INVARIANTS OF ALGEBRAIC GROUPS

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For a central simple algebra A of dimension 16 over a field F ($\operatorname{char} F \neq 2$) and exponent 2 in the Brauer group of F (hence A is a *biquaternion algebra*, i.e. the tensor product of two quaternion algebras by an old theorem of Albert [1, p.369]), M. Rost has constructed an exact sequence (cf. [13])

$$0 \longrightarrow \mathrm{SK}_1(A) \xrightarrow{r} H^4(F, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^4\big(F(X), \mathbb{Z}/2\mathbb{Z}\big),$$

where X is the Albert quadric of A. This result has been used in [14] to show that there is a field extension E/F such that $\mathrm{SK}_1(A \otimes_F E) \neq 0$ provided A is a division algebra and deduce that the variety of the algebraic group $\mathbf{SL}_1(A)$ is not rational. Since $\mathrm{SK}_1(A) = \mathrm{SL}_1(A)/[A^*, A^*]$, the map r can be viewed as a natural (with respect to field extensions) collection of homomorphisms from the group of points of the algebraic group $\mathbf{SL}_1(A)$ over field extensions of F to the fourth cohomology group with coefficients $\mathbb{Z}/2\mathbb{Z}$. The result quoted above shows that if A is division, then r is a nontrivial collection of homomorphisms, being considered for all field extensions (the group $\mathrm{SK}_1(A)$ can be trivial over the base field F even if A is a division algebra).

The aim of the present paper is to study natural with respect to field extensions group homomorphisms of a given algebraic group to cohomology-like groups. More precisely, for an algebraic group G over a field F and a cycle module M (cf. [24]; for example, M can be given by Galois cohomology groups) we introduce a notion of *invariant of G in M of dimension d* as a natural transformation of functors $G \to M_d$ from the category F-$\mathfrak{fields}$ of field extensions of F to $\mathfrak{Groups}$. For example, the Rost's map r can be considered as an invariant of $\mathbf{SL}_1(A)$ of dimension 4 in the cycle module $H^*[\mathbb{Z}/2\mathbb{Z}]$.

All invariants of G in M of degree d form an abelian group, which we denote $\mathrm{Inv}^d(G, M)$. We prove (Theorem 2.3) that if a cycle module M is of bounded exponent, then the group $\mathrm{Inv}^d(G, M)$ is isomorphic to the subgroup of multiplicative elements in the unramified group $A^0(G, M_d)$.

In section 3 we consider *cohomological invariants*, i.e. invariants in the cycle module $H^*[N]$ where N is a Galois module over F and compute the groups of invariants of dimensions 0, 1 for any N and dimension 2 for $N = \mu_n^{\otimes -1}$ (Theorems 3.1, 3.4 and 3.13). It turns out that for a simply connected groups all these invariants are trivial. In section 4 we show that the group $\mathrm{Inv}^3(G, H^*[\mu_n^{\otimes -1}])$ is also trivial for a simply connected G (Proposition 4.9). Thus, the Rost's

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invariant r is an example of a nontrivial cohomological invariant of $\mathbf{SL}_1(A)$ of the smallest dimension!

In section 5 we show that the group $\mathbf{SL}_1(A)$ has no nontrivial invariants if $\operatorname{ind}(A) \leq 2$. If $\operatorname{ind}(A) = 4$ and $\exp(A) = 2$ (i.e. A is a biquaternion division algebra), the Rost's invariant appears to be the only nontrivial invariant of $\mathbf{SL}_1(A)$ (Theorem 5.4).

In section 6 we generalize Rost's theorem to the case of arbitrary central simple algebra A of dimension 16 (without any restriction on the exponent). It turns out that the Rost's exact sequence does not exist if A is of exponent 4. Nevertheless, there exists an exact sequence (Theorem 6.6)

$$0 \longrightarrow \mathrm{SK}_1(A) \xrightarrow{r} H^4(F, \mathbb{Z}/2\mathbb{Z})/\big(2[A] \cup H^2(F, \mathbb{Z}/2\mathbb{Z})\big) \longrightarrow H^4\big(F(X), \mathbb{Z}/2\mathbb{Z}\big),$$

where X is the generalized Severi-Brauer variety $SB(2, A)$. The map r in this sequence can be considered as an invariant of $\mathbf{SL}_1(A)$ in the cycle module

$$M_d(E) = H^d(E, \mathbb{Z}/2\mathbb{Z})/\big(2[A_E] \cup H^{d-2}(E, \mathbb{Z}/2\mathbb{Z})\big),$$

which is not a cohomological cycle module. This statement motivates the definition of invariants in arbitrary cycle modules (not only cohomological ones).

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1. Notation and preliminary results

1.1. **Algebraic groups.** An *algebraic group* over a field F is a smooth affine group scheme of finite type over F. The category of all algebraic groups over F is denoted F-$\mathfrak{groups}$.

We write F-$\mathfrak{alg}$ for the category of commutative F-algebras and $\mathfrak{Groups}$ for the category of groups. We consider an algebraic group G as a functor $G : F\text{-}\mathfrak{alg} \to \mathfrak{Groups}$, taking a commutative F-algebra A to the *group of A-points* $G(A) = \operatorname{Mor}\big(\operatorname{Spec}(A), G\big)$. If A is an F-subalgebra of B, we identify $G(A)$ with a subgroup of $G(B)$.

Denote $F[G]$ the F-algebra of regular functions on G and $F(G)$ the field of rational functions if G is connected.

$G^* = \operatorname{Hom}(G, \mathbb{G}_m)$ is the character group of G over F.

Any point g of the scheme G defines an element of the group $G\big(F(g)\big)$, which we also denote g. If G is connected, the generic element of G defines an element ξ of $G\big(F[G]\big)$. We also write ξ for its image in $G\big(F(G)\big)$.

F-$\mathfrak{fields}$ is the full subcategory of F-$\mathfrak{alg}$ consisting of fields.

$\mathfrak{Ab}$ is the category of abelian groups.

Let G be an algebraic group over a field F. Denote O the power series ring $F[[t]]$ and O_m the factor ring $O/t^m O$. For any element g of $G(O)$ or $G(O_m)$, $m \geq 1$, we write $g(0)$ for its image in $G(O_1) = G(F)$.

Lemma 1.1. *For any element $g \in G(O)$ such that $g(0) = 1$ and any $n \in \mathbb{N}$, prime to* $\operatorname{char} F$, *there exists $f \in G(O)$ such that $f(0) = 1$ and $g = f^n$.*

Proof. It suffices to show that any element $h \in G(O_m)$, $m \geq 1$, such that $h(0) = 1$ and the image of g in $G(O_m)$ equals h^n, can be lifted to an element $h_1 \in G(O_{m+1})$ such that the image of g in $G(O_{m+1})$ equals h_1^n. Consider the following exact sequence

$$1 \longrightarrow \mathrm{Lie}(G) \longrightarrow G(O_{m+1}) \longrightarrow G(O_m) \longrightarrow 1.$$

The conjugation action of h on $\mathrm{Lie}(G)$ is given by the adjoint transformation $\mathrm{Ad}\big(h(0)\big)$, which is trivial since $h(0) = 1$. Hence any lifting of h to $G(O_{m+1})$ centralizes $\mathrm{Lie}(G)$, therefore the existence of the lifting we need follows from the fact that the group $\mathrm{Lie}(G)$ is n-divisible (being a vector space over F). □

1.2. **Cycle modules.** A *cycle module* over a field F is an object function $M : F\text{-}\mathfrak{fields} \to \mathfrak{Ab}$ together with a $\mathbb{Z}$-grading $M = \coprod_n M_n$ and with some data and rules (cf. [24]). The data includes a graded module structure on M under the Milnor ring $K_*(F)$, a degree 0 homomorphism $\alpha_* : M(E) \to M(K)$ for any $\alpha : E \to K$ in $F\text{-}\mathfrak{fields}$ and also a degree -1 *residue homomorphism* $\partial_v : M(E) \to M\big(\kappa(v)\big)$ for a valuation v on E (here $\kappa(v)$ is the residue field). We will always assume that $M_n = 0$ for $n < 0$.

Example 1.2. (cf. [24, Rem. 1.11]) Let N be a discrete torsion module over the absolute Galois group of a field F. Set $N[i] = \varinjlim({}_nN) \otimes \mu_n^{\otimes i}$, where μ_n is the group of n-th roots of unity. Define a *cohomological* cycle module $M_* = H^*[N]$ by

$$M_d(L) = H^d[N](L) = H^d(L, N[d]).$$

Let M be a cycle module over F and let K be a discrete valuation field over F with valuation v trivial on F and residue field L. A choice of a prime element $\pi \in K$ determines a *specialization homomorphism* $s_v^\pi : M(K) \to M(L)$ (cf. [24, p.329]).

Let X be an algebraic variety over F. For any i the homology group of the complex

$$\coprod_{x \in X^{i-1}} M_{d-i+1}\big(F(x)\big) \xrightarrow{\partial} \coprod_{x \in X^{i}} M_{d-i}\big(F(x)\big) \xrightarrow{\partial} \coprod_{x \in X^{i+1}} M_{d-i-1}\big(F(x)\big)$$

we denote $A^i(X, M_d)$. In particular, $A^0\big(\mathrm{Spec}(F), M_d\big) = M_d(F)$.

A morphism $f : X \to Y$ of smooth varieties over F induces *inverse image homomorphism* (*pull-back*)(cf. [24, Sec.12])

$$p^* : A^i(Y, M_d) \to A^i(X, M_d).$$

Lemma 1.3. *Let X be a smooth algebraic variety over a field F, having an F-point. Then the natural homomorphism $M(F) \to M\big(F(X)\big)$ is an injection.*

Proof. The homomorphism $M(F) \to M\big(F(X)\big)$ factors as

$$M(F) \longrightarrow A^0(X, M_d) \hookrightarrow M\big(F(X)\big).$$

The first homomorphism splits by the pull-back with respect to any F-rational point of X. □

1.3. **The simplicial scheme BG.** For an algebraic group G over a field F we consider the simplicial scheme BG with $BG_n = G^n$ and the face maps $\partial_i : G^n \to G^{n-1}$, $i = 0, 1, \dots, n$, given by the formulae

$$\partial_i(g_1, g_2, \dots, g_n) = \begin{cases} (g_2, g_3, \dots, g_n) & \text{if } i = 0, \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & \text{if } 0 < i < n, \\ (g_1, g_2, \dots, g_{n-1}) & \text{if } i = n. \end{cases}$$

Let $\mathcal{A}$ be a full subcategory in F-$\mathfrak{groups}$ closed under products (for example, a subcategory of reductive groups) and let $\mathcal{F} : \mathcal{A} \to \mathfrak{Ab}$ be a contravariant functor. The homology groups of the complex $C^n = \mathcal{F}(G^n)$ with the differentials

$$d_{n-1} : C^{n-1} \to C^n, \qquad d_{n-1} = \sum_{i=0}^{i=n} (-1)^i \mathcal{F}(\partial_i)$$

we denote $H^*(BG, \mathcal{F})$.

Clearly, $H^0(BG, \mathcal{F}) = \mathcal{F}(1)$. The group $H^1(BG, \mathcal{F})$ coincides with the subgroup in $\mathcal{F}(G)$ consisting of all elements x such that

$$\mathcal{F}(p_1)(x) + \mathcal{F}(p_2)(x) = \mathcal{F}(m)(x), \tag{1}$$

where $p_i : G \times G \to G$ are two projections and $m : G \times G \to G$ is the multiplication morphism. We simply denote this subgroup $\mathcal{F}(G)_{\mathrm{mult}}$ and call the *multiplicative part of* $\mathcal{F}(G)$.

Example 1.4. Let A be an abelian group. Consider the functor $\mathcal{F}(G) = \mathrm{Map}\big(G(F), A\big)$. Then $H^n(BG, \mathcal{F}) = H^n\big(G(F), A\big)$ the cohomology group of $G(F)$ with coefficients in A considered as a trivial $G(F)$-module. In particular, $\mathcal{F}(G)_{\mathrm{mult}} = \mathrm{Hom}\big(G(F), A\big)$.

We write $\mathcal{F}_{\mathrm{mult}} : \mathcal{A} \to \mathfrak{Ab}$ for the functor given by $\mathcal{F}_{\mathrm{mult}}(G) = \mathcal{F}(G)_{\mathrm{mult}}$.

A contravariant functor $\mathcal{F} : \mathcal{A} \to \mathfrak{Ab}$ is called *constant*, if $\mathcal{F}(\alpha) : \mathcal{F}(H) \to \mathcal{F}(G)$ is an isomorphism for any group homomorphism $\alpha : G \to H$ in $\mathcal{A}$ and called *additive* if for any pair of algebraic groups G and H in $\mathcal{A}$, the homomorphism

$$\mathcal{F}(p_1) \oplus \mathcal{F}(p_2) : \mathcal{F}(G) \oplus \mathcal{F}(H) \to \mathcal{F}(G \times H)$$

is an isomorphism.

Lemma 1.5. (i) *The functor $\mathcal{F}_{\mathrm{mult}}$ is additive for any $\mathcal{F}$.*
(ii) *If $\mathcal{F}$ is additive, then $\mathcal{F}_{\mathrm{mult}} = \mathcal{F}$.*

Proof. (i) Let G and H be two groups in $\mathcal{A}$. We have to show that the map

$$\mathcal{F}(p_1) \oplus \mathcal{F}(p_2) : \mathcal{F}(G)_{\mathrm{mult}} \oplus \mathcal{F}(H)_{\mathrm{mult}} \to \mathcal{F}(G \times H)_{\mathrm{mult}}$$

is an isomorphism.

Consider the embeddings $i : G \to G \times H$, $i(g) = (g, 1)$ and $j : H \to G \times H$, $j(h) = (1, h)$. The composites $p_2 \circ i$ and $p_1 \circ j$ factor through a trivial group. Since $\mathcal{F}(1)_{\mathrm{mult}} = 0$, it follows that $\mathcal{F}(i) \circ \mathcal{F}(p_2) = 0 = \mathcal{F}(j) \circ \mathcal{F}(p_1)$. Hence the map $\mathcal{F}(p_1) \oplus \mathcal{F}(p_2)$ is a split injection. It remains to show that the restriction of $\ker \mathcal{F}(i) \cap \ker \mathcal{F}(j)$ on $\mathcal{F}(G \times H)_{\mathrm{mult}}$ is trivial.

Consider the following commutative diagram:

$$\begin{array}{ccc} G \times H & \xrightarrow{p_1} & G \\ \big\downarrow{\scriptstyle k} & & \big\downarrow{\scriptstyle i} \\ G \times H \times G \times H & \xrightarrow{q_1} & G \times H, \end{array}$$

where q_1 is the first projection and $k(g,h) = (g,1,1,h)$. For any $x \in \ker \mathcal{F}(i)$, $\mathcal{F}(k) \circ \mathcal{F}(q_1)(x) = \mathcal{F}(p_1) \circ \mathcal{F}(i)(x) = 0$. Similarly, $\mathcal{F}(k) \circ \mathcal{F}(q_2)(x) = 0$, where q_2 is the second projection of $G \times H \times G \times H$ onto $G \times H$. If $x \in \mathcal{F}(G \times H)_{\mathrm{mult}}$, then $\mathcal{F}(q_1)(x) + \mathcal{F}(q_2)(x) = \mathcal{F}(m)(x)$, where m is the multiplication morphism for $G \times H$. Finally, $m \circ k = \mathrm{id}_{G \times H}$, hence

$$x = \mathcal{F}(\mathrm{id})(x) = \mathcal{F}(k) \circ \mathcal{F}(m)(x) = \mathcal{F}(k) \circ \mathcal{F}(q_1)(x) + \mathcal{F}(k) \circ \mathcal{F}(q_2)(x) = 0.$$

(ii) Now let $\mathcal{F}$ be additive. In the notation of the first part of the proof take $H = G$. Since $m \circ i = \mathrm{id}_G = m \circ j$, the composite $\big(\mathcal{F}(i), \mathcal{F}(j)\big) \circ \mathcal{F}(m)$ is the diagonal map $\mathcal{F}(G) \to \mathcal{F}(G) \oplus \mathcal{F}(G)$. But $\big(\mathcal{F}(i), \mathcal{F}(j)\big)$ is the inverse of $\mathcal{F}(p_1) \oplus \mathcal{F}(p_2)$. Hence $\mathcal{F}(p_1) + \mathcal{F}(p_2) = \mathcal{F}(m)$ and therefore $\mathcal{F}(G)_{\mathrm{mult}} = \mathcal{F}(G)$. □

Corollary 1.6. *A functor $\mathcal{F}$ is additive if and only if $\mathcal{F}_{\mathrm{mult}} = \mathcal{F}$.* □

Lemma 1.7. [8, Lemma 4.5] *Let $\mathcal{F} : \mathcal{A} \to \mathfrak{Ab}$ be a functor. Then for any G in $\mathcal{A}$,*
(i) If $\mathcal{F}$ is constant, then

$$H^i(BG, \mathcal{F}) = \begin{cases} \mathcal{F}(1) & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

(ii) If $\mathcal{F}$ is additive, then

$$H^i(BG, \mathcal{F}) = \begin{cases} \mathcal{F}(G) & \text{if } i = 1, \\ 0 & \text{if } i \neq 1. \end{cases} \quad \square$$

Corollary 1.8. *Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be an exact sequence of functors. If $\mathcal{F}_1$ and $\mathcal{F}_3$ are additive functors on $\mathcal{A}$, then $\mathcal{F}_2$ is also additive on $\mathcal{A}$.* □

Let $u : 1 \to G$ be the unit morphism. For any functor $\mathcal{F} : \mathcal{A} \to \mathfrak{Ab}$ we write $\widetilde{\mathcal{F}}(G)$ for the kernel of $\mathcal{F}(u) : \mathcal{F}(G) \to \mathcal{F}(1)$.

Lemma 1.9. *$\mathcal{F}(G)_{\mathrm{mult}} \subset \widetilde{\mathcal{F}}(G)$.*

Proof. Let $u' : 1 \to G \times G$ be the unit morphism. Applying $\mathcal{F}(u')$ to both sides of (1), we get $2\mathcal{F}(u)(x) = \mathcal{F}(u)(x)$. □

2. Definition of an invariant

In this section we give definition of invariant of an algebraic group and compute the group of invariants in simple cases.

2.1. **Definition.** Let G be an algebraic group over a field F and M be a ($\mathbb{Z}$-graded) cycle module over F. For any $d \in \mathbb{Z}$, we consider M_d as a functor from F-$\mathfrak{fields}$ to $\mathfrak{Groups}$. An *invariant of* G *in* M *of dimension* d is a natural transformation of functors $G \to M_d$ from the category F-$\mathfrak{fields}$ to $\mathfrak{Groups}$.

All invariants of G in M of degree d form an abelian group, which we denote $\operatorname{Inv}^d(G, M)$. For any $u \in \operatorname{Inv}^d(G, M)$ and any $L \in F$-$\mathfrak{fields}$, we write u_L for the corresponding homomorphism $G(L) \to M_d(L)$.

An algebraic group homomorphism $G \to H$ induces a homomorphism

$$\operatorname{Inv}^d(H, M) \to \operatorname{Inv}^d(G, M)$$

making $G \mapsto \operatorname{Inv}^d(G, M)$ a (contravariant) functor from F-$\mathfrak{groups}$ to $\mathfrak{Ab}$.

Clearly, for two groups G and H over F there is a canonical isomorphism

$$\operatorname{Inv}^d(G \times H, M) \simeq \operatorname{Inv}^d(G, M) \oplus \operatorname{Inv}^d(H, M),$$

i.e. the functor $\operatorname{Inv}^d(*, M)$ is additive.

Suppose that G is a connected group. Set $K = F(G)$, the function field of G and denote $\xi \in G(K)$ the generic point of G. For any invariant $u \in \operatorname{Inv}^d(G, M)$, $u_K(\xi)$ is an element of $M_d(K)$, so that we have a homomorphism

$$\theta : \operatorname{Inv}^d(G, M) \to M_d(K), \qquad u \mapsto u_K(\xi).$$

Lemma 2.1. *The image of* θ *is contained in* $A^0(G, M_d)$.

Proof. Let $P = F(G \times G)$ and let $\bar{p}_1$, $\bar{p}_2$ and $\bar{m}$ be the F-homomorphisms of fields $K \to P$ induced respectively by the first, the second projections and by the multiplication morphism $p_1, p_2, m : G \times G \to G$. Consider the elements $\xi_1 = G(\bar{p}_1)(\xi)$, $\xi_2 = G(\bar{p}_2)(\xi)$ and $\xi' = G(\bar{m})(\xi)$ in the group $G(P)$. Clearly, $\xi' = \xi_1 \cdot \xi_2$.

Let $u \in \operatorname{Inv}^d(G, M)$ be any invariant. We have

$$u_P(\xi') = u_P(\xi_1) + u_P(\xi_2) \in M_d(P).$$

We need to show that $\partial_g\big(u_L(\xi)\big) = 0 \in M_{d-1}\big(F(g)\big)$ for any point g in G of codimension 1. Let h be the point of codimension 1 in $G \times G$ such that $\overline{\{h\}} = \overline{\{g\}} \times G$. The projection p_2 takes h to the generic point of G. Hence by the rule R3c in [24, p.329], $\partial_h\big(u_P(\xi_2)\big) = 0 \in M_{d-1}\big(F(h)\big)$. By the same reasoning, $\partial_h\big(u_P(\xi')\big) = 0$. Hence,

$$\partial_h\big(u_P(\xi_1)\big) = \partial_h\big(u_P(\xi')\big) - \partial_h\big(u_P(\xi_2)\big) = 0 \in M_{d-1}\big(F(h)\big).$$

Let $k : F(g) \to F(h)$ be the field homomorphism induced by the projection p_1. By the rule R3a in [24, p.329], applied to the field extension $\bar{p}_1 : K \to P$, we have

$$k_*\big(\partial_g(u_K(\xi))\big) = \partial_h\big(p_{1*}(u_K(\xi))\big) = \partial_h\big(u_P(\xi_1)\big) = 0 \in M_{d-1}\big(F(h)\big).$$

The field $F(h)$ is isomorphic to $F(g)(G)$. By Lemma 1.3, the homomorphism $k_* : M_{d-1}\big(F(g)\big) \to M_{d-1}\big(F(h)\big)$ is injective and hence $\partial_g\big(u_L(\xi)\big) = 0 \in M_{d-1}\big(F(g)\big)$. $\square$

We say that a cycle module M is *of bounded exponent*, if there exists $n \in N$, prime to $\operatorname{char} F$, such that $nM = 0$.

Lemma 2.2. *Let M be a cycle module of bonded exponent and let g_1 and g_2 be two points of G such that g_2 is regular and of codimension 1 in $\overline{\{g_1\}}$. Suppose that for an invariant $u \in \operatorname{Inv}^d(G, M)$ we have $u_{F(g_1)}(g_1) = 0 \in M_d\big(F(g_1)\big)$. Then $u_{F(g_2)}(g_2) = 0 \in M_d\big(F(g_2)\big)$.*

Proof. Denote A the local ring of the point g_2 in the variety $\overline{\{g_1\}}$. By assumption, A is a discrete valuation ring with fraction field $F(g_1)$ and residue field $F(g_2)$. Denote $l : A \to F(g_2)$ the natural surjection. The image of the generic element under $G(F[G]) \to G(A)$, induced by the natural ring homomorphism $F[G] \to A$, equals $g_1 \in G(A) \subset G\big(F(g_1)\big)$. Clearly $G(l)(g_1) = g_2$. We write $\widetilde{A}$ for the completion of A with respect to the natural discrete valuation. By a theorem of Cohen (cf. [30, Ch. VIII, Th.27]), there is a section $p : F(g_2) \to \widetilde{A}$ of the natural surjection $q : \widetilde{A} \to F(g_2)$ and $\widetilde{A} \simeq F(g_2)[[t]]$. Set $\tilde{g}_1 = G(p)(g_2) \in G(\widetilde{A})$. Then

$$G(q)(g_1) = G(l)(g_1) = g_2 = G(q)(\tilde{g}_1),$$

hence $g_1 \cdot \tilde{g}_1^{-1}$ belongs to the kernel of $G(q)$.

Choose $n \in N$, prime to $\operatorname{char} F$, such that $nM_d = 0$. By Lemma 1.1, $g_1 \cdot \tilde{g}_1^{-1} = h^n$ for some $h \in G(\widetilde{A})$.

Let L be the fraction field of $\widetilde{A}$ and let i be the embedding of $F(g_2)$ into L. We have

$$0 = u_L(g_1) = u_L(h^n \cdot \tilde{g}_1) = u_L(\tilde{g}_1) = i_*\big(u_{F(g_2)}(g_2)\big) \in M_d(L).$$

Since $L \simeq F(g_2)((t))$, the specialization homomorphism s_v^t for the discrete valuation v on L splits i_*, hence i_* is injective and $u_{F(g_2)}(g_2) = 0$. □

The following statement is useful for computations of invariants.

Theorem 2.3. *Let G be a connected algebraic group over F and M be a cycle module over F of bounded exponent. Then the map θ induces an isomorphism*

$$\operatorname{Inv}^d(G, M) \xrightarrow{\sim} A^0(G, M_d)_{\mathrm{mult}}.$$

Proof. It follows from the proof of Lemma 2.1 that the image of θ belongs to $A^0(G, M)_{\mathrm{mult}}$.

Injectivity. Assume that for $u \in \operatorname{Inv}^d(G, M)$ we have $u_{F(G)}(\xi) = 0$. For a field extension L/F take any $h \in G(L)$, i.e. a morphism $h : \operatorname{Spec}(L) \to G$. We have to show that $u_L(h) = 0$. Denote $g \in G$ the only point in the image of h. There is a sequence of points $\xi = g_1, g_2, \dots, g_m = g$ such that g_{i+1} is regular and of codimension 1 in $\overline{\{g_i\}}$ for all $i = 1, 2, \dots, n-1$. By Lemma 2.2, $u_{F(g)}(g) = 0$. The element h is the image of g under $G\big(F(g)\big) \to G(L)$, induced by the natural homomorphism $F(g) \to L$, hence $u_L(h) = 0$, being the image of $u_{F(g)}(g)$ under $M_d\big(F(g)\big) \to M_d(L)$.

Surjectivity. Let $a \in A^0(G, M_d)_{\mathrm{mult}}$. For any $L \in F\text{-}\mathfrak{fields}$ we define a homomorphism $u_L : G(L) \to M_d(L)$ by the formula $u_L(g) = g^*(a)$, where

$$g^* : A^0(G, M_d) \to A^0\big(\mathrm{Spec}(L), M_d\big) = M_d(L)$$

is the inverse image map with respect to $g : \mathrm{Spec}(L) \to G$.

We show first that u_L is a homomorphism. Let $g_1, g_2 \in G(L)$. Denote g the morphism $(g_1, g_2) : \mathrm{Spec}(L) \to G \times G$. We have $p_i \circ g = g_i$ and $m \circ g = g_1 g_2$. By definition of the multiplicative part of $A^0(G, M_d)$,

$$p_1^*(a) + p_2^*(a) = m^*(a) \in A^0(G \times G, M_d).$$

Hence

$$u_L(g_1 g_2) = (g_1 g_2)^*(a) = g^*\big(m^*(a)\big) = g^*\big(p_1^*(a) + p_2^*(a)\big) = \\ g_1^*(a) + g_2^*(a) = u_L(g_1) + u_L(g_2),$$

i.e. u_L is a homomorphism.

Let $\alpha : L \to E$ be a F-homomorphism of fields. Denote $f : \mathrm{Spec}(E) \to \mathrm{Spec}(F)$ the corresponding morphism. Then for any $g \in G(L)$ one has

$$(\alpha_* \circ u_L)(g) = \alpha_*\big(g^*(a)\big) = f^*\big(g^*(a)\big) = (gf)^*(a) = u_E(gf) = \big(u_E \circ G(\alpha)\big)(g),$$

i.e. the following diagram is commutative:

$$\begin{array}{ccc} G(L) & \xrightarrow{u_L} & M(L) \\ \big\downarrow{\scriptstyle G(\alpha)} & & \big\downarrow{\scriptstyle \alpha_*} \\ G(E) & \xrightarrow{u_E} & M(E). \end{array}$$

Hence u is a functor from $F\text{-}\mathfrak{fields}$ to $\mathfrak{Groups}$, i.e. $u \in \mathrm{Inv}^d(G, M)$.

Finally it suffices to show that $\theta(u) = a$. By definition of θ, $\theta(u) = u_{F(G)}(\xi) = \xi^*(a)$ and the latter is equal to a, since the inverse image homomorphism

$$\xi^* : A^0(G, M_d) \to A^0\big(\mathrm{Spec}(F(G), M_d\big) = M_d\big(F(G)\big)$$

is the natural inclusion by [24, Cor. 12.4]. □

2.2. **Invariants of unipotent groups.** We show that unipotent groups have no nontrivial invariants.

Proposition 2.4. *Let G be a unipotent group. Then $\mathrm{Inv}^d(G, M) = 0$ for any M of bounded exponent.*

Proof. For the proof we may assume that G is connected, since G/G^0 is a p-group, $p = \mathrm{char}\, F$ and the cycle module M has exponent prime to p. If F is a perfect field, then by [6, 15.13], the variety of G is isomorphic to an affine space, hence $\widetilde{H}^0(G, M_d) = 0$ and $\mathrm{Inv}^d(G, M) = 0$ by Theorem 2.3 and Lemma 1.9. In the general case consider the perfect field $L = F^{p^{-\infty}}$, where $p = \mathrm{char}\, F$. Since the natural map $M(F) \to M(L)$ is injective (exponent of M is prime to p), and any invariant is trivial on $G(L)$, it follows that it is trivial on $G(F) \subset G(L)$. □

Let G be a connected algebraic group over a perfect field F. The unipotent radical U of G is then defined over F. Denote $\overline{G}$ the reductive factor group G/U. For any $L \in F\text{-}\mathfrak{alg}$ the natural homomorphism $G(L) \to \overline{G}(L)$ is surjective since $H^1(L, U) = 1$ (cf. [29, 18.2]). Thus, by Proposition 2.4, the natural homomorphism

$$\operatorname{Inv}^d(\overline{G}, M) \longrightarrow \operatorname{Inv}^d(G, M)$$

is an isomorphism. This reduces computation of invariants to the case of reductive groups.

2.3. **Invariants of split reductive groups.** Let G be a split reductive group over a field F. A character $\chi \in G^*$ and an element $x \in M_{d-1}(F)$ define an invariant $u^{\chi,x}$ as follows. For any $L \in F\text{-}\mathfrak{fields}$ and any $g \in G(L)$, we set $u_L^{\chi,x}(g) = i_*(x) \cdot \{\chi(g)\} \in M_d(L)$, where $i : F \to L$ is the natural homomorphism.

Proposition 2.5. *Assume that the derived subgroup G' in G is simply connected. Then the map $G^* \otimes M_{d-1}(F) \to \operatorname{Inv}^d(G, M)$, $\chi \otimes x \to u^{\chi,x}$, is an isomorphism.*

Proof. Consider first the case $G = \mathbb{G}_m$. By [24, Prop. 2.2],

$$A^0(\mathbb{G}_m, M_d) = M_d(F) \oplus M_{d-1}(F) \cdot \{t\},$$

hence $\widetilde{A}^0(\mathbb{G}_m, M_d) = M_{d-1}(F) \cdot \{t\}$. The inclusion

$$\operatorname{Inv}^d(G, M) \simeq A^0(\mathbb{G}_m, M_d)_{\text{mult}} \hookrightarrow \widetilde{A}^0(\mathbb{G}_m, M_d) \simeq M_{d-1}(F)$$

is then the inverse to the homomorphism in question. Since the functor $G \mapsto \operatorname{Inv}^d(G, M)$ is additive, the results holds for any split torus G.

Now consider an arbitrary split reductive group G over F with the simply connected derived subgroup G'. Denote S the torus G/G'. By [8, 3.20], the natural homomorphism

$$A^0(S, M_d) \longrightarrow A^0(G, M_d)$$

is an isomorphism. Hence the left vertical homomorphism in the diagram

$$\begin{array}{ccccc} A^0(S, M_d)_{\text{mult}} & \xrightarrow{\sim} & \operatorname{Inv}^d(S, M) & \longleftarrow & S^* \otimes M_{d-1}(F) \\ \downarrow & & \downarrow & & \downarrow \\ A^0(G, M_d)_{\text{mult}} & \xrightarrow{\sim} & \operatorname{Inv}^d(G, M) & \longleftarrow & G^* \otimes M_{d-1}(F) \end{array}$$

is an isomorphism. By Theorem 2.3, the middle vertical map is also an isomorphism. Then we are reduced to the considered case of a split algebraic tori, since $S^* \simeq G^*$. □

Corollary 2.6. *Let G be a split simply connected semisimple group. Then* $\operatorname{Inv}^d(G, M) = 0$*, i.e. G has no nontrivial invariants.* □

Example 2.7. The determinant map defines isomorphisms

$$\operatorname{Inv}^d(\mathbf{GL}_n, M) \xrightarrow{\sim} \operatorname{Inv}^d(\mathbb{G}_m, M) \simeq M_{d-1}(F). \quad \square$$

3. Cohomological invariants

A *cohomological invariant* of an algebraic group G defined over a filed F is an invariant in a cycle module $H^*[N]$, where N is a torsion module over the absolute Galois group Γ of F.

For a variety X over F and a torsion Galois module N we set

$$A^i\big(X, \mathcal{H}^j(N)\big) = A^i\big(X, H^j[N[-j]]\big).$$

There is a Bloch-Ogus spectral sequence (cf. [5])

$$E_2^{p,q} = A^p\big(X, \mathcal{H}^q(N)\big) \Rightarrow H_{et}^{p+q}(X, N). \tag{2}$$

3.1. **Invariants of dimension 0.** Let module N be of bounded exponent, i.e. $nN = 0$ for some n prime to $\operatorname{char} F$. The corresponding cycle module $H^*[N]$ is also of bounded exponent.

We describe all invariants of dimension 0 of an algebraic group G. Denote G^0 the connected component of the unity in G and set $\hat{G} = G/G^0$, so that $\hat{G}$ is an étale group (cf. [29]). The absolute Galois group Γ of F acts naturally on the group $\hat{G}(F_{\mathrm{sep}})$. Let $f : \hat{G}(F_{\mathrm{sep}}) \to N$ be a homomorphism of Γ-modules. We write u_F^f for the composite

$$G(F) \to \hat{G}(F) = \hat{G}(F_{\mathrm{sep}})^\Gamma \xrightarrow{f} N^\Gamma = H^0(F, N).$$

Similarly, one defines u_L^f for any $L \in F$-$\mathfrak{fields}$. Clearly, u^f is an invariant of dimension 0.

Theorem 3.1. *For a Γ-module N of bounded exponent, the map*

$$\alpha : \operatorname{Hom}_\Gamma\big(\hat{G}(F_{\mathrm{sep}}), N\big) \to \operatorname{Inv}^0(G, H^*[N]),$$

$\alpha(f) = u^f$, is an isomorphism.

Proof. Injectivity. Assume that $\alpha(f) = u^f = 0$. If $L = F_{\mathrm{sep}}$, then the map $G(L) \to \hat{G}(L)$ is surjective, hence $f = 0$.

Surjectivity. Choose $n \in \mathbb{N}$ such that $(n, \operatorname{char} F) = 1$ and $nN = 0$. The group G^0 is connected, hence $G^0(L)$ is n-divisible for $L = F_{\mathrm{sep}}$. Therefore, for any invariant $u \in \operatorname{Inv}^0(G, H^*[N])$, the map

$$u_L : G(L) \to H^0(L, N) = N$$

factors through a homomorphism $f : \hat{G}(L) = G(L)/G^0(L) \to N$. Clearly, f is Γ-equivariant and $\alpha(f) = u$. □

Corollary 3.2. *If G is connected, then* $\operatorname{Inv}^0(G, H^*[N]) = 0$. □

3.2. **Invariants of dimension** 1. Let G be a connected group over F, and let N be a Galois module of bounded exponent. The Bloch-Ogus spectral sequence (2) for the module $N[1]$ shows that the natural homomorphism

$$H_{et}^1(G, N[1]) \to A^0\big(G, \mathcal{H}^1(N[1])\big) = A^0(G, H^1[N])$$

is an isomorphism.

Theorem 2.3 can be then reformulated as follows.

Proposition 3.3. *There is a natural isomorphism*

$$\mathrm{Inv}^1(G, H^*[N]) \xrightarrow{\sim} H^1_{et}(G, N[1])_{\mathrm{mult}}. \qquad \square$$

The Hochschild-Serre spectral sequence

$$H^p\big(F, H^q(G_{\mathrm{sep}}, N[1])\big) \Rightarrow H^{p+q}(G, N[1])$$

induces an exact sequence

$$0 \longrightarrow H^1(F, N[1]) \longrightarrow H^1_{et}(G, N[1]) \longrightarrow H^0\big(F, H^1_{et}(G_{\mathrm{sep}}, N[1])\big) \longrightarrow \\ H^2(F, N[1]) \longrightarrow H^2_{et}(G, N[1]).$$

The latter homomorphism is a split injection, hence

$$\widetilde{H}^1_{et}(G, N[1]) \simeq H^0\big(F, H^1_{et}(G_{\mathrm{sep}}, N[1])\big). \tag{3}$$

The Kummer exact sequence for n prime to $\operatorname{char} F$,

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \longrightarrow 1$$

gives rise to the following exact sequence:

$$0 \longrightarrow F[G]^\times / F[G]^{\times n} \longrightarrow H^1_{et}(G, \mu_n) \longrightarrow {}_n\operatorname{Pic}(G) \longrightarrow 0.$$

By [22], $F[G]^\times = F^\times \oplus G^*$. Hence we have the following exact sequence

$$0 \longrightarrow G^*/G^{*n} \longrightarrow \widetilde{H}^1_{et}(G, \mu_n) \longrightarrow {}_n\operatorname{Pic}(G) \longrightarrow 0. \tag{4}$$

If G is reductive, by [25, Lemma 6.9(i)], the functor $G \mapsto \operatorname{Pic}(G_{\mathrm{sep}})$ is additive on the category of reductive groups. The functor $G \mapsto G^*_{\mathrm{sep}}$ is clearly also additive. Hence, by Corollary 1.8, the functor $G \mapsto \widetilde{H}^1_{et}(G, \mu_n)$ is additive. Over F_{sep} the module $N[1]$ is isomorphic to a direct sum of μ_n's for various n, hence the functor $H^1_{et}(G_{\mathrm{sep}}, N[1])$ is also additive. By (3), $\widetilde{H}^1_{et}(G, N[1])$ is an additive functor. Lemma 1.5 then gives

$$\widetilde{H}^1_{et}(G, N[1]) = \widetilde{H}^1_{et}(G, N[1])_{\mathrm{mult}} = H^1_{et}(G, N[1])_{\mathrm{mult}}. \tag{5}$$

Denote $\operatorname{Pic}^*(G_{\mathrm{sep}})$ the dual Galois module

$$\operatorname{Hom}\big(\operatorname{Pic}(G_{\mathrm{sep}}), F^\times_{\mathrm{sep}}\big).$$

Suppose that N is a free $\mathbb{Z}/n\mathbb{Z}$-module. Tensoring by N the sequence (4) over F_{sep}, we get an exact sequence

$$0 \longrightarrow G^*_{\mathrm{sep}} \otimes N \longrightarrow \widetilde{H}^1_{et}(G_{\mathrm{sep}}, N[1]) \longrightarrow \operatorname{Hom}\big(\operatorname{Pic}^*(G_{\mathrm{sep}}), N[1]\big) \longrightarrow 0.$$

Since any Galois module of bounded exponent is a direct sum of free $\mathbb{Z}/n\mathbb{Z}$-modules for all n, this sequence exists for any N of bounded exponent. Taking cohomology groups, we get in view of (3)

$$0 \longrightarrow H^0(F, G^*_{\mathrm{sep}} \otimes N) \longrightarrow \widetilde{H}^1_{et}(G, N[1]) \longrightarrow \\ \operatorname{Hom}_\Gamma\big(\operatorname{Pic}^*(G_{\mathrm{sep}}), N[1]\big) \longrightarrow H^1(F, G^*_{\mathrm{sep}} \otimes N).$$

Proposition 3.3 and (5) then give

Theorem 3.4. *Let G be a reductive group over F, N be a Galois module of bounded exponent. Then there is an exact sequence*

$$0 \longrightarrow H^0(F, G^*_{\mathrm{sep}} \otimes N) \longrightarrow \mathrm{Inv}^1(G, H^*[N]) \longrightarrow \\ \mathrm{Hom}_\Gamma\big(\mathrm{Pic}^*(G_{\mathrm{sep}}), N[1]\big) \longrightarrow H^1(F, G^*_{\mathrm{sep}} \otimes N). \quad \square$$

Corollary 3.5. *If G is a semisimple group, C is the kernel of the universal cover $\widetilde{G} \to G$, then*

$$\mathrm{Inv}^1(G, H^*[N]) \simeq \mathrm{Hom}_\Gamma\big(C(F_{\mathrm{sep}}), N[1]\big).$$

In particular, $\mathrm{Inv}^1(G, H^*[\mathbb{Z}/n\mathbb{Z}]) \simeq {}_nC^*$.

Proof. There is a natural isomorphism $\mathrm{Pic}(G_{\mathrm{sep}}) \simeq C^*_{\mathrm{sep}}$ (cf. [25, Lemma 6.9(iii)]), so that $\mathrm{Pic}^*(G_{\mathrm{sep}}) = C(F_{\mathrm{sep}})$. $\square$

Remark 3.6. Assume that the order of C is prime to $\operatorname{char} F$. Then Corollary 3.5 shows that the invariant $G(L) \to H^1(L, C)$, induced by the connecting homomorphism with respect to the exact sequence

$$1 \longrightarrow C \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1,$$

is the universal one.

Corollary 3.7. *If T is an algebraic torus, then*

$$\mathrm{Inv}^1(T, H^*[N]) \simeq H^0\big(F, T^*_{\mathrm{sep}} \otimes N\big).$$

In particular, $\mathrm{Inv}^1(T, H^*[\mathbb{Z}/n\mathbb{Z}]) \simeq H^0(F, T^*_{\mathrm{sep}}/nT^*_{\mathrm{sep}})$. $\square$

Remark 3.8. For a commutative algebraic group A the isomorphism classes of all central extensions of G by A form an abelian group $\mathrm{Ext}(G, A)$. The Kummer sequence induces the following exact sequence

$$0 \longrightarrow G^*/G^{*n} \longrightarrow \mathrm{Ext}(G, \mu_n) \longrightarrow {}_n\,\mathrm{Ext}(G, \mathbb{G}_m) \longrightarrow 0.$$

By [16, Lemma 1.6], the natural homomorphism $\mathrm{Ext}(G, \mathbb{G}_m) \to \mathrm{Pic}(G)$ is an isomorphism. Hence, in view of (4), the natural map $\mathrm{Ext}(G, \mu_n) \to \widetilde{H}^1_{et}(G, \mu_n)$, taking the class of an extension $1 \to \mu_n \to G' \to G \to 1$ to the class of μ_n-torsor G' over G, is also an isomorphism. The extension G' of G by μ_n induces an invariant in $\mathrm{Inv}^1(G, H^*[\mathbb{Z}/n\mathbb{Z}])$ via the connecting homomorphism with respect to the exact sequence. Hence

$$\mathrm{Inv}^1(G, H^*[\mathbb{Z}/n\mathbb{Z}]) \simeq \mathrm{Ext}(G, \mu_n).$$

Example 3.9. Let $2n$ be prime to $\operatorname{char} F$ and let G be an adjoint semisimple group of type ${}^2A_{n-1}$, i.e. G is the projective unitary group $\mathbf{PGU}(B, \tau)$, where B is a central simple algebra of dimension n^2 over a quadratic field extension L/F with an involution τ on B of the second kind, trivial on F. The kernel C of the universal covering of G is $\mu_{n[L]}$ (cf. [11]). The character group C^*_{sep} is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ with the action of the absolute Galois group of F via $\mathrm{Gal}(L/F)$ taking $i + n\mathbb{Z}$ to $-i + n\mathbb{Z}$. Hence, if n is odd, the group C^* is trivial and G has no nontrivial invariants in $H^*[\mathbb{Z}/k\mathbb{Z}])$ for any k prime to $\operatorname{char} F$.

If n is even, $n = 2m$, the group C^* is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, so that there is only one nontrivial invariant $u \in \operatorname{Inv}^1(G, H^*[\mathbb{Z}/2\mathbb{Z}]) \simeq {}_2C^*$. The homomorphism $u_F : G(F) \to H^1(F, \mu_2) = F^\times/F^{\times 2}$ is defined as follows. An element of $G(F)$ is represented (modulo $L^\times$) by an element $b \in B^\times$ such that $b \cdot \tau(b) = \mu \in F^\times$ (cf. [11]). The reduced norm $\beta = \operatorname{Nrd}(b) \in L^\times$ satisfies $\mathrm{N}_{L/F}(\beta) = \mu^n$. Hence $\mathrm{N}_{L/F}(\mu^{-m}\beta) = 1$ and by Hilbert's Theorem 90, $\mu^{-m}\beta = \tau(\alpha)\alpha^{-1}$ for some $\alpha \in L^\times$ uniquely determined modulo $F^\times$. Hence, the class $\mathrm{N}_{L/F}(\alpha)F^{\times 2} \in F^\times/F^{\times 2}$ is well defined and

$$u_F(bL^\times) = \mu^m \cdot \mathrm{N}_{L/F}(\alpha)F^{\times 2} \in F^\times/F^{\times 2} = H^1(F, \mu_2).$$

3.3. **Invariants of dimension 2.** We compute the group $\operatorname{Inv}^2(G, H^*[\mu_n^{\otimes -1}])$ for any reductive group G and any n prime to $\operatorname{char} F$. The direct limit of these groups for all n prime to $\operatorname{char} F$ we denote $\operatorname{Inv}^2(G, H^*[\mu^{\otimes -1}])$. Clearly,

$$\operatorname{Inv}^2(G, H^*[\mu_n^{\otimes -1}]) = {}_n\operatorname{Inv}^2(G, H^*[\mu^{\otimes -1}]).$$

For a torsion abelian group A, we write A' for the subgroup in A of all elements of order prime to $\operatorname{char} F$.

By Theorem 2.3, the group $\operatorname{Inv}^2(G, H^*[\mu^{\otimes -1}])$ is isomorphic to

$$A^0(G, H^2[\mu^{\otimes -1}])_{\mathrm{mult}} = A^0\big(G, \mathcal{H}^2(\mu)\big)_{\mathrm{mult}}$$

which, in its turn, can be identified with the subgroup $\operatorname{Br}(G)'_{\mathrm{mult}}$ of the Brauer group $\operatorname{Br}(G)$ by [7, Prop. 4.2.3(a)].

Lemma 3.10. *Let T be an algebraic torus over a separably closed field F. Then $\operatorname{Br}(T)'_{\mathrm{mult}} = 0$.*

Proof. Since F is separably closed, by Tsen's Theorem, $\operatorname{Br}\big(F(t)\big)' = 0$, hence $\operatorname{Br}(\mathbb{G}_m)'_{\mathrm{mult}} = 0$. In the general case, $T \simeq \mathbb{G}_m^n$ and the result follows by additivity from Lemma 1.5 applied to the functor $G \mapsto \operatorname{Br}(G)'$. □

This Lemma and Lemma 1.5 then imply

Corollary 3.11. *Let G be an algebraic group over a separably closed field F, T be an algebraic torus over F. Then the embedding $G \to G \times T$, $g \mapsto (g, 1)$, induces an isomorphism $\operatorname{Br}(G \times T)'_{\mathrm{mult}} \xrightarrow{\sim} \operatorname{Br}(G)'_{\mathrm{mult}}$.* □

Proposition 3.12. *Let G be a reductive group over a separably closed field F. Then $\operatorname{Br}(G)'_{\mathrm{mult}} = 0$.*

Proof. Let T be the connected center of G, $\overline{G} = G/T$. It follows from the proof of Lemma 6.12 in [25], that there is an exact sequence

$$0 \longrightarrow \operatorname{Br}(\overline{G}) \xrightarrow{\alpha} \operatorname{Br}(G) \xrightarrow{\beta} \operatorname{Br}(G \times T),$$

where α is induced by the natural epimorphism $G \to \overline{G}$ and $\beta = \beta_1 - \beta_2$ with β_1 induced by the projection $G \times T \to G$ and β_2 induced by the multiplication morphism $G \times T \to G$. Since T is a torus, by Corollary 3.11, β_1 and β_2 coincide on $\operatorname{Br}(G)'_{\mathrm{mult}}$. Hence, it suffices to show that $\operatorname{Br}(\overline{G})'_{\mathrm{mult}} = 0$, so that now we may assume that G is a semisimple group.

By [9, Th. 4.1], the group $\mathrm{Br}(G)'$ is isomorphic to $H^3(C^*,\mathbb{Z})'$ where C the kernel of the universal cover $\widetilde{G}\to G$. Thus, it suffices to show that for the functor $\mathcal{F}(H)=H^3(H,\mathbb{Z})$ the group $\mathcal{F}(H)_{\mathrm{mult}}$ is trivial for any finite abelian (constant) group H. Since H is a product of cyclic groups and $H^3(H,\mathbb{Z})$ is trivial for a cyclic group H, the result follows from Lemma 1.5. $\square$

Consider hypercohomology groups $\mathbb{H}^n_{et}(BG,\mathbb{G}_m)$ (cf. [8]). There is a spectral sequence

$$E_2^{p,q}=H^p\big(BG,H^q_{et}(*,\mathbb{G}_m)\big)\Rightarrow\mathbb{H}^{p+q}_{et}(BG,\mathbb{G}_m).$$

Since $H^0_{et}(G,\mathbb{G}_m)=F^\times\oplus G^*$, by Lemma 1.7,

$$E_2^{p,0}=\begin{cases}F^\times & \text{if } p=0,\\ G^* & \text{if } p=1,\\ 0 & \text{if } p\ge 2.\end{cases}$$

Since $H^1_{et}(G,\mathbb{G}_m)=\mathrm{Pic}(G)$ is an additive functor on the category of reductive groups, again by Lemma 1.7,

$$E_2^{p,1}=\begin{cases}0 & \text{if } p=0,\\ \mathrm{Pic}(G) & \text{if } p=1,\\ 0 & \text{if } p\ge 2.\end{cases}$$

The group $H^2_{et}(G,\mathbb{G}_m)$ is the Brauer group $\mathrm{Br}(G)$, hence

$$E_2^{p,2}=\begin{cases}\mathrm{Br}(F) & \text{if } p=0,\\ \mathrm{Br}(G)_{\mathrm{mult}} & \text{if } p=1.\end{cases}$$

Then the spectral sequence gives

$$\widetilde{\mathbb{H}}^n_{et}(BG,\mathbb{G}_m)=\begin{cases}0 & \text{if } n=0,\\ G^* & \text{if } n=1,\\ \mathrm{Pic}(G) & \text{if } n=2,\\ \mathrm{Br}(G)_{\mathrm{mult}} & \text{if } n=3.\end{cases}$$

Theorem 3.13. *Let G be a reductive group over a field F. Then there is an exact sequence*

$$H^0\big(F,\mathrm{Pic}(G_{\mathrm{sep}})\big)'\longrightarrow H^2(F,G^*_{\mathrm{sep}})'\longrightarrow \mathrm{Inv}^2(G,H^*[\mu^{\otimes -1}])\longrightarrow H^1\big(F,\mathrm{Pic}(G_{\mathrm{sep}})\big)'\longrightarrow H^3(F,G^*_{\mathrm{sep}})'.$$

Proof. Consider the Hochschild-Serre spectral sequence

$$E_2^{p,q}=H^p\big(F,\widetilde{\mathbb{H}}^n_{et}(BG_{\mathrm{sep}},\mathbb{G}_m)\big)\Rightarrow\widetilde{\mathbb{H}}^n_{et}(BG,\mathbb{G}_m).$$

We have

$$E_2^{p,q}=\begin{cases}0 & \text{if } q=0,\\ H^p(F,G^*_{\mathrm{sep}}) & \text{if } q=1,\\ H^p\big(F,\mathrm{Pic}(G_{\mathrm{sep}})\big) & \text{if } q=2.\end{cases}$$

By Proposition 3.12, $(E_2^{0,3})' \subset \operatorname{Br}(G_{\mathrm{sep}})'_{\mathrm{mult}} = 0$, so the desired exact sequence is induced by the spectral sequence and the isomorphisms

$$\operatorname{Inv}^2(G, H^*[\mu_n^{\otimes -1}]) \simeq \operatorname{Br}(G)'_{\mathrm{mult}} = \widetilde{\mathbb{H}}^3_{et}(BG, \mathbb{G}_m)'. \quad \square$$

Corollary 3.14. *If G is a semisimple group, C is the kernel of the universal cover $\widetilde{G} \to G$, then*

$$\operatorname{Inv}^2(G, H^*[\mu^{\otimes -1}]) \simeq H^1(F, C^*_{\mathrm{sep}})'. \quad \square$$

Corollary 3.15. *Let T be an algebraic torus. The pairing*

$$T(F) \otimes H^2(F, T^*_{\mathrm{sep}}) \to H^2(F, \mathbb{G}_m) = \operatorname{Br}(F)$$

induces an isomorphism

$$H^2(F, T^*_{\mathrm{sep}})' \xrightarrow{\sim} \operatorname{Inv}^2(T, H^*[\mu^{\otimes -1}]). \quad \square$$

Example 3.16. Assume that $\operatorname{char} F \neq 2$. Let G be the special orthogonal group of type B or D over F, i.e. $G = \mathbf{O}_+(A, \sigma)$, where A is a central simple algebra over F with an orthogonal involution σ. (If A splits, $A \simeq \operatorname{End}(V)$, the involution σ is adjoint with respect to a quadratic form q on V, so that G is the special orthogonal group $\mathbf{O}_+(V, q)$, cf. [11].) The kernel C of the universal covering of G is μ_2, so that C^* is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Hence, $\operatorname{Inv}^2(G, H^*[\mathbb{Z}/2\mathbb{Z}]) \simeq H^1(F, \mathbb{Z}/2\mathbb{Z}) \simeq F^\times/F^{\times 2}$. Let u be the invariant corresponding to an element $a \in F^\times$. Then for any $g \in G(F)$, $u_F(g) = \big(Sn(g), a\big)$ is the class of quaternion algebra, where Sn is the spinor norm homomorphism $G(F) \to F^\times/F^{\times 2}$ (cf. [11]).

4. Invariants of simply connected groups

We show in this section that a simply connected group has no nontrivial invariants in $H^d[\mu_n^{\otimes -1}]$ for $d \leq 3$.

4.1. **Chow groups with coefficients.** Let G be a simply connected group defined over a field F. Denote $D(G)$ the free abelian group generated by the connected component of the Dynkin diagram of G. In particular, $D(G) = \mathbb{Z}$ if G is absolutely simple. There is a natural Galois action on $D(G_{\mathrm{sep}})$.

Let G be a split simply connected group over a field F, T a split maximal torus in G, W the Weyl group of G. Let $G = G_1 \times \cdots \times G_k$ be a product of simple subgroups. Then $T^* = T_1^* \oplus \cdots \oplus T_k^*$ where T_i is a maximal torus in G_i. There is a positive definite W-invariant quadratic form on T^*, unique up to a multiple on each direct summand T_i^*. Hence the group $S^2(T^*)^W$ of W-invariant quadratic forms on T^* is canonically isomorphic to $D(G)$.

Proposition 4.1. *Let G be a split simply connected group over a field F, and M a cycle module over F. Then*
(i) $H^0(G, M_j) = M_j(F)$;
(ii) $H^1(G, M_j) = D(G) \otimes M_{j-2}(F)$.

Proof. The first part and the case $j \leq 2$ of (ii) is proved in [8]. Let $d = \dim G$ and $X = G/T$. Consider a spectral sequence (cf. [8])

$$E^1_{p,q} = \Lambda^p T^* \otimes CH^{d-p-q} X \otimes M_{j+q-d}(F) \Rightarrow A^{d-p-q}(G, M_j).$$

It suffices to show that $E^2_{k,d-k-1}$ is trivial for all $k \neq 1$ and equals $D(G) \otimes M_{j-2}(F)$ if $k = 1$. This statement is proved for $k = 0, 1$ in [8] (with $D(G)$ replaced by $S^2(T^*)^W$), so we can assume that $k \geq 2$.

The group $E^2_{k,d-k-1}$ is the homology of the complex

$$E^1_{k+1,d-k-1} \longrightarrow E^1_{k,d-k-1} \longrightarrow E^1_{k-1,d-k-1}.$$

Consider Koszul complex

$$C_i = \Lambda^i T^* \otimes S^{k+1-i} T^*.$$

The first Chern class defines an isomorphism $T^* \xrightarrow{\sim} CH^1(X)$ and therefore a homomorphism $S^l(T^*) \to CH^l(X)$ for any l. Thus, we have a commutative diagram

$$\begin{array}{ccccccc}
C_{k+1} \otimes P & \longrightarrow & C_k \otimes P & \longrightarrow & C_{k-1} \otimes P & \xrightarrow{d} & C_{k-2} \otimes P \\
\downarrow{\scriptstyle \alpha_{k+1}} & & \downarrow{\scriptstyle \alpha_k} & & \downarrow{\scriptstyle \alpha_{k-1}} & & \\
E^1_{k+1,d-k-1} & \longrightarrow & E^1_{k,d-k-1} & \longrightarrow & E^1_{k-1,d-k-1}, & &
\end{array}$$

where $P = M_{j-k-1}(F)$ with α_k and α_{k+1} being isomorphisms. Since the top row in the diagram is exact, in order to prove that the bottom row is also exact, it suffices to show that the restriction of d on the kernel of α_{k-1} is injective.

Denote H the kernel of the natural (split) surjection $S^2T^* \to CH^2(X)$. Clearly, $\ker \alpha_{k-1} = \Lambda^{k-1}T^* \otimes H \otimes P$. Hence it amounts to show that the composite

$$\Lambda^{k-1}T^* \otimes H \hookrightarrow \Lambda^{k-1}T^* \otimes S^2T^* \xrightarrow{d'} \Lambda^{k-2}T^* \otimes S^3T^* \tag{6}$$

is a split injection. The map d' factors as the following composite

$$\Lambda^{k-1}T^* \otimes S^2T^* \xrightarrow{f} \Lambda^{k-2}T^* \otimes T^* \otimes S^2T^* \xrightarrow{g} \Lambda^{k-2}T^* \otimes S^3T^*,$$

where f is the identity on S^2T^* and g is the identity on $\Lambda^{k-2}T^*$. Hence the composite (6) factors as follows:

$$\Lambda^{k-1}T^* \otimes H \longrightarrow \Lambda^{k-2}T^* \otimes T^* \otimes H \hookrightarrow \Lambda^{k-2}T^* \otimes T^* \otimes S^2T^* \xrightarrow{g} \Lambda^{k-2}T^* \otimes S^3T^*.$$

The first homomorphism is a split injection, hence it suffices to show that the restriction of g on $\Lambda^{k-2}T^* \otimes T^* \otimes H$ is also a split injection. But g is the identity on $\Lambda^{k-2}T^*$, hence we need to show that the restriction h of $T^* \otimes S^2T^* \to S^3T^*$ on $T^* \otimes H$ is a split injection.

By [8, 4.8], the group H coincides with $(S^2T^*)^W$. Clearly, $H = H_1 \oplus \cdots \oplus H_k$ where $H_i = (S^2T_i^*)^{W_i}$ for the Weyl group W_i of G_i. The problem reduces to showing that the restriction of $T_i^* \otimes S^2T_i^* \to S^3T_i^*$ on $T_i^* \otimes H_i$ is a split injection, so we can assume that G is a simple group. In this case H is a cyclic group, generated by an integral quadratic form $q \in S^2T^*$. Then h is the multiplication by q and it is an injection since the symmetric algebra S^*T^* is a domain and h splits since q is not zero modulo any prime number and therefore the cokernel of the restriction of h has no torsion. $\square$

Now assume that G is a simply connected (not necessarily split) group over F. By [8, Cor. B.3],

$$A^1(G, K_2) \simeq H^0\big(F, A^1(G_{\mathrm{sep}}, K_2)\big).$$

Proposition 4.1 then gives

Corollary 4.2. $A^1(G, K_2) \simeq H^0\big(F, D(G_{\mathrm{sep}})\big)$ $\square$.

By [12], $K_0(G) = \mathbb{Z}$, hence the first term $K_0(G)^{(1)}$ of the topological filtration is trivial. The natural homomorphisms $\mathrm{CH}^i(G) \to K_0(G)^{(i/i+1)}$ are split by the Chern classes up to multiplication by $(i-1)!$. Hence, $(i-1)! \cdot \mathrm{CH}^i(G) = 0$. In particular, $\mathrm{Pic}(G) = \mathrm{CH}^1(G) = 0 = \mathrm{CH}^2(G)$ and $2 \cdot \mathrm{CH}^3(G) = 0$.

The only possibly nontrivial differential in the Brown-Gersten-Quillen spectral sequence (cf. [21, Prop. 5.8])

$$E_2^{p,q} = A^p(G, K_{-q}) \Rightarrow K_{-p-q}(G)$$

arriving to $\mathrm{CH}^3(G)$ is a surjective homomorphism

$$A^1(G, K_2) \longrightarrow \mathrm{CH}^3(G),$$

hence $\mathrm{CH}^3(G)$ is a cyclic group of at most two elements, if G is absolutely simple, since in this case $D(G_{\mathrm{sep}}) = \mathbb{Z}$ and $A^1(G, K_2) = \mathbb{Z}$.

Proposition 4.3. *Let A be a central simple algebra over F, $G = \mathbf{SL}_1(A)$. Then*

$$\mathrm{CH}^3(G) = \begin{cases} 0 & \textit{if } \mathrm{ind}(A) \textit{ is odd,} \\ \mathbb{Z}/2\mathbb{Z} & \textit{if } \mathrm{ind}(A) \textit{ is even.} \end{cases}$$

Proof. Assume first that index of A is odd. Choose a splitting field extension L/F of A of odd degree. In the split case the group $\mathrm{CH}^3(G_L)$ is trivial by [27, Th. 2.7]. Since $\mathrm{CH}^3(G)$ is a 2-torsion group, the natural homomorphism $\mathrm{CH}^3(G) \to \mathrm{CH}^3(G_L)$ is injective, hence $\mathrm{CH}^3(G) = 0$.

Now let $\mathrm{ind}(A)$ be even number. Assume first that A is a quaternion (and hence division) algebra. The variety of G is an affine quadric. Let X be a projective quadric containing G as an open subset, $Z = X \setminus G$. We have an exact sequence

$$\mathrm{CH}^2(Z) \longrightarrow \mathrm{CH}^3(X) \longrightarrow \mathrm{CH}^3(G) \longrightarrow 0.$$

Since X has a rational point, $\mathrm{CH}^3(X) = \mathbb{Z}$. By assumption, the quadric Z has no rational point, hence $\mathrm{CH}^2(Z) = 2\mathbb{Z}$ (cf. [10] and [28]) and therefore $\mathrm{CH}^3(G) = \mathbb{Z}/2\mathbb{Z}$.

Assume now that $\mathrm{ind}(A) = 2$, i.e. $A = M_n(Q)$, where Q is quaternion division algebra. Consider $H = \mathbf{SL}_1(Q)$ as a subgroup in $G = \mathbf{SL}_n(Q)$ and the variety $X = G/H = \mathbf{GL}_n(Q)/\mathbf{GL}_1(Q)$. By Hilbert Theorem 90, $X(L) = \mathrm{GL}_n(Q_L)/\mathrm{GL}_1(Q_L)$ for any field extension L/F, hence the natural map $G(L) \to X(L)$ is surjective and therefore the fiber of $\pi : G \to X$ over any geometric point is isomorphic to $\mathbf{SL}_1(Q)$. In particular, the generic fiber Y of π is isomorphic to H_E, where $E = F(X)$. Since X is a smooth variety having a rational point, we have $\operatorname{ind}(Q_E) = \operatorname{ind}(Q) = 2$. The surjectivity of the natural homomorphism

$$\mathrm{CH}^3(G) \longrightarrow \mathrm{CH}^3(Y) \simeq \mathrm{CH}^3(H_E) = \mathbb{Z}/2\mathbb{Z}$$

shows that $\mathrm{CH}^3(G) \neq 0$, hence $\mathrm{CH}^3(G) = \mathbb{Z}/2\mathbb{Z}$.

Consider the general case. Find a field extension L/F such that $\operatorname{ind}(A_L) = 2$. For example, L can be taken as the function field of the generalized Severi-Brauer variety $SB(2, A)$ (cf. [4]). In the commutative diagram

$$\begin{array}{ccc} H^1(G, K_2) & \longrightarrow & \mathrm{CH}^3(G) \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle j} \\ H^1(G_L, K_2) & \longrightarrow & \mathrm{CH}^3(G_L) \end{array}$$

the horizontal homomorphisms are surjective and i is an isomorphism, hence j is surjective. By the case considered above, $\mathrm{CH}^3(G_L) = \mathbb{Z}/2\mathbb{Z}$, hence $\mathrm{CH}^3(G)$ is not trivial, and therefore $\mathrm{CH}^3(G) = \mathbb{Z}/2\mathbb{Z}$. □

Remark 4.4. The proof shows that if L/F is a field extension such that the algebra A_L is of even index, then the natural homomorphism $\mathrm{CH}^3(G) \to \mathrm{CH}^3(G_L)$ is an isomorphism of groups of order 2.

Lemma 4.5. *Let n be prime to $\operatorname{char} F$. For any $i \geq 0$ there exists an exact sequence*

$$A^i(G, K_{i+1}) \xrightarrow{n} A^i(G, K_{i+1}) \longrightarrow A^i\big(G, \mathcal{H}^{i+1}(\mu_n^{\otimes i+1})\big) \longrightarrow {}_n\mathrm{CH}^{i+1}(G) \longrightarrow 0.$$

In particular,

$$A^i\big(G, \mathcal{H}^{i+1}(\mu_n^{\otimes i+1})\big) = A^i(G, K_{i+1})/nA^i(G, K_{i+1})$$

for $i = 0$ and 1.

Proof. Follows by diagram chase in

$$\begin{array}{ccccc} \coprod_{X^{i-1}} K_2F(x) & \longrightarrow & \coprod_{X^i} K_1F(x) & \longrightarrow & \coprod_{X^{i+1}} K_0F(x) \\ \big\downarrow{\scriptstyle n} & & \big\downarrow{\scriptstyle n} & & \big\downarrow{\scriptstyle n} \\ \coprod_{X^{i-1}} K_2F(x) & \longrightarrow & \coprod_{X^i} K_1F(x) & \longrightarrow & \coprod_{X^{i+1}} K_0F(x) \\ \big\downarrow{\scriptstyle h_2} & & \big\downarrow{\scriptstyle h_1} & & \big\downarrow{\scriptstyle h_0} \\ \coprod_{X^{i-1}} H^2F(x) & \longrightarrow & \coprod_{X^i} H^1F(x) & \longrightarrow & \coprod_{X^{i+1}} H^0F(x) \end{array}$$

where the cohomology groups are taken with coefficients $\mu_n^{\otimes 2}$, μ_n and $\mathbb{Z}/n\mathbb{Z}$ respectively, and surjectivity of h_2 (cf. [17]). □

4.2. **Étale cohomology.** Let G be a simply connected group defined over a field F and let N be a Galois module of bounded exponent. We have the following computation of étale cohomology groups over F_{sep}.

Lemma 4.6. *There are following isomorphisms of Galois modules:*

$$H^m_{et}(G_{\mathrm{sep}}, N) \simeq \begin{cases} N & \text{if } m = 0, \\ 0 & \text{if } m = 1, 2, 4, \\ D(G_{\mathrm{sep}}) \otimes N[-2] & \text{if } m = 3. \end{cases}$$

Proof. The case $m = 0$ is trivial. Since $G^*_{\mathrm{sep}} = 0$ and $\mathrm{Pic}(G_{\mathrm{sep}}) = 0$, it follows from (4), that $H^1_{et}(G_{\mathrm{sep}}, \mu_n) = 0$ for any n prime to $\operatorname{char} F$ and hence $H^1_{et}(G_{\mathrm{sep}}, N) = 0$ for any N of bounded exponent.

The Kummer sequence induces an exact sequence

$$0 \longrightarrow \mathrm{Pic}(G_{\mathrm{sep}})/n \longrightarrow H^2_{et}(G_{\mathrm{sep}}, \mu_n) \to {}_n\mathrm{Br}(G_{\mathrm{sep}}).$$

Since ${}_n\mathrm{Br}(G_{\mathrm{sep}}) = 0$ by [9, Th. 4.1], $H^2_{et}(G_{\mathrm{sep}}, \mu_n) = 0$ for any n prime to $\operatorname{char} F$ and hence $H^2_{et}(G_{\mathrm{sep}}, N) = 0$ for any N of bounded exponent.

The Bloch-Ogus spectral sequence (2) for N gives an exact sequence

$$0 \longrightarrow A^1\big(G_{\mathrm{sep}}, \mathcal{H}^2(N)\big) \longrightarrow H^3_{et}(G_{\mathrm{sep}}, N) \longrightarrow A^0\big(G_{\mathrm{sep}}, \mathcal{H}^3(N)\big).$$

By Proposition 4.1, the last group in this sequence is trivial and the first is isomorphic to $D(G_{\mathrm{sep}}) \otimes N[-2]$.

In order to show that $H^4_{et}(G_{\mathrm{sep}}, N)$ is trivial, it suffices to prove that the groups $A^2\big(G_{\mathrm{sep}}, \mathcal{H}^2(N)\big)$, $A^1\big(G_{\mathrm{sep}}, \mathcal{H}^3(N)\big)$ and $A^0\big(G_{\mathrm{sep}}, \mathcal{H}^4(N)\big)$ staying on the corresponding diagonal in the Bloch-Ogus spectral sequence (2), are trivial. The first group is isomorphic to $CH^2(G_{\mathrm{sep}}) \otimes N[-2]$ and hence trivial. The last two groups are trivial by Proposition 4.1. □

Since the natural homomorphisms $H^p(F, N) \to H^p_{et}(G, N)$ are injective, all the differentials in the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p\big(F, H^q_{et}(G_{\mathrm{sep}}, N)\big) \Rightarrow H^{p+q}_{et}(G, N),$$

arriving to $E_*^{p,0}$, are trivial. By Lemma 4.6, $E_2^{p,q} = 0$ if $q = 1$ or 2 and $E_2^{p,3} = H^p\big(F, D(G_{\mathrm{sep}}) \otimes N[-2]\big)$.

Thus, we have proved

Proposition 4.7. *Let G be a simply connected group defined over a field F and let N be a Galois module of bounded exponent. Then*

$$\widetilde{H}^m_{et}(G, N) = \begin{cases} 0 & \text{if } m = 0, 1, 2, \\ H^0\big(F, D(G_{\mathrm{sep}}) \otimes N[-2]\big) & \text{if } m = 3 \\ H^1\big(F, D(G_{\mathrm{sep}}) \otimes N[-2]\big) & \text{if } m = 4. \quad \square \end{cases}$$

Let $n \in \mathbb{N}$ be prime to $\operatorname{char} F$. The Bloch-Ogus spectral sequence for the Galois module $\mu_n^{\otimes 2}$ gives the following exact sequence

$$0 \longrightarrow A^1\big(G, \mathcal{H}^2(\mu_n^{\otimes 2})\big) \xrightarrow{e_2} \widetilde{H}^3_{et}\big(G, \mu_n^{\otimes 2}\big) \longrightarrow \widetilde{A}^0\big(G, \mathcal{H}^3(\mu_n^{\otimes 2})\big) \xrightarrow{d_2} A^2\big(G, \mathcal{H}^2(\mu_n^{\otimes 2})\big)$$

Lemma 4.8. *Let G be a simply connected group. Then e_2 is an isomorphism and $\widetilde{A}^0\big(G, \mathcal{H}^3(\mu_n^{\otimes 2})\big) = 0$.*

Proof. In the commutative diagram

$$\begin{array}{ccccc} A^1(G, K_2) & \longrightarrow & A^1\big(G, \mathcal{H}^2(\mu_n^{\otimes 2})\big) & \xrightarrow{e_2} & \widetilde{H}^3_{et}\big(G, \mu_n^{\otimes 2}\big) \\ \downarrow \wr & & & & \downarrow \wr \\ H^0\big(F, D(G_{\mathrm{sep}})\big) & & \longrightarrow & & H^0\big(F, D(G_{\mathrm{sep}})/n\big) \end{array}$$

the vertical isomorphisms are given by Corollary 4.2 and by Proposition 4.7. The group $H^0\big(F, D(G_{\mathrm{sep}})/n\big)$ is isomorphic to $H^0\big(F, D(G_{\mathrm{sep}})\big)/n$ since the Galois module $D(G_{\mathrm{sep}})$ is permutation. Hence the bottom homomorphism and therefore e_2 are surjective. Thus, d_2 is injective and $\widetilde{A}^0\big(G, \mathcal{H}^3(\mu_n^{\otimes 2})\big) = 0$ since $A^2\big(G, \mathcal{H}^2(\mu_n^{\otimes 2})\big) = \mathrm{CH}^2(G)/n = 0$. □

An application of Lemma 4.8 is the following

Proposition 4.9. *For a simply connected group G, $\mathrm{Inv}^d(G, H^*[N]) = 0$ for any Galois module N of bounded exponent, $d = 0, 1$, and for $N = \mu_n^{\otimes -1}$ (n prime to $\mathrm{char}\, F$), $d \leq 3$.*

Proof. The case $d = 0$ has been considered in Corollary 3.2. The cases $d = 1$ and $d = 2$ follow from Corollaries 3.5 and 3.14. Finally, if $d = 3$, by Lemmas 1.9, 4.8 and Theorem 2.3, $\mathrm{Inv}^3(G, H^*[\mu_n^{\otimes -1}]) = 0$. □

5. Cohomological invariants of $\mathbf{SL}_1(A)$

It is proved in section 4 that simply connected groups have no nonzero invariants in $H^*[\mu_n^{\otimes -1}]$ of dimension at most 3 and split simply connected groups have no nontrivial invariants at all. We show that the group $\mathbf{SL}_1(A)$ still has no invariant if index of A is 2. For a biquaternion algebra A of index 4 we prove that there is the only nontrivial invariant in $H^4[\mathbb{Z}/2\mathbb{Z}]$, namely the Rost's invariant.

5.1. Invariants of $\mathbf{SL}_1(A)$ with $\mathrm{ind}(A) = 2$.

Lemma 5.1. *Let $G = \mathbf{SL}_1(Q)$, where Q is a quaternion algebra over F. Then $A^0(G, M) = M(F)$ for any cycle module M over F.*

Proof. Let X and Z be projective quadrics as in the proof of Proposition 4.3 and let C be projective conic curve, corresponding to Q. Then $Z \simeq C \times C$ and X (being a projective quadric with a rational point) contains a hyperplane section Y, such that $X \setminus Y$ is isomorphic to an affine space $\mathbb{A}^3_F$. We have the localization exact sequence (cf. [24, Sec. 5])

$$0 \longrightarrow A^0(X, M_d) \longrightarrow A^0(G, M_d) \longrightarrow A^0(Z, M_{d-1}) \longrightarrow A^1(X, M_d).$$

Another localization sequence for the pair (X, Y) immediately gives

$$A^0(X, M_d) = A^0(\mathbb{A}^3_F, M_d) = M_d(F) \text{ and } A^1(X, M_d) \simeq A^1(Y, M_{d-1}).$$

A spectral sequence ([24, Cor. 8.2]) associated to a projection $Z \to C$ identifies $A^0(Z, M_{d-1})$ with $A^0(C, M_{d-1})$.

On the other hand, $Y - pt$ is a vector bundle over C. Hence, by [24, Prop. 8.6],

$$A^1(X, M_d) \simeq A^0(Y, M_{d-1}) = A^0(Y - pt, M_{d-1}) \simeq A^0(C, M_{d-1}).$$

Therefore, it suffices to show commutativity of the diagram

$$\begin{tikzcd} A^0(Z, M_{d-1}) \arrow[r] \arrow[d, "\wr"] & A^1(X, M_d) \arrow[d, "\wr"] \\ A^0(C, M_{d-1}) \arrow[r, equal] & A^0(C, M_{d-1}). \end{tikzcd}$$

Since all homomorphisms in the diagram are natural (given by four basic maps in [24]), it is sufficient to prove the statement in the case $d = 1$ and M is given by Milnor K-groups. In other words, we have to check that the classes of Z and Y coincide in $A^1(X, K_1) = \mathrm{CH}^1(X)$. But these subvarieties in X are hyperplane sections, hence are rationally equivalent. □

Theorem 5.2. *Let $G = \mathbf{SL}_1(A)$, where A is a central simple algebra of index at most 2. Then $\mathrm{Inv}^d(G, M) = 0$ for any d and a cycle module M.*

Proof. The case of a split algebra A was considered in Corollary 2.6. Hence we may assume that $G = \mathbf{SL}_n(Q)$, where Q is a quaternion division algebra. The case $n = 1$ follows from Theorem 2.3 and Lemma 5.1. In the general case the group of F-points of G is generated by $SL_1(Q)$, embedded to G by $a \mapsto \mathrm{diag}(a, 1, \ldots, 1)$ and unipotent subgroups of elementary matrices. By Proposition 2.4, the restriction of any invariant of G on all there subgroups is trivial, hence so is the invariant. □

5.2. **Invariants of dimension 4.** Let A be a biquaternion algebra (tensor product of two quaternion algebras) over a field F of characteristic different from 2. There is a 6-dimensional quadratic form q associated to A (unique up to a scalar multiple), called an *Albert form* (cf. [11, 16.A]). This form defines a 4-dimensional projective quadric hypersurface, which we call the *Albert quadric* of A. The Albert quadric is the only 4-dimensional projective homogeneous variety of the spinor group $\mathbf{Spin}(q)$ (cf. [19]). This simply connected absolutely simple group of type D_3 is naturally isomorphic to $\mathbf{SL}_1(A)$, the one of type A_3 (cf. [11, 26.B]). The only 4-dimensional projective homogeneous variety of $\mathbf{SL}_1(A)$ is the generalized Severi-Brauer variety $SB(2, A)$ of right 8-dimensional ideals in A, so that the Albert quadric of A is isomorphic to $SB(2, A)$.

M. Rost (cf. [13]) has constructed an invariant of $G = \mathbf{SL}_1(A)$ in $H^4[\mathbb{Z}/2\mathbb{Z}]$. He also computed the kernel and the image of the invariant. More precisely, the following holds (with $\mathrm{SK}_1(A) = \mathrm{SL}_1(A)/[A^\times, A^\times]$).

Theorem 5.3. (Rost) *Let X be an Albert quadric of A. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{SK}_1(A) \longrightarrow H^4(F, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^4\big(F(X), \mathbb{Z}/2\mathbb{Z}\big). \quad \square$$

An elementary construction of Rost's invariant is given in [11, §17].

If A is not a division algebra, then Rost's invariant is trivial by Theorem 5.2. If A is division, then it is shown in [14] that the group $\mathrm{SK}_1(A_{F(G)})$ is not trivial, hence the Rost's invariant is not trivial.

Our aim is to prove that Rost's invariant is the only nontrivial degree 4 invariant of G. More precisely, we prove

Theorem 5.4. *Let $G = \mathbf{SL}_1(A)$, where A is a biquaternion division algebra. Then $\mathrm{Inv}^4(G, H^*[\mathbb{Z}/2\mathbb{Z}]) = \mathbb{Z}/2\mathbb{Z}$ is generated by the Rost's invariant.*

Since $A^2(G, \mathcal{H}^2) = \mathrm{CH}^2(G)/2 = 0$ (cohomology groups are taken with coefficients $\mathbb{Z}/2\mathbb{Z}$), the Bloch-Ogus spectral sequence for $\mathbb{Z}/2\mathbb{Z}$ gives the following exact sequence

$$0 \longrightarrow A^1(G, \mathcal{H}^3) \xrightarrow{e_3} \widetilde{H}^4_{et}(G) \longrightarrow \widetilde{A}^0(G, \mathcal{H}^4) \xrightarrow{d_3} A^2(G, \mathcal{H}^3).$$

Lemma 5.5. *e_3 is an isomorphism. In particular, $d_3 : \widetilde{A}^0(G, \mathcal{H}^4) \to A^2(G, \mathcal{H}^3)$ is injective.*

Proof. In the commutative diagram

$$\begin{array}{ccccc}
A^1(G, \mathcal{H}^2) \otimes F^\times & \xrightarrow{e_2 \otimes \mathrm{id}} & \widetilde{H}^3_{et}(G) \otimes F^\times & \xrightarrow{\sim} & H^0\big(F, D(G_{\mathrm{sep}})/2\big) \otimes F^\times \\
\downarrow{\scriptstyle m} & & \downarrow & & \downarrow \\
A^1(G, \mathcal{H}^3) & \xrightarrow{e_3} & \widetilde{H}^4_{et}(G) & \xrightarrow{\sim} & H^1(F, D(G_{\mathrm{sep}})/2)
\end{array}$$

the vertical homomorphisms are the natural product maps and the right horizontal isomorphisms are given by Proposition 4.7. Since G is absolutely simple, $D(G_{\mathrm{sep}}) = \mathbb{Z}$ and hence the right vertical homomorphism is an isomorphism. By Lemma 4.8, e_2 is an isomorphism, hence e_3 is surjective and therefore is an isomorphism. Note that m is also an isomorphism. $\square$

Corollary 5.6. *The natural homomorphism $A^1(G, K_3) \to A^1(G, \mathcal{H}^3)$ is surjective.*

Proof. In the commutative diagram

$$\begin{array}{ccc}
A^1(G, K_2) \otimes F^\times & \longrightarrow & A^1(G, \mathcal{H}^2) \otimes F^\times \\
\downarrow & & \downarrow{\scriptstyle m} \\
A^1(G, K_3) & \longrightarrow & A^1(G, \mathcal{H}^3)
\end{array}$$

the upper horizontal homomorphism is surjective by Lemma 4.5 and m is an isomorphism (cf. the proof of Lemma 5.5), hence the result. $\square$

Lemma 5.7. *$A^2(G, K_3)$ is an infinite cyclic group.*

Proof. By [27, Th. 2.7], $A^2(G,K_3)=\mathbb{Z}$ in the split case. Hence, it suffices to show that in the general case $A^2(G,K_3)$ has no torsion. Since there is a splitting field of A of degree 4, it is sufficient to prove that ${}_2A^2(G,K_3)=0$. Consider the following diagram with exact columns (cf. [17])

$$\begin{tikzcd}
& \mu_2\otimes\coprod_{G^{(1)}}K_1F(g) \arrow[r, "d"] \arrow[d] & \mu_2\otimes\coprod_{G^{(2)}}K_0F(g) \arrow[d] & \\
& \coprod_{G^{(1)}}K_2F(g) \arrow[r] \arrow[d, "2"] & \coprod_{G^{(2)}}K_1F(g) \arrow[r] \arrow[d, "2"] & \coprod_{G^{(3)}}K_0F(g) \arrow[d, "2"] \\
K_3F(G) \arrow[r] \arrow[d, "h_3"] & \coprod_{G^{(1)}}K_2F(g) \arrow[r] \arrow[d, "h_2"] & \coprod_{G^{(2)}}K_1F(g) \arrow[r] \arrow[d, "h_1"] & \coprod_{G^{(3)}}K_0F(g) \\
H^3F(G) \arrow[r] & \coprod_{G^{(1)}}H^2F(g) \arrow[r] & \coprod_{G^{(2)}}H^1F(g). &
\end{tikzcd}$$

The homomophism d is surjective since $\mathrm{CH}^2(G)=0$. The surjectivity of h_3 is proved in [18] and [23]. Now the result follows from Corollary 5.6 by diagram chase . □

Proof of Theorem 5.4. By Lemma 5.5,

$$d_3:\widetilde{A}^0(G,\mathcal{H}^4)\longrightarrow A^2(G,\mathcal{H}^3)$$

is injective. Denote j the injective composite

$$\mathrm{Inv}^4(G,H^*[\mathbb{Z}/2\mathbb{Z}])\hookrightarrow\widetilde{A}^0(G,\mathcal{H}^4)\xrightarrow{d_3}A^2(G,\mathcal{H}^3).$$

Let L/F be a field extension such that $\mathrm{ind}(A_L)=2$. Consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Inv}^4(G,H^*[\mathbb{Z}/2\mathbb{Z}]) \arrow[r, "j"] \arrow[d] & A^2(G,\mathcal{H}^3) \arrow[r, "k"] \arrow[d, "i"] & {}_2\,\mathrm{CH}^3(G) \arrow[d, "l"] \\
\mathrm{Inv}^4(G_L,H^*[\mathbb{Z}/2\mathbb{Z}]) \arrow[r, "j_L"] & A^2(G_L,\mathcal{H}^3) \arrow[r, "k_L"] & {}_2\,\mathrm{CH}^3(G_L).
\end{tikzcd}$$

where k is defined in Lemma 4.5. The group $\mathrm{Inv}^4(G_L,H^*[\mathbb{Z}/2\mathbb{Z}])$ is trivial by Theorem 5.2, hence $i\circ j=0$. Lemma 4.5 shows that k is surjective and l is an isomorphism of groups of order 2 by Remark 4.4. Therefore, $i\neq 0$ and j is not surjective.

By Lemma 4.5, the sequence

$$0\longrightarrow A^2(G,K_3)/2A^2(G,K_3)\longrightarrow A^2(G,\mathcal{H}^3)\xrightarrow{k}{}_2\,\mathrm{CH}^3(G)\longrightarrow 0$$

is exact. Proposition 4.3 and Lemma 5.7 show that $A^2(G,\mathcal{H}^3)$ is a group of order 4. Since j is not surjective, the group $\mathrm{Inv}^4(G,H^*[\mathbb{Z}/2\mathbb{Z}])$ is of at most 2 elements. On the other hand, the Rost invariant is not trivial, hence $\mathrm{Inv}^4(G,H^*[\mathbb{Z}/2\mathbb{Z}])=\mathbb{Z}/2\mathbb{Z}$. □

Remark 5.8. It follows from Lemma 5.5 that the homomorphism $\widetilde{H}^4_{et}(G) \to \widetilde{A}^0(G, \mathcal{H}^4)$ is trivial. This implies that the Rost's invariant is not "global", i.e. cannot be extended to a natural transformation of functors G and $H^4(*, \mathbb{Z}/2\mathbb{Z})$ from the category F-$\mathfrak{alg}$ to $\mathfrak{Groups}$. That is why we define an invariant as a natural transformation of functors defined on a smaller category F-$\mathfrak{fields}$.

6. Degree four algebras

The aim of this section is to generalize Theorem 5.3 to the case of arbitrary central simple algebra of dimension 16. We assume that $\operatorname{char} F \neq 2$ for the base field F.

Let A and B be two central simple algebras over F. If they are similar (determine the same element in the Brauer group $\operatorname{Br}(F)$), then the groups $K_1(A)$ and $K_1(B)$ are canonically isomorphic.

Any anti-automorphism φ of A induces the identity automorphism on $K_1(A)$ since for any $a \in A^\times$ the elements a and $\varphi(a)$ have the same minimal polynomials and therefore are conjugate. Hence, if A and B are anti-similar (i.e. A is similar to B^{op}), then $K_1(A)$ and $K_1(B)$ are also canonically isomorphic.

Let A be a central simple algebra of degree 4. The algebra $A \otimes_F A$ is then similar to a quaternion algebra Q. Denote C the corresponding conic curve. Since A is of exponent dividing 4, the algebra $A \otimes_F Q$ is similar to A^{op} and hence is anti-similar to A. Denote i the canonical isomorphism

$$K_1(A \otimes_F Q) \xrightarrow{\sim} K_1(A).$$

For a variety X over F we write $K_*(X, A)$ for the K-groups of the category of locally free right $\mathcal{O}_X \otimes_F A$-modules of finite rank.

By a computation of K-theory of Severi-Brauer varieties (cf. [21], [20]),

$$K_1(C, A) \xrightarrow{\sim} K_1(A) \oplus K_1(A \otimes Q) \xrightarrow{\sim} K_1(A) \oplus K_1(A). \tag{7}$$

Consider the case when Q splits, i.e. C is a projective line. The inverse isomorphism to (7) is given by the inverse image with respect to the structure morphism $p : C \to \operatorname{Spec}(F)$ on the first component and on the second component by $u \mapsto p^*\big(u \otimes \mathcal{O}(-1)\big)$. Since the class of a rational point in $K_0(C)$ equals $[\mathcal{O}] - [\mathcal{O}(-1)]$, for any rational point $x \in C$ the direct image homomorphism

$$K_1(A) = K_1(\operatorname{Spec} F(x), A) \to K_1(C, A) \xrightarrow{\sim} K_1(A) \oplus K_1(A)$$

takes a to $([a], -[a])$.

In the general case (when Q is not necessarily split) consider any closed point $x \in C$. The direct image homomorphism

$$K_1(A_{F(x)}) \xrightarrow{i_{x*}} K_1(C, A) \xrightarrow{\sim} K_1(A) \oplus K_1(A)$$

factors as follows:

$$K_1(A_{F(x)}) \longrightarrow K_1(C_{F(x)}, A_{F(x)}) \xrightarrow{N_{F(x)/F}} K_1(C, A) \xrightarrow{\sim} K_1(A) \oplus K_1(A).$$

Hence the computation above proves the following

Lemma 6.1. *For any closed point $x \in C$, the sum of $K_1(A)$-coordinates in the image of i_{x*} is zero.* $\square$

Consider the inverse image homomorphism with respect to the generic point $q : \operatorname{Spec} F(C) \to C$ of C:

$$K_1(A) \oplus K_1(A \otimes Q) = K_1(C, A) \xrightarrow{q^*} K_1\big(A_{F(C)}\big).$$

The restriction of q^* on the first component $K_1(A)$ is the field extension homomorphism. The restriction of q^* on the second component $K_1(A \otimes Q)$ is given by the following composite of natural homomorphisms (we use the fact that Q splits over $F(C)$):

$$K_1(A \otimes Q) \longrightarrow K_1(A_{F(C)} \otimes Q_{F(C)}) \xrightarrow{\sim} K_1(A_{F(C)}).$$

Hence the composite of the second restriction with the natural identification $K_1(A \otimes Q) = K_1(A)$ is also the field extension homomorphism. Thus, we have proved

Lemma 6.2. *The image of q^* coincides with the image of the natural field extension homomorphism $K_1(A) \longrightarrow K_1(A_{F(C)})$.* $\square$

Now consider the localization sequence (cf. [21]):

$$\coprod_{x \in C} K_1(A_{F(x)}) \xrightarrow{i_{x*}} K_1(C, A) \xrightarrow{q^*} K_1\big(A_{F(C)}\big) \longrightarrow \coprod_{x \in C} K_0(A_{F(x)}).$$

Proposition 6.3. *The natural homomorphism $\mathrm{SK}_1(A) \to \mathrm{SK}_1(A_{F(C)})$ is an isomorphism.*

Proof. By 6.1 and 6.2, the rows of the following diagram with reduced norm homomorphisms

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_1(A) & \longrightarrow & K_1\big(A_{F(C)}\big) & \longrightarrow & \coprod_{x \in C} K_0(A_{F(x)}) \\
 & & \Big\downarrow{\scriptstyle \mathrm{Nrd}} & & \Big\downarrow{\scriptstyle \mathrm{Nrd}} & & \Big\downarrow{\scriptstyle \mathrm{Nrd}} \\
0 & \longrightarrow & K_1(F) & \longrightarrow & K_1\big(F(C)\big) & \longrightarrow & \coprod_{x \in C} K_0\,\big(F(x)\big).
\end{array}$$

are exact. The result follows from the injectivity of the right vertical homomorphism and the snake lemma. $\square$

The Bloch-Ogus spectral sequence for C becomes a long exact sequence of cohomology groups with coefficients $\mathbb{Z}/2\mathbb{Z}$:

$$(8) \qquad \dots \to \coprod_{x \in C} H^{n-2}F(x) \longrightarrow H^n_{et}C \longrightarrow H^nF(C) \xrightarrow{d} \coprod_{x \in C} H^{n-1}F(x) \to \dots$$

Since Q splits over $F(C)$, the class of $A_{F(C)}$ belongs to $H^2F(C)$. It follows from $d(A_{F(C)}) = 0$ that there is $\theta \in H^2_{et}C$ such that $\theta_{F(C)} = A_{F(C)}$.

Assume that Q is a division algebra. Then θ is not in the image of H^2F. By [26], there is an exact sequence

$$(9) \qquad \dots \longrightarrow H^nF \longrightarrow H^n_{et}C \longrightarrow H^{n-2}F \xrightarrow{\partial} H^{n+1}F \longrightarrow \dots,$$

where ∂ is the multiplication by $(-1) \cup [Q] \in H^3F$. In our case the class of Q is divisible by 2 in the Brauer group, hence $(-1) \cup [Q] = 0$. Thus, for any n there is an exact sequence

$$0 \longrightarrow H^nF \longrightarrow H^n_{et}C \longrightarrow H^{n-2}F \longrightarrow 0.$$

Since the class θ does not come from H^2F, its image in H^0F is not trivial. Hence any element in $H^n_{et}C$ can be written in the form $v_C + u_C \cup \theta$ for $v \in H^nF$ and $u \in H^{n-2}F$.

It follows from exactness of (8) that the natural map

$$H^n_{et}C \longrightarrow A^0(C, \mathcal{H}^n)$$

is surjective. We have proved

Lemma 6.4. *Any element in $A^0(C, \mathcal{H}^n)$ is of the form $v_{F(C)} + u_{F(C)} \cup A_{F(C)}$ for $v \in H^nF$ and $u \in H^{n-2}F$.* □

Let X be the generalized Severi-Brauer variety $SB(2, A)$. For any point $x \in X$, the index of A over $F(x)$ is at most 2. Hence the algebra Q splits over $F(x)$ and therefore $C_{F(x)}$ is a projective line.

By [2, Ch.XI, Th.9], there is a quadratic subfield $L \subset A$. Since $\operatorname{ind}(A_L) = 2$, there is an L-rational point in X_L. The image of this point to X gives a closed point of degree 2.

Lemma 6.5.

$$\operatorname{Ker}\big(A^0(C, \mathcal{H}^4) \to A^0(C_{F(X)}, \mathcal{H}^4)\big) \subset \operatorname{Im}\big(H^4F \to H^4F(C)\big).$$

Proof. Let w be in the kernel. By Lemma 6.4, $w = v_{F(C)} + u_{F(C)} \cup A_{F(C)}$ for some $v \in H^4F$ and $u \in H^2F$. Choose a closed point $x \in X$ of degree 2. The element $t = v_{F(x)} + u_{F(x)} \cup A_{F(x)} \in H^4F(x)$ is split by the extension $F(x)(C \times X)$. Since $C \times X$ has a point over $F(x)$, the map $H^4F(x) \to H^4F(x)(C \times X)$ is injective and hence $t = 0 \in H^4F(x)$, i.e. $u_{F(x)} \cup A_{F(x)} = v_{F(x)}$.

Choose an element $u' \in H^2(F, \mu_4^{\otimes 2})$ in the inverse image of u under the surjection $H^2(F, \mu_4^{\otimes 2}) \to H^2F$ (cf. [17]) and consider the cup-product $s = u' \cup [A]$ with respect to the pairing

$$H^2(F, \mu_4^{\otimes 2}) \otimes H^2(F, \mu_4) \longrightarrow H^4(F, \mu_4^{\otimes 3}).$$

We know that $s_{F(x)} = v_{F(x)}$ and v is an element of order 2. Hence

$$0 = 2v = N_{F(x)/F}(v_{F(x)}) = N_{F(x)/F}(s_{F(x)}) = 2s = u' \cup [Q] \in H^4(F, \mu_4^{\otimes 3}).$$

Since the natural homomorphism $H^4(F) \to H^4(F, \mu_4^{\otimes 3})$ is injective (cf. [18], [23]), we have $u \cup [Q] = 0 \in H^4F$. By [15, Prop.3.15], u belongs to the image of the norm map $\coprod_{x \in C} H^2F(x) \to H^2F$. The exactness of the sequences (8) and (9) shows then that $u_{F(C)} \cup A_{F(C)} \in \operatorname{Im}\big(H^4F \to H^4F(C)\big)$, hence $w \in \operatorname{Im}\big(H^4F \to H^4F(C)\big)$. □

Theorem 6.6. *Let A be a central simple algebra of dimension 16 over F, $X = SB(2, A)$. Then there exists an exact sequence*

$$0 \longrightarrow \mathrm{SK}_1(A) \longrightarrow H^4F/(2[A] \cup H^2F) \longrightarrow H^4F(X).$$

Proof. By Theorem 5.3 such a sequence exists if $2[A] = 0 \in \mathrm{Br}(F)$. In the general case, A is a division algebra and $2[A_{F(C)}] = 0 \in \mathrm{Br}\big(F(C)\big)$, hence there is a sequence

$$0 \longrightarrow \mathrm{SK}_1(A_{F(C)}) \longrightarrow H^4F(C) \longrightarrow H^4F(C \times X).$$

Proposition 6.3 shows that the first term of this sequence if isomorphic to $\mathrm{SK}_1(A)$. Hence it suffices to prove the exactness of the sequence

$$H^2F \xrightarrow{\cup 2[A]} \mathrm{Ker}\big(H^4F \to H^4F(X)\big) \longrightarrow \mathrm{Ker}\big(H^4F(C) \longrightarrow H^4F(C \times X)\big) \longrightarrow 0.$$

By [15, Prop.3.15], the kernel of $H^4F \to H^4F(C)$ equals $2[A] \cup H^2F$, whence the exactness in the second term. Finally take any $w \in \mathrm{Ker}\big(H^4F(C) \longrightarrow H^4F(C \times X)\big)$. For any closed point $x \in C$ consider the following commutative diagram

$$\begin{array}{ccc} H^4F(C) & \longrightarrow & H^4F(C \times X) \\ \Big\downarrow{\scriptstyle \partial_x} & & \Big\downarrow{\scriptstyle \partial_y} \\ H^3F(x) & \longrightarrow & H^3F(x)(X) \end{array}$$

where $\overline{\{y\}} = x \times X$. Since Q is split over $F(x)$, X is an Albert quadric over $F(x)$. Anisotropic Albert form cannot be a subform of a 3-fold Pfister form, hence by a theorem of Arason [3], the bottom homomorphism is injective; therefore $\partial_x(w) = 0$ for any closed $x \in C$, i.e. $w \in A^0(C, \mathcal{H}^4)$. By Lemma 6.5, $w = v_{F(C)}$ for some $v \in H^4F$. It remains to notice that, since C has a point over $F(X)$, the natural homomorphism $H^4F(X) \to H^4F(C \times X)$ is injective and therefore $v \in \ker\big(H^4F \to H^4F(X)\big)$. $\square$

References

[1] A.A. Albert. Normal division algebras of degree four over an algebraic field. *Trans. Amer. Math. Soc.* **34** (1932), 363–372.

[2] A.A. Albert. *Structure of Algebras.* Amer. Math. Soc. Coll. Pub. XXIV, Providence R. I. 1961.

[3] J. Arason. Cohomologische Invarianten quadratischer Formen. *J. Algebra* **36** (1975), 448–491.

[4] A. Blanchet. Function fields of generalized Brauer-Severy varieties. *Comm. Algebra.* **19** (1991), 97–118.

[5] S. Bloch, A. Ogus. Gersten's conjecture and the homology of schemes. Ann. Sci. Ec. Norp. Sup. Ser. 4. **7** (1974), 181–202.

[6] A. Borel. Linear Algebraic Groups, Second Enlarged Edition. *Graduate Texts in Mathematics.* Vol. 126, Springer, Berlin, 1991.

[7] J.-L. Colliot–Thélène. Birational invariants, purity, and the Gersten conjecture. *Proceedings of the 1992 Summer Research Institute on Quadratic Forms and Division Algebras,* (W. Jacob and A. Rosenberg, eds), *Symposia Pure Math.* **58.1** (1995) 1–64, Amer. Math. Soc., Providence, R.I.

[8] H. Esnault, B. Kahn., M. Levine, E. Viehweg. The Arason invariant and mod 2 algebraic cycles. *J. Amer. Math. Soc.* **11** (1998), no. 1, 73–118.

[9] B. Iversen. Brauer group of a linear algebraic group. *J. Algebra* **42** (1976), no. 2, 295–301.

[10] N.A. Karpenko. Algebro-geometric invariants of quadratic forms. *Leningrad (St.Petersburg) Math. J.* **2** (1991), 119–138.
[11] M.-A. Knus, A.S. Merkurjev, M. Rost, J.-P. Tignol. *The Book of Involutions,* Coll. Publ. AMS, v.**44**, 1998.
[12] M. Levine. The algebraic K-theory of the classical groups and some twisted forms. *Duke Math. J.* **70** (1993), 405–443.
[13] A.S. Merkurjev. K-theory of simple algebras. *Proceedings of the 1992 Summer Research Institute on Quadratic Forms and Division Algebras,* (W. Jacob and A. Rosenberg, eds), *Symposia Pure Math.* **58.1** (1995) 65–83, Amer. Math. Soc., Providence, R.I.
[14] A.S. Merkurjev. Generic element in SK_1 for simple algebras. *K-theory.* **7** (1993), 1–3.
[15] A.S. Merkurjev. On the norm residue homomorphism for fields. *Mathematics in St. Petersburg*, 49–71, Amer. Math. Soc. Transl. Ser. 2, **174**, *Amer. Math. Soc., Providence, RI*, 1996.
[16] A.S. Merkurjev. Index reduction formula. *J.Ramanujan Math. Soc.* **12**, no. 1 (1997), 49–95.
[17] A.S. Merkurjev, A.A. Suslin. $\mathcal{K}$-cohomology of Severi-Brauer varieties and the norm residue homomorphism. *Math. USSR Izvestiya.* **21** (1983), 307–340.
[18] A.S. Merkurjev, A.A. Suslin. The norm residue homomorphism of degree 3. *Izv. Akad. Nauk SSSR Ser. Mat.* **54** (1990), 339–356.
[19] A.S. Merkurjev, I.A. Panin, A. Wadsworth. Index reduction formulas for twisted flag varieties, I. *K-Theory* **10** (1996), 517–596.
[20] I.A.Panin,*On the algebraic K-theory of twisted flag varieties*,K-Theory, **8**, (1994), 541–585.
[21] D. Quillen. Higher algebraic K-theory I. *Lect. Notes Math.* **341** (1972), Springer-Verlag, Berlin, 85–147.
[22] M. Rosenlicht. Maxwell Toroidal algebraic groups. *Proc. Amer. Math. Soc.* **12** (1961), 984–988.
[23] M. Rost. Hilbert 90 for K_3 for degree-two extensions. *Preprint Regensburg.*, 1986.
[24] M. Rost. Chow groups with coefficients. *Doc. Math.* **1** (1996), No. 16, 319–393 (electronic).
[25] J.-J. Sansuc. Group de Brauer et arithmétique des groupes algébrique linéaires sur un corps de nombres. *J. Reine Angew. Math.* **327** (1981), 12–80.
[26] A.A. Suslin. Quaternion homomorphism for the field of functions on a conic. (Russian) *Dokl. Akad. Nauk SSSR* **265** (1982), no. 2, 292–296.
[27] A.A. Suslin. K-theory and $\mathcal{K}$-cohomology of certain group varieties. *Advances in Soviet Math.* **4** (1991), 53–74.
[28] R.G. Swan. Zero cycles on quadric hypersurfaces. *Proc. Amer. Math. Soc.* **107** (1989), 43–46.
[29] W.C. Waterhouse. *Introduction to Affine Group Schemes.* , Springer, New York, 1979.
[30] O. Zariski, P. Samuel. *Commutative Algebra. II.* Graduate Texts in Mathematics, vol. 29 Springer-Verlag, 1976.

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