = \dim W, \\ 0 & \text{otherwise.} \end{cases}$$

For a scheme X denote by $\mathrm{CH}_n(X)$ the *Chow group* of classes of cycles in X of dimension n. For a proper morphism $f : Y \to X$ there is *push-forward homomorphism*

$$f_* : \mathrm{CH}_n(Y) \to \mathrm{CH}_n(X)$$

defined by

$$f_*\big([V]\big) = \deg\big(V \to f(V)\big) \cdot [f(V)].$$

The group $\mathrm{CH}_0(X)$ is generated by classes $[x]$ of closed points $x \in X$. If X is complete, the push-forward homomorphism with respect to the structure morphism $X \to \operatorname{Spec} F$ is called the *degree map*

$$\deg : \mathrm{CH}_0(X) \to \mathbb{Z}, \quad [x] \mapsto \deg(x) = [F(x) : F].$$

If X is a smooth scheme, the ring $\mathrm{CH}_*(X)$ is a commutative ring with the identity $1 = [X]$.

For a vector bundle $E \to X$ there are *operational Chern classes*

$$c_i(E) : \mathrm{CH}_n(X) \to \mathrm{CH}_{n-i}(X), \quad \alpha \mapsto c_i(E) \cap \alpha.$$

If X is a smooth variety, $\mathrm{CH}_*(X)$ is a commutative ring and $c_i(E)$ is the multiplication by $c_i(E) \cap [X]$. The latter element we will also denote by $c_i(E)$ Thus, $c_i(E) \in \mathrm{CH}_{d-i}(X)$ where $d = \dim X$.

For a smooth X the Chern classes extend to a map

$$c_i : K_0(X) \to CH_*(X).$$

In particular, for a vector bundle $E \to X$, the classes $c_i(-E) \in CH_{d-i}(X)$ are defined.

3. Definition of the invariant $\eta_p(X)$

Let p be a prime number, F a field. We assume that $\mathrm{char}(F) \neq p$ and $\mu_p \subset F$. We fix a primitive p-th root of unity ξ.

Let X be quasi-projective scheme over F. The group $G = \mathbb{Z}/p\mathbb{Z}$ acts by cyclic permutations on the product

$$X^p = X \times X \times \cdots \times X.$$

The factor scheme X^p/G we denote by C^pX. The image $\overline{X}$ of the diagonal $X \subset X^p$ under the natural morphism $X^p \to C^pX$ is a closed subscheme in C^pX, isomorphic to X. In particular, $n_{\overline{X}} = n_X$.

Consider a G-action on the trivial linear bundle $X^p \times \mathbb{A}^1$ over X^p by

$$(x_1, x_2, \dots, x_p, t) \mapsto (x_2, \dots, x_p, x_1, \xi t).$$

The projection $X^p \setminus X \to C^pX \setminus \overline{X}$ is unramified, hence the restriction of the factor vector bundle $(X^p \times \mathbb{A}^1)/G$ to $C^pX \setminus \overline{X}$ is a linear bundle over $C^pX \setminus \overline{X}$. Denote it by L_X.

Let $d = \dim X$. Set

$$l_X = c_1(L_X)^{pd} \cap [C^pX \setminus \overline{X}] \in \mathrm{CH}_0(C^pX \setminus \overline{X}).$$

Remark 3.1. The element l_X is known as the *Euler class* of the bundle $L_X^{\oplus pd}$, i.e. l_X is the image of $[X]$ under the composition

$$\mathrm{CH}_{pd}(C^pX \setminus \overline{X}) \to^{s_*} \mathrm{CH}_{pd}\big(L_X^{\oplus pd}\big) \to^{p^*} \mathrm{CH}_0(C^pX \setminus \overline{X}),$$

where s is the zero section of the vector bundle $L_X^{\oplus pd}$ and p^* is the pull-back with respect to the (flat) morphism $p : L_X^{\oplus pd} \to C^pX \setminus \overline{X}$.

If X is projective, we have degree homomorphism

$$\deg : \mathrm{CH}_0(C^pX \setminus \overline{X}) \to \mathbb{Z}/n_X\mathbb{Z}.$$

Thus, the degree of l_X defines an element $\eta_p(X) \in \mathbb{Z}/n_X\mathbb{Z}$.

Thus, for a projective scheme we have defined an invariant $\eta_p(X) \in \mathbb{Z}/n_X\mathbb{Z}$. Note that X is projective but not necessarily smooth scheme over F.

Lemma 3.2. *$p \cdot l_X = 0 \in \mathrm{CH}_0(C^pX \setminus \overline{X})$. In particular,*

$$p \cdot \eta_p(X) = 0 \in \mathbb{Z}/n_X\mathbb{Z}.$$

Proof. Let

$$f : X^p \setminus X \to C^pX \setminus \overline{X}$$

be the projection (of degree p). Since the linear bundle $f^*(L_X)$ is trivial, by the projection formula,

$$0 = f_*\big(c_1(f^*L_X)^{pd} \cap [X^p \setminus X]\big) = c_1(L_X)^{pd} \cap f_*[X^p \setminus X] =$$

$$p \cdot c_1(L_X)^{pd} \cap [C^pX \setminus \overline{X}] = p \cdot l_X.$$

□

4. Degree formula

We prove two degree formulas due to M. Rost.

Theorem 4.1. (Regular Degree Formula) *Let $f : Y \to X$ be a morphism of projective schemes of the same dimension d. Then n_X divides n_Y and*

$$\eta_p(Y) = \deg(f) \cdot \eta_p(X) \in \mathbb{Z}/n_X\mathbb{Z}.$$

Proof. Denote by $\widetilde{Y}$ the inverse image of $\overline{X}$ under $C^p f : C^p Y \to C^p X$, so that $\overline{Y} \subset \widetilde{Y}$. In particular,

$$n_X = n_{\overline{X}} \mid n_{\widetilde{Y}} \mid n_{\overline{Y}} = n_Y.$$

We have open embedding

$$j : C^p(Y) \setminus \widetilde{Y} \hookrightarrow C^p(Y) \setminus \overline{Y}$$

and proper morphism

$$i : C^p(Y) \setminus \widetilde{Y} \to C^p(X) \setminus \overline{X}$$

of degree d^p.

Denote by L' the restriction of the vector bundle L_Y on $C^p Y \setminus \widetilde{Y}$. Clearly, L' is the inverse image of L_X with respect to i, $L' = i^*(L_X)$. We have by projection formula,

$$i_* j^*(l_Y) = i_* j^*\big(c_1(L_Y)^{pd} \cap [C^p Y \setminus \overline{Y}]\big) =$$
$$i_*\big(c_1(L')^{pd} \cap [C^p Y \setminus \widetilde{Y}]\big) = i_*\big(c_1(i^* L_X)^{pd} \cap [C^p Y \setminus \widetilde{Y}]\big) =$$
$$c_1(L_X)^{pd} \cap i_*[C^p Y \setminus \widetilde{Y}] = \deg(i) \cdot \big(c_1(L_X)^{pd} \cap [C^p X \setminus \overline{X}]\big) = \deg(f)^p \cdot l_X.$$

Note, that by Lemma 3.2,

$$\deg(f)^p \cdot l_X = \deg(f) \cdot l_X.$$

Finally, taking degree,

$$\eta_p(Y) = \deg(l_Y) = \deg\big(i_* j^*(l_Y)\big) = \deg(f) \cdot \eta_p(X) \in \mathbb{Z}/n_X\mathbb{Z}.$$

□

Theorem 4.2. (Rational Degree Formula) *Let $f : Y \to X$ be a rational morphism of projective varieties of dimension d with Y smooth. Then n_X divides n_Y and*

$$\eta_p(Y) = \deg(f) \cdot \eta_p(X) \in \mathbb{Z}/n_X\mathbb{Z}.$$

Proof. By Proposition 1.4(iv), n_X divides n_Y. Let Y_1 be closure of the graph of f in $Y \times X$. Applying Theorem 4.1 to the birational isomorphism $Y_1 \to Y$ (of degree 1), we get

$$\eta_p(Y_1) = \eta_p(Y) \in \mathbb{Z}/n_Y\mathbb{Z}.$$

On the other hand, applying Theorem 4.1 to the projection $Y_1 \to X$(of degree $= \deg(f)$), we get

$$\eta_p(Y_1) = \deg(f) \cdot \eta_p(X) \in \mathbb{Z}/n_X\mathbb{Z}.$$

Thus,

$$\eta_p(Y) = \deg(f) \cdot \eta_p(X) \in \mathbb{Z}/n_X\mathbb{Z}.$$

□

Corollary 4.3. *The class $\eta_p(X) \in \mathbb{Z}/n_X\mathbb{Z}$ is a birational invariant of a smooth projective variety X.*

Problem 4.4. How to define the invariant $\eta_p(X) \in \mathbb{Z}/n_X\mathbb{Z}$ out of the function field $L = F(X)$? (Note that $n_X = n_L$ by Proposition 1.4.)

Here is the main application:

Theorem 4.5. *Let X and Y be two projective varieties with Y smooth. Assume that X has a rational point over the field $F(Y)$. Then n_X divides n_Y and if $\eta_p(Y) \not\equiv 0 \pmod{n_X}$, the following holds:*
(i) $\dim(X) \geq \dim(Y)$.
(ii) *If* $\dim(X) = \dim(Y)$, *then Y has a closed point over $F(X)$ of degree prime to p.*

Proof. By assumption, there exists a rational morphism $f : Y \to X$, hence n_X divides n_Y by Proposition 1.4(iv).

(i) Assume that $m = \dim(Y) - \dim(X) > 0$. Consider the composition

$$g : Y \to^f X \hookrightarrow X \times \mathbb{P}^m_F.$$

Clearly, $\deg(g) = 0$ and $n_{X\times\mathbb{P}^m_F} = n_X$. By Theorem 4.2, applied to g,

$$\eta_p(Y) \equiv 0 \pmod{n_X},$$

a contradiction.

(ii) By the degree formula, applied to f, and Lemma 3.2, the degree $\deg(f)$ is not divisible by p. Hence the generic point of Y determines a point over $F(X)$ of degree $= \deg(f)$. □

Remark 4.6. The first statement of Theorem 4.5 shows that a variety Y cannot be "compressed" to a variety X of smaller dimension if

$$\eta_p(Y) \neq 0 \in \mathbb{Z}/n_X\mathbb{Z}.$$

5. Computation of η_2 for a quadric

Let $Q = Q(V, q)$ be anisotropic smooth projective quadric of dimension d, so that $n_Q = 2$.

Proposition 5.1.

$$\eta_2(Q) = \begin{cases} 1 + 2\mathbb{Z}, & \text{if } d = 2^k - 1 \text{ for some } k, \\ 2\mathbb{Z}, & \text{otherwise.} \end{cases}$$

Proof. Each pair of distinct points in Q determines a line in $\mathbb{P}(V)$, i.e. a plane in V. Thus we have a rational morphism

$$\alpha : C^2Q \to \mathrm{Gr}(2, V).$$

Clearly, α is a birational isomorphism! Indeed, let U be the open subvariety in C^2Q consisting of all pairs of points $([v],[u])$ such that $[v]\neq[u]$ and the restriction of the quadratic form q on the 2-dimensional subspace generated by v and u is nondegenerate. Then the restriction $\alpha|_U$ is an open immersion identifying U with the open subvariety $U'\subset \mathrm{Gr}(2,V)$ consisting of all planes $W\subset V$ such that the restriction $q|_W$ is nondegenerate. The inverse rational morphism α^{-1} takes a plane $W\subset V$ to the intersection $\mathbb{P}(W)\cap Q$. If $q|_W$ is nondegenerate, this intersection is an effective 0-cycle of degree 2, i.e. is a point of C^2X.

Remark 5.2. If $\dim(Q)=1$, i.e. if Q is a conic, α is an isomorphism between C^2Q and $\mathrm{Gr}(2,V)=\mathbb{P}(V^*)$. In the split case this isomorphism looks as follows: $C^2\mathbb{P}^1_F\simeq\mathbb{P}^2_F$.

Let E be the tautological linear bundle over $\mathrm{Gr}(2,V)$ (the second exterior power of rank 2 tautological vector bundle). Denote by L' the restriction of the linear bundle L_Q to the open subvariety $U\subset C^2Q$.

Lemma 5.3. $(\alpha|_U)^*(E)\simeq L'$.

Proof. Let $\widetilde{U}$ be the inverse image of U under the natural morphism $Q^2\to C^2Q$. We have the following morphism of vector bundles:

$$\beta:\widetilde{U}\times\mathbb{A}^1_F\to E,\quad ([v],[u],t)\mapsto(\langle v,u\rangle, t\,\frac{v\wedge u}{b(v,u)})$$

where b is the polar form of q. The action of $G=\mathbb{Z}/2\mathbb{Z}$ on $\widetilde{U}\times\mathbb{A}^1_F$ by $([v],[u],t)\mapsto([u],[v],-t)$ commutes with the trivial action of G on E. Hence β induces an isomorphism $L'\to E$ over $\alpha|_U$. □

At every point of "bad loci" $C^2Q\backslash U$ and $Gr(2,V)\backslash U'$ the form q is isotropic. Hence the numbers $n_{C^2Q\backslash U}$ and $n_{Gr(2,V)\backslash U'}$ are even. Lemma 5.3 implies that

$$\eta_2(Q)\equiv\deg\big(c_1(E)\big)^{2d}\ \ (mod\ 2).$$

Let

$$i:\mathrm{Gr}(2,V)\hookrightarrow\mathbb{P}(\wedge^2V)$$

be the canonical closed embedding. The vector bundle E on $\mathrm{Gr}(2,V)$ is the inverse image under i of the tautological vector bundle on $\mathbb{P}(\wedge^2V)$. The first Chern class of the tautological vector bundle is negative of the class of a hyperplane section. Thus, the degree $\deg(c_1E)^{2d}$ is equal to $(-1)^d$ times the degree of the subvariety $i(\mathrm{Gr}(2,V))$ in the projective space $\mathbb{P}(\wedge^2V)$. The later degree is known as the Catalan number

$$\frac{1}{d+1}\binom{2d}{d}.$$

One easily checks that this number is odd iff $d=2^k-1$ for some k. □

Theorem 5.4. *Let $Q = Q(V, q)$ be a smooth projective quadric of dimension $d \geq 2^k - 1$ and let X be a variety over F such that n_X is even and X has a point over $F(Q)$. Then*
(i) $\dim(X) \geq 2^k - 1$.
(ii) *If* $\dim(X) = 2^k - 1$, *then Q has a point over $F(X)$.*

Proof. Let $Q \to X$ be a rational morphism. Choose a subquadric $Q' \subset Q$ of dimension $2^k - 1$ such that the composition $f : Q' \hookrightarrow Q \to X$ is defined.

(i) Since $\eta_2(Q') \neq 0 \in \mathbb{Z}/2\mathbb{Z}$ and hence in $\mathbb{Z}/n_X\mathbb{Z}$, by Theorem 4.5(1),

$$\dim(X) \geq \dim(Q') = 2^k - 1.$$

(ii) By Theorem 4.5(2), Q' has a point of odd degree over $F(X)$. By Springer's Theorem, Q' and hence Q has a point over $F(X)$. $\square$

Corollary 5.5. *(Hoffmann) Let Q_1 and Q_2 be two anisotropic quadrics. If* $\dim(Q_1) \geq 2^k - 1$ *and Q_2 is isotropic over $F(Q_1)$, then* $\dim(Q_2) \geq 2^k - 1$.

Corollary 5.6. *(Izhboldin) Let Q_1 and Q_2 be two anisotropic quadrics. If* $\dim(Q_1) = \dim(Q_2) = 2^k - 1$ *and Q_2 is isotropic over $F(Q_1)$, then Q_1 is isotropic over $F(Q_2)$.*

6. Computing η_2 for smooth varieties

For a smooth variety X let T_X be the tangent bundle over X.

Theorem 6.1. *Let X be a proper smooth variety of dimension $d > 0$. Then the degree of the 0-cycle $c_d(-T_X)$ is even and*

$$\eta_2(X) = \frac{\deg c_d(-T_X)}{2} \in \mathbb{Z}/n_X\mathbb{Z}.$$

Proof. We would like first to compactify smoothly $C^2X \setminus \overline{X}$ and extend the line bundle L_X to the compactification. Note that C^2X is smooth only if $\dim(X) = 1$.

Let W be the blow-up of X^2 along the diagonal $X \subset X^2$. The G-action on X^2 extends to one on W. The subvariety W^G coincides with the exceptional divisor

$$\mathbb{P}\big((T_X \oplus T_X)/T_X\big) = \mathbb{P}(T_X) :$$

$$\begin{tikzcd} \mathbb{P}(T_X) \arrow[r] \arrow[d] & W \arrow[d] & W \setminus \mathbb{P}(T_X) \arrow[l] \arrow[d] \\ X \arrow[r] & X^2 & X^2 \setminus X \arrow[l] \end{tikzcd}$$

Since W^G is of codimension 1 in W, W/G is smooth and therefore can be taken as a smooth compactification of $C^2X \setminus \overline{X}$.

Now we would like to construct a canonical extension L' of L_X to the whole W/G. There is a canonical linear bundle L_{can} over the blow-up W with the induced G-action. The group G acts by -1 on the restriction of L_{can} on the exceptional divisor $\mathbb{P}(T_X)$. The restriction of L_{can} to the complement of the exceptional divisor $W \setminus \mathbb{P}(T_X) \simeq X^2 \setminus X$ is the trivial vector bundle with the trivial G-action. Now we modify the G-action on L_{can} by -1. The new G-action of the restriction of L_{can} on the exceptional divisor $\mathbb{P}(T_X)$ is trivial, hence L_{can} descends to a linear vector bundle L' on W/G which is a desired extension of L_X on $C^2X \setminus \overline{X}$.

Now we compute $\deg c_1(L')^{2d}$. At one hand, since L_X is the restriction of L' on $C^2X \setminus \overline{X}$, we have

$$\deg c_1(L')^{2d} = \deg c_1(L_X)^{2d} = \eta_2(X) \in \mathbb{Z}/n_X\mathbb{Z}.$$

On the other hand, consider morphisms

$$\begin{tikzcd} \mathbb{P}(T_X) \arrow[r, "i"] \arrow[d, "q"'] & W \arrow[r, "p"] & W/G \\ X. & & \end{tikzcd}$$

We have

$$p^*c_1(L') = c_1(L_{can}) = [\mathbb{P}(T_X)] = i_*(1) \in \mathrm{CH}^1(W).$$

In particular, $p^*c_1(L')^{2d} = c_1(L_{can})^{2d}$ and hence

$$2\deg c_1(L')^{2d} = \deg c_1(L_{can})^{2d}.$$

The restriction of L_{can} on $\mathbb{P}(T_X)$ is the canonical line bundle E_{can} with the sheaf of sections $O_{\mathbb{P}(T_X)}(1)$. Hence, by the projection formula (note that $d > 0$),

$$c_1(L_{can})^{2d} = i_*(1) \cdot c_1(L_{can})^{2d-1} = i_*\big(c_1(E_{can})^{2d-1}\big) \in \mathrm{CH}_0(W),$$

hence

$$\deg c_1(L_{can})^{2d} = \deg c_1(E_{can})^{2d-1}$$

The class

$$q_*\big(c_1(E_{can})^{2d-1}\big) \in \mathrm{CH}_0(X)$$

is known as the *Segre class* of T_X and coincides with $c_d(-T_X)$. (In general, $q_*\big(c_1(E_{can})^{i+d-1}\big) = c_i(-T_X)$.)

Finally,

$$2\deg c_1(L')^{2d} = \deg c_1(L_{can})^{2d} = \deg c_1(E_{can})^{2d-1} = \deg c_d(-T_X).$$

□

Remark 6.2. The integer $\frac{\deg c_d(-T_X)}{2}$ does not change under field extensions and therefore can be computed over an algebraically closed field.

Remark 6.3. If $p > 2$, the blow up W of the diagonal in X^p does not have smooth orbit space W/G and the linear bundle L_X cannot be extended to a linear bundle on W/G.

Remark 6.4. The number $\deg c_d(-T_X)$ does not change under field extensions and hence can be computed over algebraically closed fields.

Remark 6.5. The Theorem shows that the degree m of the cycle $c_d(-T_X)$ is always even. Hence this cycle defines a cycle on the symmetric square C^2X of degree $\frac{m}{2}$. In the proof we construct this class "canonically" (it is $c_1(L_X)^{2d}$). The degree formula follows from "canonical" nature of this class. One should study other cases of divisibility of characteristic numbers.

7. Computing η_2 for projective spaces

Proposition 7.1. *Let $X = \mathbb{P}^d_F$. Then*

$$\deg c_d(-T_X) = (-1)^d \binom{2d}{d}.$$

Proof. It is known that $[-T_X] = -(d+1)[E_{can}] + 1 \in K_0(X)$. We can compute the total Chern class:

$$c(-T_X) = \frac{1}{c(E_{can})^{d+1}} = \frac{1}{(1+t)^{d+1}}.$$

Hence

$$\deg c_d(-T_X) = \binom{-d-1}{d} = (-1)^d \binom{2d}{d}.$$

□

Let v_p be the p-adic discrete valuation and $s_p(a)$ be the sum of digits of a written in base p.

Corollary 7.2. *Let Y be a Severi-Brauer variety of dimension d. Then*

$$v_2\big(\deg c_d(-T_Y)\big) = s_2(d), \quad \eta_2(Y) = 2^{s_2(d)-1} + n_Y\mathbb{Z} \in \mathbb{Z}/n_Y\mathbb{Z}.$$

Corollary 7.3. *(Karpenko) Let A be a division algebra with orthogonal involution σ, Y the Severi-Brauer variety of A. Then the involution σ is anisotropic over $F(Y)$.*

Proof. Let $X = I(A, \sigma) \subset Y$ be the involution variety, $\deg(A) = 2^k$. Then $2^k = n_Y$ divides n_X. If σ is isotropic over $F(Y)$ then there is a rational morphism $Y \to X$. By Corollary 7.2, $\eta_2(Y) = 2^{k-1} + n_X\mathbb{Z}$ is nontrivial in $\mathbb{Z}/n_X\mathbb{Z}$. A contradiction by Theorem 4.5(1). □

Remark 7.4. In fact, we proved that the Severi-Brauer variety of a division algebra of degree 2^k cannot be compressed to a variety X of smaller dimension with $I(X) = 2^k\mathbb{Z}$.

Remark 7.5. Let p be any prime integer. One can show that for a Severi-Brauer variety Y of dimension d divisible by $p-1$,

$$\eta_p(X) = p^{\frac{s_p(d)}{p-1}-1} \in \mathbb{Z}/n_Y\mathbb{Z}.$$

As in Corollary 7.3 one can prove that the Severi-Brauer variety of a division algebra of degree p^k cannot be compressed to a variety X of smaller dimension with $n_X = p^k\mathbb{Z}$.

8. Fundamental class of a smooth variety

A *partition* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ is a sequence of integers (possibly empty) $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_k > 0$. The *degree* of α is the integer

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_k.$$

Denote by $\pi(d)$ the set of all partitions α with $|\alpha| = d$ and by $|\pi(d)|$ the number of all partitions in $\pi(d)$.

Consider the polynomial ring $\mathbb{Z}[b_1, b_2, \dots] = \mathbb{Z}[\mathbf{b}]$ as a graded ring with $\deg b_i = i$. For every partition α set

$$b_\alpha = b_{\alpha_1} b_{\alpha_2} \dots b_{\alpha_k}.$$

The monomials b_α form a basis of the polynomial ring over $\mathbb{Z}$, and more precisely, the b_α with $|\alpha| = d$ form a basis of $\mathbb{Z}[\mathbf{b}]_d$.

Example 8.1. $\mathbb{Z}[\mathbf{b}]_0 = \mathbb{Z}$, $\mathbb{Z}[\mathbf{b}]_1 = \mathbb{Z}b_1$, $\mathbb{Z}[\mathbf{b}]_2 = \mathbb{Z}b_2 \oplus \mathbb{Z}b_1^2$, $\mathbb{Z}[\mathbf{b}]_3 = \mathbb{Z}b_3 \oplus \mathbb{Z}b_1 b_2 \oplus \mathbb{Z}b_1^3$.

Consider another polynomial ring $\mathbb{Z}[c_1, c_2, \dots] = \mathbb{Z}[\mathbf{c}]$ with similar grading $\deg c_i = i$. The elements of $\mathbb{Z}[\mathbf{c}]$ we call *characteristic classes* and the c_n - the *Chern classes.*

For any partition α we define the "smallest" symmetric polynomial

$$P_\alpha(x_1, x_2, \dots) = \sum_{(i_1, i_2, \dots, i_k)} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_k}^{\alpha_k} = Q_\alpha(\sigma_1, \sigma_2, \dots),$$

containing the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_k^{\alpha_k}$ where the σ_i are the standard symmetric functions, and set

$$c_\alpha = Q_\alpha(c_1, c_2, \dots).$$

Example 8.2. 1. $c_n = c_{(1,1,\dots,1)}$ (n units).

2. The classes $s_d = c_{(d)}$ are called the *additive* characteristic classes. Note that $Q_{(d)}$ is the *Newton polynomial.*

The characteristic classes c_α with $|\alpha| = d$ form a basis of $\mathbb{Z}[\mathbf{c}]_d$. We consider a pairing between $\mathbb{Z}[\mathbf{b}]_d$ and $\mathbb{Z}[\mathbf{c}]_d$ such that the b_α and c_α form dual bases. In particular, the additive class s_d corresponds to b_d.

For a smooth complete variety X of dimension d we define the *fundamental class*

$$[X] = \sum_{|\alpha|=d} \deg c_\alpha(-T_X) b_\alpha \in \mathbb{Z}[\mathbf{b}]_d.$$

Thus, $[X]$ is a homogeneous polynomial in the b_i of degree d. Note that $[X]$ does not change under field extensions. The numbers $\deg c_\alpha(-T_X)$ are called *characteristic numbers.* They are well known in topology. The class $-T_X$ plays the role of the normal bundle of an embedding of X into $\mathbb{R}^n$.

Example 8.3. 1. $[pt] = 1$;

2. For a smooth projective curve X of genus g, $[X] = 2(g-1)b_1$. In particular, $[\mathbb{P}^1] = -2b_1$.

3. $[\mathbb{P}^2] = -3b_2 + 6b_1^2$, $[\mathbb{P}^1 \times \mathbb{P}^1] = 4b_1^2$.
4. $[\mathbb{P}^3] = -4b_3 + 20b_1b_2 - 20b_1^3$, $[Q_3] = 6b_3 - 10b_1^3$.

Exercise 8.4. Show that $[X \times Y] = [X] \cdot [Y]$ for every smooth complete varieties X and Y.

9. Formal group laws

Let R be a commutative ring. A (one-dimensional, commutative) *formal group law* over R is a power series

$$\Phi(x, y) = x + y + \sum_{i,j\geq 1} r_{ij} x^i y^j,$$

with $r_{ij} \in R$ such that

(1) $\Phi\big(\Phi(x, y), z\big) = \Phi\big(x, \Phi(y, z)\big)$,
(2) $\Phi(x, y) = \Phi(y, x)$.

Example 9.1. $\Phi(x, y) = x + y$ (additive group law), $\Phi(x, y) = x + y + rxy$, $r \in R$.

Let $f : R \to S$ be a commutative ring homomorphism, Φ is a formal group law over R. Then $f(\Phi)$ is a formal group law over S. A formal group law Φ_{univ} over a ring L is called *universal* if for a any group law Φ over a ring R there is unique ring homomorphism $f : L \to R$ such that $\Phi = f(\Phi_{univ})$.

Proposition 9.2. *A universal group law exists.*

Proof. Consider the polynomial ring $\mathbb{Z}[a_{ij}]$, $i, j \geq 1$, and let L be the factor ring modulo the ideal generated by elements encoded in the definition of a formal group law. $\square$

The ring L is unique (up to canonical isomorphism) and it represents the functor

$$R \mapsto \{\text{formal group laws over } R\}$$

from the category of commutative rings to the category of sets. The ring L is called the *Lazard ring*. Note that L is generated by the coefficients of Φ_{univ}.

We would like to find a convenient model for L.

Consider the power series

$$\exp t = t + b_1t^2 + b_2t^3 + \cdots \in \mathbb{Z}[\mathbf{b}][[t]]$$

and its formal inverse

$$\log t = t + m_1t^2 + m_2t^3 + \cdots \in \mathbb{Z}[\mathbf{b}][[t]].$$

Exercise 9.3. Prove that $m_d = [\mathbb{P}^d]/(d + 1)$.

Consider the formal group law over $\mathbb{Z}[\mathbf{b}]$:

$$\Phi = \exp(\log x + \log y) = x + y + \sum_{i,j\geq 1} r_{ij} x^i y^j.$$

By the universal property, there is unique ring homomorphism $L \to \mathbb{Z}[\mathbf{b}]$ taking Φ_{univ} to Φ. The following algebraic result is well known.

Theorem 9.4. *The homomorphism $L \to \mathbb{Z}[\mathbf{b}]$ is injective and its image is generated by the coefficients of Φ.*

We shall identify the Lazard ring L with its image in $\mathbb{Z}[\mathbf{b}]$. Note that L is a graded subring in $\mathbb{Z}[\mathbf{b}]$ generated by the coefficients a_{ij}.

Example 9.5.

$$\begin{aligned}
a_{11} &= 2b_1 = -[\mathbb{P}^1],\\
a_{12} &= 3b_2 - 2b_1^2 = -[\mathbb{P}^2] - 2[\mathbb{P}^1 \times \mathbb{P}^1],\\
a_{13} &= 4b_3 - 8b_1b_2 + 4b_1^3,\\
a_{22} &= 6b_3 - 6b_1b_2 + 2b_1^3, \dots
\end{aligned}$$

Remark 9.6. The power series log is called the *logarithm* of the formal group law Φ.

It turns out that the Lazard ring is a polynomial ring $\mathbb{Z}[M_1, M_2, \dots]$ with $\deg M_i = i$, where the $M_i \in \mathbb{Z}[\mathbf{b}]$ can be chosen as follows:

$$M_i = \begin{cases} pb_i + \{\text{decomposable terms}\}, & \text{if } i = p^k - 1 \\ & \text{for some prime } p \text{ and integer } k > 0, \\ b_i + \{\text{decomposable terms}\}, & \text{otherwise.} \end{cases}$$

Example 9.7.

$$\begin{aligned}
M_1 &= 2b_1 = -[\mathbb{P}^1],\\
M_2 &= 3b_2 - 2b_1^2 = -[\mathbb{P}^2],\\
M_3 &= 2b_3 + 10b_1b_2 - 204b_1^3 = -[\mathbb{P}^3] + [Q_3],
\end{aligned}$$

Set

$$r_i = \begin{cases} p, & \text{if } i = p^k - 1 \\ & \text{for some prime } p \text{ and integer } k > 0, \\ 1, & \text{otherwise.} \end{cases}$$

For any partition α of degree d set

$$r_\alpha = r_{\alpha_1} r_{\alpha_2} \dots r_{\alpha_k} \in \mathbb{Z}.$$

Proposition 9.8. *The factor group $\mathbb{Z}[\mathbf{b}]_d / L_d$ is finite and isomorphic to $\coprod_\alpha \mathbb{Z}/r_\alpha\mathbb{Z}$, where the coproduct is taken over all partitions α of degree d. In particular, L_d is a free group of rank $|\pi(d)|$.*

Thus, $\mathbb{Z}[\mathbf{b}]_d / L_d$ is a finite group of order

$$r = \prod_{|\alpha| = d} r_\alpha = \prod_{p \text{ prime}} p^{[\frac{d}{p-1}]}.$$

This number coincides with the denominator of the Todd class td_d.

Example 9.9.

$$\begin{aligned} r_1 &= 2, \\ r_2 &= 12, \\ r_3 &= 24, \\ r_4 &= 720. \end{aligned}$$

10. Fundamental classes and Lazard ring

Let L' be the subgroup in $\mathbb{Z}[\mathbf{b}]$ generated by the classes $[X]$ for all smooth varieties X over F. Clearly, L' is a graded subring in $\mathbb{Z}[\mathbf{b}]$.

Theorem 10.1. *If the base field F is infinite, $L' = L$.*

Remark 10.2. In topology the fundamental class of a manifold takes values in L, the cobordism group of the point.

We first prove that $L \subset L'$. The Lazard ring L is generated by the coefficients a_{ij} of the universal group law Φ. Hence it is sufficient to prove that $a_{ij} \in L'$.
Let $H_{n,m}$ be a smooth hypersurface of type $(1,1)$ in $\mathbb{P}^n \times \mathbb{P}^m$.

Proposition 10.3.

$$[H_{n,m}] = \sum_{i,j \geq 0} a_{ij}[\mathbb{P}^{n-i}][\mathbb{P}^{m-j}].$$

It follows from the formula by induction on $n+m$ that $a_{nm} \in L'$.
We need to show now that $L \subset L'$. The idea (due to Stong) is to construct a fundamental class of a smooth complete variety in K-theory. We define the first Chern class of a linear bundle L over X by

$$c_1^K(L) = 1 - [L^\vee] \in K_0(X)$$

and then extend the definition to get classes $c_\alpha^K(E) \in K_0(X)$ for arbitrary partition α and vector bundle E over X.
We define the fundamental class in K-theory for a smooth projective variety X of dimension d by

$$[X]^K = \sum_\alpha f_*\big(c_\alpha^K(-T_X)\big)b_\alpha \in \mathbb{Z}[\mathbf{b}],$$

where $f_* : K_0(X) \to K_0(pt) = \mathbb{Z}$ is the push-forward homomorphism with respect to $f : X \to pt$. The class $[X]^K$ is a polynomial over $\mathbb{Z}$ of degree at most d.

Example 10.4.

$$\begin{aligned} [\mathbb{P}^1]^K &= 1 - 2b_1, \\ [\mathbb{P}^2]^K &= 1 - 3b_1 + (-3b_2 + 6b_1b_2). \end{aligned}$$

We compare two fundamental classes of X using the Chern character

$$\mathrm{ch} : K_0(X) \to \mathrm{CH}_*(X) \otimes \mathbb{Q}$$

The Riemann-Roch theorem gives

Proposition 10.5. *For a smooth projective variety X,*

$$f_*\big(c_\alpha^K(-T_X)\big) = \big(\mathrm{ch}(c_\alpha^K)td^{-1}\big)(-T_X)$$

for all α. In particular, $[X] = \big([X]^K\big)_d$.

Denote by L_d^K the subgroup of $\mathbb{Z}[\mathbf{b}]$ generated by the classes $[X]^K$ for all projective smooth varieties X of dimension d. By Proposition 10.5, the restriction on L_d^K of the projection $\mathbb{Z}[\mathbf{b}] \to \mathbb{Z}[\mathbf{b}]_d$ is injective with image L'_d. Hence L_d^K is a free abelian group of rank at most $|\pi(d)|$.

Proposition 10.6. *For every prime p and every integer d, there are varieties M_α^p for $\alpha \in \pi(d)$ such that the classes $[M_\alpha^p]^K$ in $\mathbb{Z}[\mathbf{b}]/p\mathbb{Z}[\mathbf{b}]$ are linearly independent.*

It follows from Proposition that dimension of the image of L_d^K in $\mathbb{Z}[\mathbf{b}]/p\mathbb{Z}[\mathbf{b}]$ is at least $|\pi(d)|$. Hence, L_d^K is a free abelian group of rank exactly $|\pi(d)|$ and hence L_d^K is generated by $[M_\alpha^p]^K$ for all p and α. Also, the map

$$L_d^K/pL_d^K \to \mathbb{Z}[\mathbf{b}]/p\mathbb{Z}[\mathbf{b}]$$

is an isomorphism, hence L_d^K is a direct summand of $\mathbb{Z}[\mathbf{b}]$ (Hattori-Stong theorem).

Note that $[M_\alpha^p]$ generate L'. Computing classes $[\mathbb{P}^n]^K$ and $[H_{n,m}]^K$ in terms of the classes of generators $[M_\alpha^p]^K$ we get

Corollary 10.7. *The ring L' is generated by the classes $[\mathbb{P}^n]$ and $[H_{n,m}]$ for all n and m.*

Finally one shows that the classes $[\mathbb{P}^n]$ and $[H_{n,m}]$ can be expressed in terms of the coefficients a_{ij} of the formal group law Φ, hence $L' \subset L$.

Proposition 10.8. *Let X be a smooth projective variety of dimension $d = p^k - 1$ where p is prime. Then $\deg s_d(-T_X)$ is divisible by p.*

11. General degree formulas

Let X be an algebraic proper smooth variety over F of dimension d. We define the (graded) ideal $M(X)$ in L generated by the classes $[Y] \in L$ for all smooth proper varieties Y with $\dim(Y) < d$ and such that there exists a morphism $Y \to X$. Thus, as an abelian group, $M(X)$ is generated by the classes $[Y \times Z]$ of varieties of dimension d, where $\dim Z > 0$ and there is a morphism $Y \to X$.

Conjecture 11.1. (General Degree Formula) *For any morphism $f : Y \to X$ of proper smooth varieties of dimension d,*

$$[Y] = \deg(f) \cdot [X] \in L_d/M(X)_d.$$

Remark 11.2. If $f : Y \to X$ is étale morphism, then $T_Y = f^*(T_X)$ and hence

$$[Y] = \deg(f) \cdot [X] \in L_d,$$

i.e. the degree formula holds in a stronger form. In general, the difference $[Y] - \deg(f) \cdot [X]$ can be non-trivial and is caused by the ramification of f.

Remark 11.3. In topology the statement of the conjecture is known as Quillen's theorem.

The group L_d is a subgroup in $\mathbb{Z}[\mathbf{b}]_d$ of finite index. Hence, the inclusion of dual groups

$$\mathbb{Z}[\mathbf{c}]_d = \big(\mathbb{Z}[\mathbf{b}]_d\big)^* \subset \big(L_d\big)^* = \mathrm{Hom}(L_d, \mathbb{Z})$$

is an equality after tensoring with $\mathbb{Q}$. Thus, the group $\big(L_d\big)^*$ can be identified with the group of rational characteristic classes $c \in \mathbb{Z}[\mathbf{c}]_d \otimes \mathbb{Q}$ such that $c(L_d) \subset \mathbb{Z}$, i.e. $c(-T_X) \in \mathbb{Z}$ for every smooth complete X of dimension d. We call such rational characteristic classes c *special*.

Example 11.4. By Riemann-Roch theorem, for a smooth complete variety X,

$$\deg td(T_X) = \sum_{i \geq 0} (-1)^i \dim H^i(X, \mathcal{O}_X),$$

thus, the inverse Todd class td^{-1} is special.

Let $c \in \mathbb{Z}[\mathbf{c}]_d \otimes \mathbb{Q}$ be a special characteristic class. We can consider c as a homomorphism $c : L_d \to \mathbb{Z}$. Applying c to both sides of the degree formula, we get a degree formula

$$c[Y] = \deg(f) \cdot c[X] \in \mathbb{Z}/c(M(X)_d).$$

Thus, to every special characteristic class c we have associated special degree formula.

Example 11.5. Let $c = \frac{1}{2} c_d$ (c is the Chern class). The group $M(X)_d$ is generated by the classes $[Y \times Z]$ of varieties of dimension d, where $k = \dim Z > 0$ and there is a morphism $Y \to X$. Let p and q be two projections of $Y \times Z$ on Y and Z respectively. Then

$$T_{Y \times Z} = p^*(T_Y) \otimes q^*(T_Z).$$

The highest Chern class (the Euler class) is multiplicative, hence

$$\frac{1}{2} \deg c_d(-T_{Y \times Z}) = \deg c_{d-k}(-T_Y) \cdot \frac{1}{2} \deg c_k(-T_Z).$$

Note that the second factor is integer by Theorem 6.1. Clearly, $\deg c_k(-T_Y)$ is divisible by n_Y. Since there is a morphism $Y \to X$, $n_X \mid n_Y$ by Proposition 1.4 and hence $\deg c_k(-T_Y)$ is divisible by n_X. Thus $c(M(X)_d) = n_X\mathbb{Z})$ and we get the degree formula (4.1) in the case of smooth varieties and $p = 2$.

Example 11.6. The additive integral class $s_d = c_{(d)}$ vanishes on decomposable elements in L_d. Hence if $d = p^k - 1$, we get

$$\frac{s_d(Y)}{p} \equiv \deg(f) \frac{s_d(X)}{p} \quad mod\ I(X).$$

Recently, F. Morel and M. Levine prove the conjecture over fields of characteristic zero. They used heavily resolution of singularities.

References