COHOMOLOGICAL INVARIANTS OF SIMPLY CONNECTED GROUPS OF RANK 3

ALEXANDER MERKURJEV

Let G be a linear algebraic group defined over a field F. One can define an equivalence relation (called *R-equivalence*) on the group $G(F)$ of points over F as follows (cf., [4], [9] and [14]). Two points $g_0, g_1 \in G(F)$ are *R-equivalent*, if there is a rational morphism $f : \mathbb{A}^1_F \to G$ of algebraic varieties over F defined at points 0 and 1 such that $f(0) = g_0$ and $f(1) = g_1$. The group of R-equivalence classes is denoted by $G(F)/R$. For example, if $G = \mathbf{SL}_1(A)$ is the special linear group of a central simple F-algebra A, then the group of R-equivalence classes $G(F)/R$ is equal to the reduced Whitehead group in algebraic K-theory (cf., [27])

$$\mathrm{SK}_1(A) = \mathrm{SL}_1(A)/[A^\times, A^\times].$$

We say that the group G is *R-trivial* if $G(E)/R = 1$ for any field extension L/F (cf., [14]). The group $G(F)/R$ measures complexity of $G(F)$. One of the major properties of $G(F)/R$ is that this group is *rigid*, i.e. $G(F)/R = G\big(F(t)\big)/R$. In other words, any rational family of elements in $G(F)/R$ is constant, whereas any R-trivial element (i.e. an element in the kernel of $G(F) \to G(F)/R$) can be connected to the identity of the group by a rational family of elements in G.

It is known that an algebraic group G of rank (dimension of a maximal torus) at most 2 is rational (cf., [2, Th. 7.9], [27, Th.4.74]) and hence G is R-trivial by [4], [14, Prop. 1]. In the present paper we consider the group $G(F)/R$ for a simple simply connected group of type A_3 (rank 3). The starting point is the following two results:

Theorem 0.1. (M. Rost, [12, Th. 4], [7, §17]) *Let A be a central simple algebra of dimension* 16 *over a field F* ($\operatorname{char} F \neq 2$) *and exponent* 2 *in the Brauer group of F. Then there is an exact sequence*

$$1 \longrightarrow \mathrm{SK}_1(A) \xrightarrow{r} H^4(F, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^4\big(F(X), \mathbb{Z}/2\mathbb{Z}\big),$$

where X is the Albert quadric of A.

Thus, Rost's theorem gives a cohomological description of the group of equivalence classes $\mathrm{SL}_1(A)/R = \mathrm{SK}_1(A)$ of the simple simply connected group $\mathbf{SL}_1(A)$ of rank 3.

Date: April, 1999.

Partially supported by the N.S.F.

Theorem 0.2. [10] *Let A be a central division algebra of dimension* 16 *over a field F* ($\operatorname{char} F \neq 2$). *Then there exists a field extension E/F such that* $\mathrm{SK}_1(A \otimes_F E) \neq 1$. *In particular, the group $\mathbf{SL}_1(A)$ is not R-trivial and hence is not rational.*

In other words, if A is a division algebra of exponent 2, the family of homomorphisms $r \otimes_F E$ in Theorem 0.1 for all field extensions E/F, being considered as a morphism of functors $E \mapsto \mathrm{SK}_1(A \otimes_F E)$ and $E \mapsto H^4(E, \mathbb{Z}/2\mathbb{Z})$ from the category of field extensions of F to the category of groups, is not trivial (whereas the group $\mathrm{SK}_1(A)$ can be trivial).

The homomorphism r can be considered as a dimension four *cohomological invariant* of the group $\mathbf{SL}_1(A)$ as defined in [17]. Note that cohomological invariants of dimension at most 3 of simply connected groups are trivial by [17, Prop. 4.9]. Thus the Rost's invariant r is an example of a nontrivial invariant of the smallest dimension of a simply connected group of the smallest rank.

The main result of the paper (Theorem 4.5) is a generalization of the Rost's Theorem 0.1 to the case of arbitrary simple simply connected group G of type A_3. It is known (cf., [29], [7, Th. 26.9]) that G is isomorphic to the *special unitary group* $\mathbf{SU}(B, \tau)$ of a (uniquely determined up to isomorphism) semisimple algebra B of degree 4 over a quadratic extension L/F with a unitary involution τ. We define a dimension four invariant of G with coefficients in a certain cycle module and describe its image. If L splits, i.e. $L = F \times F$, then $B = A \times A^{\mathrm{op}}$ with the exchange involution τ and $\mathbf{SU}(B, \tau) = \mathbf{SL}_1(A)$. The special case $\exp(A) = 2$ is the Rost's Theorem 0.1.

Note that the root system A_3 coincides with D_3, hence G is canonically isomorphic to the *spinor group* $\mathbf{Spin}(D, \sigma)$ for the discriminant algebra D of the pair (B, τ) with canonical involution σ (cf., [7, §26]).

We proceed in several steps. First we define a variety $X = J(B, \tau)$ which plays a role of the Albert quadric in Theorem 0.1. It is the variety of right ideals J in B of L-dimension 8 such that $\tau(J) \cdot J = 0$. Let $\Gamma_1(F)$ be the inverse image of $K_1(F) \subset K_1(L)$ under the reduced norm homomorphism $\mathrm{Nrd} : K_1(B) \to K_1(L)$. Denote by $\Sigma_1(F)$ the subgroup in $\Gamma_1(F)$ generated by the classes of the elements represented by $b \in B^\times$ such that $\tau(b) = b$. We use a computation of $G(F)/R$ in [3, Th. 5.4] as the factor group $\Gamma_1(F)/\Sigma_1(F)$. Using a computation of algebraic K-theory of projective homogeneous varieties given in [22] we relate the group $\Gamma_1(F)/\Sigma_1(F)$ to the factor group $K_1(X)^{(2/3)}$ of the topological filtration on $K_1(X)$. Then, considering the Brown-Gersten-Quillen spectral sequence for X we prove that $K_1(X)^{(2/3)}$ is isomorphic to the K-cohomology group $H^2(X, \mathcal{K}_3)$ using computation of the Chow group $\mathrm{CH}^4(X)$ given in [11]. Finally we use a theorem of B. Kahn relating the latter group to the degree four Galois cohomology groups.

We use the following notation.

For an algebraic variety X defined over a field F and a field extension L/F, by X_L we denote $X \times_{\operatorname{Spec} F} \operatorname{Spec} L$ and by X_{sep} the variety $X_{F_{\mathrm{sep}}}$ over a separable closure F_{sep} of F.

By $H^p(X,\mathcal{K}_q)$ we denote the $\mathcal{K}$-cohomology groups and by $\mathrm{CH}^p(X) = H^p(X,\mathcal{K}_p)$ the Chow groups (cf., [24, §7.5]).

For a field F, $\mathrm{char} F \neq 2$, denote by $H^p(F)$ the Galois cohomology group $H^p\big(\mathrm{Gal}(F_{\mathrm{sep}}/F), \mathbb{Z}/2\mathbb{Z}\big)$. For an element $a \in F^\times$, (a) is the element of $H^1(F)$ corresponding to a under the canonical isomorphism $H^1(F) \simeq F^\times/F^{\times 2}$.

We denote by μ_n the Galois module of n-th roots of unity and for any $q \in \mathbb{Z}$ and by $\mathbb{Q}/\mathbb{Z}(q)$ the direct limit of $\mu_n^{\otimes q}$ over all n prime to $\mathrm{char} F$.

We will consider a central semisimple algebra B with a unitary involution τ over a quadratic Galois extension L/F, i.e. τ acts on L as the generator of the Galois group of L over a field F. If L is a field, then B is a simple algebra over L. If L splits, i.e. $L \simeq F \times F$, then $B \simeq A \times A^{\mathrm{op}}$ for a central simple F-algebra A with the *exchange* involution τ (cf., [7, Prop. 2.14]). By *index* $\mathrm{ind}\, B$ we mean the standard index of the central simple algebra B if L is a field and index of A if L splits.

For (B,τ) as above, $\mathbf{SU}(B,\tau)$ denotes the special unitary group of a central simple algebra B with a unitary involution τ (cf., [7, §23]). It is a simple simply connected algebraic group of type A_{n-1}, where n is the degree of B over L. The group $\mathrm{SU}(B,\tau)$ of its F-points consists of all $b \in B^\times$ such that $\tau(b)\cdot b = 1$ and $\mathrm{Nrd}(b) = 1$.

1. K-theory of X

In this section we introduce a variety X, which replaces the Albert quadric in the Rost's theorem and compute its algebraic K-theory.

1.1. **Split case.** Let V be a vector space of dimension 4 over a field F. Consider the Grassmannian variety $X = \mathrm{Gr}(2,V)$ of 2-dimensional subspaces $U \subset V$. Denote by $\mathcal{E}$ the locally free sheaf of sections of the canonical vector bundle on X (of the subvariety in $V \times X$ consisting of all pairs (v,U) such that $v \in U$). We denote by ρ the class of $\mathcal{E}$ in $K_0(X)$ and by ξ the class of the second exterior power $\Lambda^2\mathcal{E}$, so that $\xi = \lambda^2\rho$, where λ^2 is the second exterior power operation on $K_0(X)$. By [21, 5.9 and 7.1], $K_*(X)$ is a free module over $K_*(F)$ with the basis

$$\{1, \xi, \xi^2, \rho, \rho\xi, s^2\rho\}, \tag{1}$$

where s^2 is the symmetric square.

We will need a slightly different basis. The second exterior power gives a closed embedding

$$X \hookrightarrow \mathbb{P}_F(\Lambda^2 V),$$

identifying X with a projective quadric in $\mathbb{P}_F(\Lambda^2 V)$. Let $Y \subset X$ be a hyperplane section in X. Since the class of the sheaf $\mathcal{O}_X(-1)$ in $K_0(X)$ equals ξ, the existence of an exact sequence

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0,$$

where $i : Y \hookrightarrow X$ is the closed embedding, implies that the class $h \in K_0(X)$ of $i_*\mathcal{O}_Y$ is equal to $1 - \xi$.

A linear form $\varphi \in V^*$ can be viewed as a section of the sheaf $\mathcal{E}^*$ and hence as a morphism of sheaves $\mathcal{E} \to \mathcal{O}$. The variety $P \subset X$ of zeros of a section $\varphi \neq 0$ consists of all 2-dimensional subspaces $U \subset V$ such that $\varphi(U) = 0$ and hence is equal to the projective plane $\mathrm{Gr}(2, \mathrm{Ker}\,\varphi) \subset X$. There is an exact sequence

$$0 \to \Lambda^2\mathcal{E} \to \mathcal{E} \xrightarrow{\varphi} \mathcal{O}_X \to j_*\mathcal{O}_P \to 0,$$

where $j : P \hookrightarrow X$ is the closed embedding. Denote the class of $j_*\mathcal{O}_P$ in $K_0(X)$ by p, so that

$$p = 1 + \xi - \rho. \tag{2}$$

Dualizing the natural inclusion $\mathcal{E} \hookrightarrow V \otimes_F \mathcal{O}_X$ we get an exact sequence

$$0 \to \mathcal{E}' \to V^* \otimes_F \mathcal{O}_X \to \mathcal{E}^* \to 0. \tag{3}$$

The class of $\mathcal{E}'$ in $K_0(X)$ we denote by ρ'. A vector $v \in V$ can be viewed as a section of the sheaf $\mathcal{E}'^*$ and hence as a morphism $\mathcal{E}' \to \mathcal{O}$. The variety $P' \subset X$ of zeros of a section $v \neq 0$ consists of all 2-dimensional subspaces $U \subset V$ such that $v \in U$ and hence is isomorphic to another projective plane $\mathbb{P}(V/Fv)$ in the quadric X.

There is an exact sequence

$$0 \to \Lambda^2\mathcal{E}' \to \mathcal{E}' \xrightarrow{v} \mathcal{O}_X \to j'_*\mathcal{O}'_P \to 0,$$

where $j' : P' \hookrightarrow X$ is the closed embedding. Denote the class of $j'_*\mathcal{O}_{P'}$ in $K_0(X)$ by p', so that

$$p' = 1 + \xi - \rho'. \tag{4}$$

It follows from (3) that $\lambda^2\rho' \cdot \lambda^2\rho^* = 1$, hence $\lambda^2\rho' = \lambda^2\rho = \xi$. The natural pairing

$$\mathcal{E}' \otimes \mathcal{E}' \to \Lambda^2(\mathcal{E}')$$

induces an isomorphism between sheaves $\mathcal{E}'^*$ and $\mathcal{E}' \otimes \Lambda^2(\mathcal{E}')^{\otimes -1}$, hence $\rho'^* = \rho'\xi^{-1}$. The exact sequence dual to (3) gives $\rho'^* = 4 - \rho$, hence

$$\rho' = 4\xi - \rho\xi. \tag{5}$$

Thus, the element $\rho\xi$ in the basis (1) can be replaced by ρ'.

Similarly, the sequence (3) gives $\rho^* = 4 - \rho'$ and hence

$$\rho = 4\xi - \rho'\xi. \tag{6}$$

If $\varphi \in V^*$ and $v \in V$ are chosen so that $\varphi(v) \neq 0$, then $P \cap P' = \emptyset$, hence $pp' = 0 \in K_0(X)$. Using (2), (4), (5) and (6) we then get

$$\rho^2\xi^{-1} = 1 - 6\xi + \xi^2 + 4\rho. \tag{7}$$

Since $\rho^2 = s^2\rho + \lambda^2\rho = s^2\rho + \xi$, by (5) and (7),

$$s^2\rho = -6\xi^2 + \xi^3 + 4\rho\xi = 16\xi - 6\xi^2 + \xi^3 - 4\rho'.$$

Thus, we can replace $s^2\rho$ in the basis (1) by ξ^3.

We have proved

Proposition 1.1. *The group $K_*(X)$ is a free module over $K_*(F)$ with the basis $\{1, \xi, \xi^2, \xi^3, \rho, \rho'\}$.* $\square$

We will also need a basis of $K_*(X)$ which is compatible with the topological filtration $K_*(X)^{(i)}$ by codimension of support (cf., [24, §5]). Clearly, p and p' are the elements of codimension 2 and h^i is of codimension i. The variety X is cellular (cf., [5]), hence

$$H^i(X, \mathcal{K}_{*+i}) = K_*(F) \otimes \mathrm{CH}^i(X).$$

Also for $i = 0, 1$, the composition

$$K_*(F) \xrightarrow{h^i} K_*(X)^{(i/i+1)} \hookrightarrow H^i(X, \mathcal{K}_{*+i})$$

is then injective since $\mathrm{CH}^i(X)$ is an infinite cyclic group generated by the i-th power of the class of a hyperplane section. A simple computation using (2) and (4) then gives

Corollary 1.2. *The set of elements $\{1, h, h^2, h^3, p, p'\}$ is a basis of $K_*(X)$ over $K_*(F)$, $\{h, h^2, h^3, p, p'\}$ is a basis of $K_*(X)^{(1)}$ and $\{h^2, h^3, p, p'\}$ is a basis of $K_*(X)^{(2)}$.* $\square$

1.2. **Nonsplit case.** We consider in this section algebraic K-theory of twisted forms of $\mathrm{Gr}(2, 4)$.

Let B be a central semisimple algebra of degree 4 with a unitary involution τ over a quadratic extension L of a field F of characteristic different from 2 (we don't exclude the split case $L = F \times F$). Consider the variety $X = J(B, \tau)$ of right ideals J in B of L-dimension 8 such that $\tau(J) \cdot J = 0$. This is a projective homogeneous variety of the special unitary group $\mathbf{SU}(B, \tau)$ of dimension 4 being a twisted form of the Grassmannian variety $\mathrm{Gr}(2, 4)$ considered in section (1.1) (cf., [19]). In other words, X is isomorphic to $\mathrm{Gr}(2, 4)$ in the split case, i.e. when B and L split.

Let D be the discriminant algebra of (B, τ) (cf., [7, §10]). It is a central simple algebra of degree 6 over F with canonical orthogonal involution σ. The algebra with involution (B, τ) can be reconstructed from (D, σ) as the Clifford algebra with canonical involution [7, §15 D]. The variety X is isomorphic to a unique projective homogeneous variety of dimension 4 of the spinor group $\mathbf{Spin}(D, \sigma)$ which is canonically isomorphic to $\mathbf{SU}(B, \tau)$ (cf., [7, Prop. 15.27]). Thus, X is canonically isomorphic to the *involution variety* $I(D, \sigma)$ of right ideals $I \subset D$ of dimension 6 such that $\sigma(I) \cdot I = 0$. In particular, D splits over any field extension E/F such that $X(E) \neq \emptyset$, for example, over the function field $F(X)$. In the split case X is isomorphic to a quadric in $\mathbb{P}^5$.

Consider the following locally free sheaves on X. Denote by $\mathcal{J}$ the sheaf of sections of the canonical vector bundle on X (of the subvariety in $B \times X$ consisting of all pairs (b, J) such that $b \in J$). It is a sheaf of rank 16 having a natural structure of a right B-module.

In the split case, $B = \mathrm{End}_F(V) \times \mathrm{End}_F(V^*)$ for a vector space V over F of dimension 4. Any rank 8 over $L = F \times F$ right ideal $J \subset B$ such that

$\tau(J) \cdot J = 0$ is of the form

$$J = \mathrm{Hom}(V, U) \times \mathrm{Hom}\big(V^*, (V/U)^*\big)$$

for some 2-dimensional subspace $U \subset V$. Hence $X = \mathrm{Gr}(2, V)$ and

$$\mathcal{J} \simeq (V^* \otimes_F \mathcal{E}) \oplus (V \otimes_F \mathcal{E}'). \tag{8}$$

Viewing X as the involution variety $I(D, \sigma)$ we can consider the sheaf $\mathcal{I}$ of sections of the canonical vector bundle on $I(D, \sigma)$ (of the subvariety in $D \times I(D, \sigma)$ consisting of all pairs (a, I) such that $a \in I$). Clearly, $\mathcal{I}$ is a sheaf of rank 6 and there is a natural structure of a right D-module on $\mathcal{I}$.

In the split case, $D = \mathrm{End}_F(\Lambda^2 V)$ and hence any 6-dimensional right ideal $I \subset D$ is of the form $\mathrm{Hom}(\Lambda^2 V, W)$ for a 1-dimensional subspace $W \subset \Lambda^2 V$. Hence

$$\mathcal{I} \simeq \Lambda^2 V^* \otimes_F \Lambda^2 \mathcal{E} = \Lambda^2(V^* \otimes_F \mathcal{E}). \tag{9}$$

The exact functor $j : M \mapsto \mathcal{J} \otimes_B M$ from the category $\mathcal{P}(B)$ of finitely generated left B-modules to the category $\mathcal{P}(X)$ of locally free sheaves on X of finite rank induces a graded group homomorphism

$$g : K_*(B) \longrightarrow K_*(X).$$

In the split case the category $\mathcal{P}(B)$ is equivalent to $\mathcal{P}(F) \times \mathcal{P}(F)$ and we identify $K_*(B)$ with $K_*(F) \oplus K_*(F)$. By (8), the functor j is then identified with

$$N \times N' \mapsto (\mathcal{E} \otimes_F N) \oplus (\mathcal{E}' \otimes_F N')$$

and hence

$$g(a, a') = a \cdot \rho + a' \cdot \rho'. \tag{10}$$

Let $D^{\otimes i}$ be the i-th tensor power of D over F, so that the tensor power $\mathcal{I}^{\otimes i}$ is a right $D^{\otimes i}$-module. The exact functor $i : \mathcal{P}(D^{\otimes i}) \to \mathcal{P}(X)$, taking a module N to $\mathcal{I}^{\otimes i} \otimes_{D^{\otimes i}} N$, induces a graded group homomorphism

$$f_i : K_*(D^{\otimes i}) \longrightarrow K_*(X).$$

In the split case the category $\mathcal{P}(D)$ is equivalent to $\mathcal{P}(F)$ and we identify $K_*(D)$ with $K_*(F)$. By (9), the functor i is then identified with

$$N \mapsto (\Lambda^2 \mathcal{E})^{\otimes i} \otimes_F N$$

and hence

$$f_i(a) = a \cdot \xi^i. \tag{11}$$

Proposition 1.3. *The map*

$$h = g + \sum_{i=0}^{i=3} f_i : K_*(B) \oplus K_*(F) \oplus K_*(D) \oplus K_*(D^{\otimes 2}) \oplus K_*(D^{\otimes 3}) \longrightarrow K_*(X)$$

is an isomorphism.

Proof. By Proposition 1.1, (10) and (11), h is an isomorphism in the split case. The general case follows from the split one by the splitting principle (cf., [22], [16, 2.8]). ☐

We'd like to find a similar presentation of $K_n(X)$ for $n = 0, 1$, which is compatible with the topological filtration as in Corollary 1.2 in the split case. Consider the reduced norm homomorphism

$$N = \mathrm{Nrd} : K_n(D) \longrightarrow K_n(F).$$

If $n = 0$, N takes the canonical generator of $K_0(D)$ to $\mathrm{ind}(D)$ times the generator of $K_0(F)$. Also consider the composition

$$\widetilde{N} : K_n(B) \xrightarrow{\mathrm{Nrd}} K_n(L) \xrightarrow{N_{L/F}} K_n(F).$$

The algebra D carries the involution σ of the first kind, hence the index of D is a power of 2. Since D is of degree 6, it is isomorphic to the matrix algebra $M_3(Q)$ for a quaternion algebra Q over F. Denote by m the composition

$$K_n(F) \xrightarrow{\sim} K_n M_3(F) \longrightarrow K_n M_3(Q) \xrightarrow{\sim} K_n(D).$$

The reduced norm homomorphism $\mathrm{Nrd} : K_n(D) \to K_n(F)$ is injective (if $n = 1$, for all algebras D of squarefree index by [28]) and its image is generated by the norms in finite field extensions of F which split D. The group $K_n(B)$ is also generated by the norms in finite field extensions of L. If B and its center L split, then D is also split since by [7, Prop. 10.30] $D \otimes_F L$ is similar to $B^{\otimes 2}$. Hence the image of $\widetilde{N}$ is contained in the image of N. Thus we have a well defined homomorphism

$$j : K_n(B) \longrightarrow K_n(D)$$

such that $N \circ j = \widetilde{N}$.

The exponent of the algebra D is at most 2, hence $D^{\otimes 2} \sim F$ and $D^{\otimes 3} \sim D$. We identify $K_n(D^{\otimes 2})$ with $K_n(F)$ and $K_n(D^{\otimes 3})$ with $K_n(D)$.

We define the following homomorphisms:

$$\begin{aligned}
\overline{g} &: K_n(B) \longrightarrow K_n(X), & \overline{g} &= f_0 \circ \widetilde{N} + f_1 \circ j - g \\
\overline{f}_0 &: K_n(F) \longrightarrow K_n(X), & \overline{f}_0 &= f_0; \\
\overline{f}_1 &: K_n(D) \longrightarrow K_n(X), & \overline{f}_1 &= f_0 \circ N - f_1; \\
\overline{f}_2 &: K_n(F) \longrightarrow K_n(X), & \overline{f}_2 &= f_0 - f_1 \circ m + f_2; \\
\overline{f}_3 &: K_n(D) \longrightarrow K_n(X), & \overline{f}_3 &= f_0 \circ N - 3f_1 + 3f_2 \circ N - f_3.
\end{aligned}$$

Thus, we have a group homomorphism

$$\overline{h} = \overline{g} + \sum_{i=0}^{i=3} \overline{f}_i : K_n(B) \oplus K_n(F) \oplus K_n(D) \oplus K_n(F) \oplus K_n(D) \longrightarrow K_n(X).$$

If B and L split, we identify $\widetilde{N} : K_*(F) \oplus K_*(F) \to K_*(F)$ and j with $(a, a') \mapsto a + a'$. It follows from (2), (4), (10) that

$$\overline{g}(a, a') = (a + a') \cdot (1 + \xi) - a \cdot \rho - a' \cdot \rho' = a \cdot p + \ a' \cdot p'.$$

In particular,

$$\operatorname{Im}(\overline{g}) \subset K_n(X)^{(2)}. \tag{12}$$

If just D splits, we identify N with the identity endomorphism of $K_*(F)$ and m with the twice of the identity of $K_*(F)$. It follows from (11) that

$$\begin{aligned}
\overline{f}_0(a) &= a \cdot 1;\\
\overline{f}_1(a) &= a \cdot 1 - a \cdot \xi = a \cdot h;\\
\overline{f}_2(a) &= a \cdot 1 - 2a \cdot \xi + a \cdot \xi^2 = a \cdot h^2;\\
\overline{f}_3(a) &= a \cdot 1 - 3a \cdot \xi + 3a \cdot \xi^2 - a \cdot \xi^3 = a \cdot h^3.
\end{aligned}$$

In particular,

$$\operatorname{Im}(\overline{f}_i) \subset K_n(X)^{(i)}. \tag{13}$$

We claim that the inclusions (12) and (13) also hold in the nonsplit case. It is trivial for $\overline{f}_0$. Since the homomorphisms $\overline{f}_i$ (resp. $\overline{g}$) commute with the norms in finite field extensions and the groups $K_n(D)$ (resp. $K_n(B)$) for $n = 0, 1$ are generated by the norms in field extensions which split D (resp. split B and L), the inclusions hold also for $\overline{f}_1$, $\overline{f}_3$, $\overline{g}$ and it remains to show the statement in the case $i = 2$ and $n = 0$. It follows from the fact that the natural homomorphisms of factor groups of the terms of the topological filtration (for $k = 0, 1$)

$$K_0(X)^{(k/k+1)} = \mathrm{CH}^k(X) \longrightarrow \mathrm{CH}^k(X_{\mathrm{sep}}) = K_0(X_{\mathrm{sep}})^{(k/k+1)}$$

are injective.

Proposition 1.4. *The following homomorphisms:* $\overline{h}$,

$$\overline{h}_1 = \overline{g} + \overline{f}_1 + \overline{f}_2 + \overline{f}_3 : K_n(B) \oplus K_n(D) \oplus K_n(F) \oplus K_n(D) \to K_n(X)^{(1)},$$

$$\overline{h}_2 = \overline{g} + \overline{f}_2 + \overline{f}_3 : K_n(B) \oplus K_n(F) \oplus K_n(D) \to K_n(X)^{(2)}$$

are isomorphisms for $n = 0, 1$. *The image of* $\overline{f}_3$ *is contained in* $K_n(X)^{(3)}$.

Proof. The homomorphisms h and $\overline{h}$ are related by the invertible matrix

$$\begin{pmatrix}
-1 & \widetilde{N} & j & 0 & 0\\
0 & 1 & 0 & 0 & 0\\
0 & N & -1 & 0 & 0\\
0 & 1 & -m & 1 & 0\\
0 & N & -3 & 3N & -1
\end{pmatrix}$$

hence by Proposition 1.3, $\overline{h}$ is an isomorphism.

It suffices to prove surjectivity of $\overline{h}_1$ and $\overline{h}_2$. The factor group $K_n(X)^{(0/1)}$ is canonically embedded into $K_n\big(F(X)\big)$. The composition

$$K_n(F) \xrightarrow{\overline{f}_0} K_n(X) \longrightarrow K_n(X)^{(0/1)} \hookrightarrow K_n\big(F(X)\big)$$

is the natural injection, whence the surjectivity of $\overline{h}_1$.

In order to prove surjectivity of $\overline{h}_2$ it suffices to show that the composition

$$l : K_n(D) \xrightarrow{\overline{f}_1} K_n(X)^{(1)} \longrightarrow K_n(X)^{(1/2)}$$

is injective. By Corollary 1.2, l is an isomorphism over a splitting field extension E/F. Now the statement follows from the fact that l is compatible with field extensions and the natural homomorphism $K_n(D) \to K_n(D \otimes_F E)$ is injective since D is similar to a quaternion algebra. $\square$

It follows from the Proposition 1.4 that the composition

$$\varepsilon_n : K_n(B) \oplus K_n(F) \xrightarrow{\overline{g}+\overline{f}_2} K_n(X)^{(2)} \longrightarrow K_n(X)^{(2/3)} \tag{14}$$

is surjective for $n = 0, 1$.

For $n = 0, 1$, define a subgroup $\Gamma_n(F)$ in $K_n(B)$:

$$\Gamma_n(F) = \{b \in K_n(B) \quad \text{such that} \quad \mathrm{Nrd}(b) \in K_n(F) \subset K_n(L)\},$$

where $\mathrm{Nrd} : K_n(B) \to K_n(L)$ is the reduced norm homomorphism and we identify $K_n(F)$ with a subgroup in $K_n(L)$. Clearly, if B splits, then $\Gamma_n(F) = K_n(F) \subset K_n(L) = K_n(B)$.

We define injective homomorphism

$$\rho_n : \Gamma_n(F) \longrightarrow K_n(B) \oplus K_n(F), \qquad \rho_n(b) = \big(b, -\mathrm{Nrd}(b)\big).$$

Lemma 1.5. *In the split case, the sequence*

$$0 \longrightarrow \Gamma_n(F) \xrightarrow{\rho_n} K_n(B) \oplus K_n(F) \xrightarrow{\varepsilon_n} K_n(X)^{(2/3)}$$

is exact.

Proof. We identify $K_n(B)$ with $K_n(F) \oplus K_n(F)$. For an element $c = (b, b', a) \in K_n(B) \oplus K_n(F)$ we have

$$\varepsilon_n(c) = \bar{b} \cdot p + \bar{b}' \cdot p' + \bar{a} \cdot h^2$$

("bar" means the residue modulo $K_n(X)^{(3)}$). By [6, §3.2],

$$K_n(X)^{(2/3)} = K_n(F) \cdot \overline{p} \oplus K_n(F) \cdot \overline{p}'$$

and $p + p' - h^2$ in $K_0(X)$ is the class of a projective line in the split quadric X, hence $\overline{h} = \overline{p} + \overline{p}'$. Therefore, $c \in \operatorname{Ker} \varepsilon_n$ if and only if $b = b' = -a$ if and only if $c \in \operatorname{Im} \rho_n$. $\square$

In the general case denote by θ_n the composition $\varepsilon_n \circ \rho_n$, $(n = 0, 1)$.

Corollary 1.6. *In the general case, the sequence*

$$\Gamma_n(F) \xrightarrow{\theta_n} K_n(X)^{(2/3)} \longrightarrow K_n(X_{\mathrm{sep}})^{(2/3)}$$

is exact.

Proof. We have the following commutative diagram

$$\begin{array}{ccc} K_n(B)\oplus K_n(F) & \xrightarrow{\varepsilon_n} & K_n(X)^{(2/3)} \\ \Big\downarrow i & & \Big\downarrow \\ K_n(B_{\mathrm{sep}})\oplus K_n(F_{\mathrm{sep}}) & \xrightarrow{\varepsilon_n} & K_n(X_{\mathrm{sep}})^{(2/3)}. \end{array}$$

The statement follows now from Lemma (1.5) and the obvious equality $i^{-1}\big(\Gamma_n(F_{\mathrm{sep}})\big)=\Gamma_n(F)$. □

2. Brown-Gersten-Quillen spectral sequence for X

In this section we compare certain $\mathcal{K}$-cohomology groups with factors of the topological filtration on the K-groups of the variety $X=J(B,\tau)$ as defined in the first section. We still assume that $\mathrm{char}F\neq 2$.

Consider the Brown-Gersten-Quillen spectral sequence [24, Th. 5.4]

$$E_2^{p,q}=H^p(X,\mathcal{K}_{-q})\Longrightarrow K_{-p-q}(X).$$

The composition of the canonical homomorphism $K_1(F)\to K_1(X)$, with the edge map $i_1:K_1(X)\to H^0(X,\mathcal{K}_1)$ is clearly an isomorphism, hence i_1 is surjective and therefore all differentials starting at $E^{0,-1}$ are trivial. Therefore, all differential arriving in $E^{2,-2}$ are trivial. Hence

$$\mathrm{CH}^2(X)=H^2(X,\mathcal{K}_2)=E_2^{2,-2}=E_\infty^{2,-2}=K_0(X)^{(2/3)}.$$

The composition of the canonical homomorphism $K_2(F)\to K_2(X)$ with the edge map $i_2:K_2(X)\to H^0(X,\mathcal{K}_2)$ is an isomorphism by [26], hence i_2 is surjective and therefore all differentials starting at $E^{0,-2}$ are trivial. Therefore, all differential arriving in $E^{2,-3}$ are trivial.

The only possibly nontrivial differential starting at $E^{2,-3}$ is

$$d:E_2^{2,-3}\to E_2^{4,-4}=H^4(X,\mathcal{K}_4)=\mathrm{CH}^4(X).$$

The image of this homomorphism is torsion (by Grothendieck's theorem, the image of d has exponent $3!=6$). It is shown in [11] that the Chow group of zero-dimensional cycles $\mathrm{CH}_0(X)=\mathrm{CH}^4(X)$ is torsion free (it is true for all involution varieties X for central simple algebras of index at most 2 with orthogonal involution). Hence all the differentials starting at $E^{2,-3}$ are trivial. Thus, we have proved

Proposition 2.1. *There are canonical isomorphisms*

$$\mathrm{CH}^2(X)\simeq K_0(X)^{(2/3)},\qquad H^2(X,\mathcal{K}_3)\simeq K_1(X)^{(2/3)}.\quad\square$$

We will need an information about the torsion part of $\mathrm{CH}^2(X)$. Since X_{sep} is split, the group $\mathrm{CH}^2(X_{\mathrm{sep}})$ is torsion free, hence the standard transfer argument (cf., the proof of Corollary 2.3) shows that

$$\mathrm{CH}^2(X)_{\mathrm{tors}}=\mathrm{Ker}\big(\mathrm{CH}^2(X)\longrightarrow\mathrm{CH}^2(X_{\mathrm{sep}})\big).$$

Lemma 2.2. *Assume that B splits (over L) and the involution τ is isotropic. Then* $\mathrm{CH}^2(X)_{\mathrm{tors}}=0$.

Proof. By Propositions 1.6 and 2.1, there is canonical surjective homomorphism

$$\Gamma_0(F) \xrightarrow{\theta_0} \operatorname{Ker}\big(K_0(X)^{(2/3)} \to K_0(X_{\mathrm{sep}})^{(2/3)}\big) \xrightarrow{\sim} \mathrm{CH}^2(X)_{\mathrm{tors}}.$$

The group $\Gamma_0(F)$ is infinite cyclic with the canonical generator γ which corresponds to 1 under isomorphism $\Gamma_0(F) = K_0(F) \simeq \mathbb{Z}$. Note that $\Gamma_0(F)$ does not change under field extensions since B is split over L.

We will prove that the image of γ under θ_0 is trivial in $K_0(X)^{(2/3)}$. In the split case, $\theta_0(\gamma) = \overline{p}+\overline{p}'-\overline{h}^2$, and by [6, §3.2], $p+p'-h^2 = l$, there l is the class of a projective line in the quadric X. Thus, since the map $K_0(X) \longrightarrow K_0(X_{\mathrm{sep}})$ is injective, it suffices to show that l belongs to the image of the natural homomorphism

$$K_0(X)^{(3)} \longrightarrow K_0(X_{\mathrm{sep}})^{(3)}.$$

The algebra B splits, i.e. $B = \mathrm{End}_L(V)$, where V is a vector space over L of dimension 4. The involution τ is then the adjoint involution τ_h with respect to a nondegenerate hermitian form h on V uniquely determined up to a scalar multiple by [7, Th. 4.2]. The variety X can be identified with the variety of isotropic (with respect to h) subspaces $U \subset V$ of L-dimension 2. Since τ is isotropic, then so is h. Let $W \subset V$ be a 1-dimensional isotropic subspace. Then the orthogonal complement $W^\perp$ is of dimension 3 and contains W. Consider the subvariety $Y \subset X$ consisting of isotropic subspaces $U \subset V$ of dimension 2 such that $W \subset U \subset W^\perp$. In the split case, Y is the subvariety of 2-dimensional subspaces U in $\mathrm{Gr}(2,4)$ such that $W_1 \subset U \subset W_3$ where W_1 and W_3 are some fixed subspaces of dimensions 1 and 3 respectively. It is easy to check that Y corresponds to a projective line on X considered as a quadric, hence the class $[\mathcal{O}_Y] \in K_0(X)$ is equal to l over F_{sep}. □

Corollary 2.3. *If B splits, then* $|\,\mathrm{CH}^2(X)_{\mathrm{tors}}| \le 2$.

Proof. Since $\Gamma_0(F)$ is a cyclic group, $\mathrm{CH}^2(X)_{\mathrm{tors}}$ is also cyclic. On the other hand, by [11], there is a quadratic extension E/F such that τ is isotropic. By Lemma (2.2), the composition of the multiplication by 2

$$\mathrm{CH}^2(X)_{\mathrm{tors}} \longrightarrow \mathrm{CH}^2(X_E)_{\mathrm{tors}} \xrightarrow{N_{E/F}} \mathrm{CH}^2(X)_{\mathrm{tors}}$$

is zero. □

3. A theorem of B. Kahn

In this section we compare $\mathcal{K}$-cohomology groups $H^2(X, \mathcal{K}_3)$ for $X = J(B,\tau)$ and dimension four Galois cohomology groups of F.

Since X has a closed point x of degree 2, the transfer argument for the quadratic field extension $F(x)/F$ shows the kernel of the natural homomorphism

$$H^p\big(F, \mathbb{Q}/\mathbb{Z}(q)\big) \longrightarrow H^p\big(F(X), \mathbb{Q}/\mathbb{Z}(q)\big)$$

is of exponent 2. Since $H^p(F)$ is naturally isomorphic to the subgroup of elements of exponent 2 in $H^p\big(F, \mathbb{Q}/\mathbb{Z}(p-1)\big)$ for $p \leq 4$ (cf., [20, 25]), the kernels of two homomorphisms

$$H^p\big(F, \mathbb{Q}/\mathbb{Z}(p-1)\big) \longrightarrow H^p\big(F(X), \mathbb{Q}/\mathbb{Z}(p-1)\big),$$

$$H^p(F) \longrightarrow H^p\big(F(X)\big)$$

are naturally isomorphic. We denote the kernel of the second homomorphism by $\widetilde{H}^p(F)$.

Since D splits over $F(X)$, the cup-product with the class $[D] \in H^2(F)$ gives a well defined homomorphism

$$i : H^{p-2}(F) \xrightarrow{[D]} \widetilde{H}^p(F).$$

It is proved in [18] that if $p = 2$, the homomorphism i is surjective. A generalization of this result to the case $p = 3$ is given in the following theorem. The original formulation of this theorem involves Galois cohomology groups with coefficients in $\mathbb{Q}/\mathbb{Z}(2)$. Using remark above we formulate it in terms of the group $\widetilde{H}^3(F)$.

Theorem 3.1. [13], [23] *There is an exact sequence*

$$H^1(F) \xrightarrow{i} \widetilde{H}^3(F) \longrightarrow CH^2(X)_{\mathrm{tors}}. \quad \square$$

This Theorem and Corollary 2.3 imply

Corollary 3.2. *If B splits, then* $|\widetilde{H}^3(F)/[D] \cup H^1(F)| \leq 2$. $\square$

B. Kahn found a further generalization to the case $p = 4$. He uses a theory of V. Voevodsky which requires resolution of singularities. That is why he assumes that F is of characteristic zero.

Theorem 3.3. [5, Cor. 5.5] *Assume that* $\mathrm{char} F = 0$*. Then there is an exact sequence*

$$H^2(F) \xrightarrow{i} \widetilde{H}^4(F) \longrightarrow H^2(X, \mathcal{K}_3) \longrightarrow H^2(X_{\mathrm{sep}}, \mathcal{K}_3). \quad \square$$

Thus, by Corollary 1.6, Proposition 2.1, Theorems 3.1 and 3.3, the composition

$$\varphi_n : \Gamma_n(F) \xrightarrow{\theta_n} \mathrm{Ker}\big(K_n(X)^{(2/3)} \longrightarrow K_n(X_{\mathrm{sep}})^{(2/3)}\big) \xrightarrow{\sim}$$

$$\mathrm{Ker}\big(H^2(X, \mathcal{K}_{n+2}) \longrightarrow H^2(X_{\mathrm{sep}}, \mathcal{K}_{n+2})\big) \xrightarrow{\sim} \widetilde{H}^{n+3}(F)/[D] \cup H^{n+1}(F)$$

is a surjective homomorphism for $n = 0, 1$. We have proved

Proposition 3.4. *For $n = 0, 1$ there is an exact sequence*

$$\Gamma_n(F) \xrightarrow{\varphi_n} H^{n+3}(F)/[D] \cup H^{n+1}(F) \longrightarrow H^{n+3}\big(F(X)\big). \quad \square$$

4. Invariant $e_3(B,\tau)$

Let (B,τ) be a central semisimple algebra of degree 4 with a unitary involution τ over a quadratic extension L/F. We assume that the algebra B is split, i.e. $B = \mathrm{End}_L(V)$ for a 4-dimensional vector space V over L. Hence $\tau = \tau_h$ is the adjoint involution with respect to some nondegenerate hermitian form h on V (uniquely determined up to a scalar multiple) with respect to the quadratic extension L/F. Consider the quadratic form q on V (viewed now as a vector space of dimension 8 over F) defined by $q(v) = h(v,v)$. The Clifford algebra of q is equivalent to the quaternion algebra $Q = (L,d)_F$, where d is the discriminant of h (cf., [8, Ch. V]) and hence, by [7, Cor. 10.35], to the discriminant algebra D of (B,τ). Denote by g be the 2-fold Pfister form corresponding to the quaternion algebra Q. Then for any $a \in F^\times$ the form $f = q \perp ag$ has split Clifford algebra and hence belongs to I^3 where I is the fundamental ideal in the Witt ring of F. One can choose a such that f is represented by a 8-dimensional form which is then necessarily similar to a 3-fold Pfister form. Its Arason invariant in $H^3(F)$ we denote by $e_3(B,\tau)$. A different choice of an element a leads to the change of $e_3(B,\tau)$ by a multiple of the class $[Q] = [D] \in H^2(F)$. Thus, we have a well defined invariant

$$e_3(B,\tau) \in H^3(F)/[D] \cup H^1(F).$$

The invariant $e_3(B,\tau)$ reflects splitting properties of the involution τ.

Lemma 4.1. *The invariant $e_3(B,\tau)$ is trivial if and only if the involution τ is isotropic. In particular, $e_3(B,\tau)$ is trivial over $F(X)$.*

Proof. If the invariant $e_3(B,\tau)$ is trivial, then $q \perp ag + \langle 1,-b\rangle g \in I^4$ for some $b \in F^\times$ since I^3/I^4 is canonically isomorphic to $H^3(F)$ (cf., [20, 25]). Then the form $f' = q \perp abg$ also belongs to I^4. Since $\dim(f') < 16$, f' is hyperbolic by [8, Th. X.3.1], and hence the form q is equal to $(-ab)g$ in the Witt ring $W(F)$ and hence is isotropic. Therefore the hermitian form h is isotropic, i.e. the involution $\tau = \tau_h$ is also isotropic.

Conversely, if τ is isotropic, then q is isotropic and since the Clifford algebras of q and g are similar, q is a scalar multiple of g in $W(X)$. Thus, $e_3(B,\tau) = 0$. □

One can describe the invariant $e_3(B,\tau)$ in terms of the homomorphism

$$\varphi_0 : \Gamma_0(F) \longrightarrow \widetilde{H}^3(F)/[D] \cup H^1(F).$$

Lemma 4.2. *If B splits, then $\varphi_0(\gamma) = e_3(B,\tau)$.*

Proof. If τ is isotropic, then Lemma 2.2 shows that $\varphi_0(\gamma) = 0$. By Lemma 4.1, $e_3(B,\tau)$ is also trivial.

Assume that τ is anisotropic. Again by Lemma 4.1, $e_3(B,\tau)$ is a nontrivial element in the image of φ_0. On the other hand, by Corollary 3.2, $|\operatorname{Im}(\varphi_0)| \leq 2$. Since γ is a generator of Γ_0 and φ_0 is surjective, we must have $\varphi_0(\gamma) = e_3(B,\tau)$. □

Corollary 4.3. *If B splits, then for any $a \in F^\times$, considered as an element of $\Gamma_1(F) = K_1(F)$, one has $\varphi_1(a) = (a) \cup e_3(B,\tau) \in H^4(F)$.*

Proof. The statement follows from Lemma 4.2 and the commutativity of the diagram

$$\begin{array}{ccccc} K_0(F) & = & \Gamma_0(F) & \xrightarrow{\varphi_0} & \widetilde{H}^3(F)/[D] \cup H^1(F) \\ \Big\downarrow (a) & & & & \Big\downarrow (a) \\ K_1(F) & = & \Gamma_1(F) & \xrightarrow{\varphi_1} & \widetilde{H}^4(F)/[D] \cup H^2(F) \end{array}$$

where the vertical homomorphisms are the products with the element a. □

Denote by $H^4(B,\tau)$ the subgroup in $H^4(F)$ generated by $[D] \cup H^2(F)$ and the norms $N_{E/F}\big(E^\times \cup e_3(B_E, \tau_E)\big)$ for all finite field extensions E/F such that the algebra $B_E = B \otimes_F E$ is split. Also let $\Sigma_1(F)$ be the subgroup in $\Gamma_1(F)$ generated by the classes of all $b \in B^\times$ such that $\tau(b) = b$.

Lemma 4.4. *Assume that* $\operatorname{ind} B = 4$. *Then*
1. $\varphi_1\big(\Sigma_1(F)\big) \subset H^4(B,\tau)$.
2. $H^4(B,\tau)$ *splits over* $F(X)$.

Proof. 1. Let $b \in B^\times$ such that $\tau(b) = b$. Choose a τ-invariant subfield $E \subset B$ of degree 4 over F containing b. The algebra B_E splits and the element $b \in \Gamma_1(F)$ is the image of $\tilde{b} = 1 \otimes b$ under the norm map

$$K_1(L \otimes_F E) = K_1(B_E) \xrightarrow{N_{E/F}} K_1(B).$$

Corollary 4.3 and the commutativity of the diagram

$$\begin{array}{ccc} \Gamma_1(E) & \xrightarrow{\varphi_1} & \widetilde{H}^4(E)/[D \otimes_F E] \cup H^2(E) \\ \Big\downarrow N_{E/F} & & \Big\downarrow N_{E/F} \\ \Gamma_1(F) & \xrightarrow{\varphi_1} & \widetilde{H}^4(F)/[D] \cup H^2(F) \end{array}$$

implies that

$$\varphi_1(b) = N_{E/F}\big((\tilde{b}) \cup e_3(B_E, \tau_E)\big) \in H^4(B,\tau).$$

The second statement follows from Lemma 4.1 since τ is isotropic over $F(X)$ and D is split over X. □

Now we can prove the main result of the paper. If $\operatorname{ind}(B) \le 2$, then by [3, Cor. 6.5], the group $\mathbf{SU}(B,\tau)$ is rational, hence $\mathrm{SU}(B,\tau)/R = 1$. Thus we may assume that $\operatorname{ind} B = 4$.

Theorem 4.5. *Let (B,τ) be a central semisimple algebra of index 4 over a quadratic extension L of a field F of characteristic zero, X the variety of right ideals in B of L-dimension 8 such that $\tau(J) \cdot J = 0$. Then there is an exact sequence*

$$\mathrm{SU}(B,\tau)/R \longrightarrow H^4(F)/H^4(B,\tau) \longrightarrow H^4\big(F(X)\big).$$

Proof. By Lemma 4.4 and Proposition 3.4 there is an exact sequence

$$\Gamma_1(F)/\Sigma_1(F) \longrightarrow H^4(F)/H^4(B,\tau) \longrightarrow H^4\big(F(X)\big).$$

On the other hand, there is a natural isomorphism [3, Th. 5.4]

$$\mathrm{SU}(B,\tau)/R \xrightarrow{\sim} \Gamma_1(F)/\Sigma_1(F).$$

□

Remark 4.6. One can derive from the proof that injectivity of the first homomorphism of the exact sequence in Theorem 4.5 is equivalent to the fact that $\operatorname{Ker}\theta_1 \subset \Sigma_1(F)$. It is an easy consequence of the following theorem of Rost (unpublished, private communication) applied to X:
If for a smooth proper variety X of dimension d over a perfect field F the degree map $\mathrm{CH}^d(X_E) \to K_0(X_E)$ is injective for any field extension E/F, then the group $H^{d-1}(X, \mathcal{K}_d)$ is generated by the images of the compositions

$$E^\times \otimes \mathrm{CH}^{d-1}(X_E) \longrightarrow H^{d-1}(X_E, \mathcal{K}_d) \xrightarrow{N_{E/F}} H^{d-1}(X, \mathcal{K}_d)$$

for all finite extensions E/F.

Consider two special cases.

1. Assume that the quadratic extension L/F splits. Then $B = A \times A^{\mathrm{op}}$ for some central simple algebra A of degree 4 over F with the exchange involution τ and $G = \mathbf{SU}(B,\tau) = \mathbf{SL}_1(A)$ is the special linear groups of A. The variety X is the generalized Severi-Brauer variety $\mathrm{SB}(2, A)$ (cf., [7, 1.16]).

The group of R-equivalence classes $G(F)/R$ is naturally isomorphic to $SK_1(A)$. The exchange involution τ is isotropic (even hyperbolic), hence by Lemma 4.1, the invariant $e_3(B,\tau)$ is trivial. The discriminant algebra D is similar to $A^{\otimes 2}$ by [7, 10.31]. The exact sequence in the Theorem 4.5 looks as follows:

$$SK_1(A) \longrightarrow H^4(F)/(2[A]) \cup H^2(F) \longrightarrow H^4\big(F(X)\big).$$

Exactness of this sequence (including injectivity of the first homomorphism) has been proven in [17]. In particular, if A is biquaternion algebra, we get Rost's exact sequence in Theorem 0.1.

2. Assume that the discriminant algebra D splits, i.e. $\mathbf{SU}(B,\tau) = \mathbf{Spin}(q)$ for some 6-dimensional quadratic form q over F unique up to a scalar multiple. The algebra B is the even Clifford algebra of q by [7, Th. 15.24]. The variety X is the projective quadric associated to q.

Assume that B is split, i.e. q is similar to a subform of a 3-fold Pfister form p. If q is isotropic, then the kernel of $i : H^3(F) \to H^3\big(F(X)\big)$ is trivial and hence the Arason class $e_3(p)$ and $e_3(B,\tau)$ in $H^3(F)$ are trivial. If q is anisotropic, then $e_3(p)$ is the only nontrivial element of the kernel of i (cf., [1]) and so is $e_3(B,\tau)$ by Corollary 3.2. Hence in any case $e_3(B,\tau) = e_3(p)$.

Thus the subgroup $H^4(B,\tau) \subset H^4(F)$ in the Theorem 4.5 is generated by the norms $N_{E/F}\big(E^\times \cup e_3(p)\big)$ for all finite field extensions E/F such that the form q_E is similar to a subform of a 3-fold Pfister form p. Denote this subgroup by $H^4(q)$. Theorem 4.5 reduces to

Corollary 4.7. *For a quadratic form q of dimension 6 over F there is an exact sequence*

$$\mathrm{Spin}(q)/R \longrightarrow H^4(F)/H^4(q) \longrightarrow H^4\big(F(X)\big),$$

where X is the projective quadric of q. □

If the discriminant of q is trivial (q is an Albert form), then q_E cannot be anisotropic Pfister neighbor over any field extension E/F, hence $H^4(q) = 0$ in this case and we again get back to the Rost's sequence.

Remark 4.8. If $\mathrm{ind}(B) = 4$, then the group $\mathbf{SU}(B, \tau)$ is not R-trivial (and hence is not stably rational) by [15, Th. 9.1], i.e. the group $\mathrm{SU}(B_E, \tau_E)$ is not trivial for some field extension E/F. Thus, in general the kernel of the natural homomorphism $H^4(F) \to H^4\big(F(X)\big)$ in larger than the subgroup of "obvious" elements $H^4(B, \tau)$.

Remark 4.9. We have considered in the paper simply connected groups of types $A_3 = D_3$. Other types of rank 3 are B_3 and C_3. A simply connected group G of type C_3 (symplectic groups of a central simple algebra of degree 6 with a symplectic involution) is rational by [29] and hence $G(F)/R = 1$. The only case left is the one of a simply connected group G of type B_3. By [29] and [7, Th. 26.12], $G = \mathbf{Spin}(q)$ for a nondegenerate quadratic form q of dimension 7.

Conjecture 4.10. *For a non-degenerate quadratic form q of dimension 7 there is an exact sequence*

$$1 \longrightarrow \mathrm{Spin}(q)/R \longrightarrow H^4(F)/H^4(q) \longrightarrow H^4\big(F(X)\big),$$

where X is a quadric of q and $H^4(q)$ is the subgroup in $H^4(F)$ generated by the norms $N_{E/F}\big(E^\times \cup e_3(p)\big)$ for all finite field extensions E/F such that the form q_E is similar to a subform of a 3-fold Pfister form p.

References

[1] J. Arason. Cohomologische Invarianten quadratischer Formen. *J. Algebra* **36** (1975), 448–491.

[2] A. Borel, T.A. Springer. Rationality properties of linear algebraic groups II. *Tôhoku Math. J.* **20** (1968), 443–497.

[3] V.I. Chernousov, A.S. Merkurjev. R-equivalence and special unitary groups. *Journal of algebra* **209** (1998), no. 1, 175–198.

[4] J.-L. Colliot-Thélène, J.-J. Sansuc. La R-équivalence sur les tores. *Ann. Scient. Éc. Norm. Sup., 4e série.* **10**, (1977), 175–230.

[5] B. Kahn. Motivic cohomology of smooth geometrically cellular varieties. To appear in the Proceedings of the Seattle K-theory conference (Proc. Symp. Pure Math., AMS).

[6] N.A. Karpenko. Algebro-geometric invariants of quadratic forms. (Russian) *Algebra i Analiz* **2** (1990), no. 1, 141–162, translation in St. Petersburg Math. J. **2** (1991), 119–138.

[7] M.-A. Knus, A.S. Merkurjev, M. Rost, J.-P. Tignol. *The Book of Involutions.* American Mathematical Society Colloquium Publications, **44**. American Mathematical Society, Providence, RI, 1998.

[8] T.-Y. Lam. *The Algebraic theory of Quadratic Forms.* Benjamin, Reading, Massachusetts, 1973.
[9] Yu.I. Manin. *Cubic forms.* North-Holland, Amsterdam, 1974.
[10] A.S. Merkurjev. Generic element in SK_1 for simple algebras. *K-theory.* **7** (1993), 1–3.
[11] A.S. Merkurjev. Zero-dimensional cycles on some involution varieties. (Russian) *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **227** (1995), Voprosy Teor. Predstav. Algebr i Grupp. 4, 93–105, 158.
[12] A.S. Merkurjev. K-theory of simple algebras. *Proceedings of the 1992 Summer Research Institute on Quadratic Forms and Division Algebras,* (W. Jacob and A. Rosenberg, eds), *Symposia Pure Math.* **58.1** (1995) 65–83, Amer. Math. Soc., Providence, R.I.
[13] A.S. Merkurjev. The group $H^1(X, \mathcal{K}_2)$ for projective homogeneous varieties. (Russian) *Algebra i Analiz.* **7** (1995), 136–164. *translation in St.Petersburg Math.J.* **7** (1996), no. 3, 421–444.
[14] A.S. Merkurjev. R-equivalence and rationality problem for semisimple adjoint classical algebraic groups. *Inst. Hautes Études Sci. Publ. Math.* **84** (1996), 189–213 (1997).
[15] A.S. Merkurjev. K-theory and algebraic groups. *Proceedings of the Second European Congress of Mathematics, Progress in Math.* **169**, (1998) Birkhauser Verlag Basel/Switzerland, 43–72.
[16] A.S. Merkurjev. Comparison of the equivariant and the standard K-theory of algebraic varieties. (Russian) *Algebra i Analiz* **9** (1997), no. 4, 175–214, translation in St. Petersburg Math. J. **9** (1998), no. 4, 815–850.
[17] A.S. Merkurjev. Invariants of algebraic groups. *J. Reine Angew. Math.* **508** (1999), 127–156.
[18] A.S. Merkurjev, J.-P. Tignol. The multipliers of similitudes and the Brauer group homogeneous varieties. *J. Reine Angew. Math.* **461** (1995), 13–47.
[19] A.S. Merkurjev, I.A. Panin, A. Wadsworth. Index reduction formulas for twisted flag varieties I. *K-Theory* **10** (1996)), no. 6, 517–596.
[20] A.S. Merkurjev, A.A. Suslin. The norm residue homomorphism of degree 3. *Izv. Akad. Nauk SSSR Ser. Mat.* **54** (1990), 339–356.
[21] I.A. Panin. On algebraic K-theory of generalized flag fibre bundles and some of their twisted forms. *Algebraic K-Theory* (ed. A. A. Suslin), Advances in Soviet Math., Vol. 4, Amer. Math. Soc., Providence, R.I., 1991, 21–46.
[22] I.A.Panin. On the algebraic K-theory of twisted flag varieties. *K-Theory*, **8**, (1994), 541–585.
[23] E. Peyre. Corps de fonctions de variétés homogènes et cohomologie galoisienne. *C. R. Acad. Sci. Paris* **321** (1995), 891–896.
[24] D. Quillen. *Higher algebraic K-theory I.* Lect. Notes Math., Springer-Verlag, Berlin **341**, (1972), 85–147.
[25] M. Rost. Hilbert 90 for K_3 for degree-two extensions. *Preprint Regensburg*, 1986.
[26] A.A. Suslin. Algebraic K-theory and norm-residue homomorphism. *J. Soviet Math.* **30** (1985), 2556–2611.
[27] V.E. Voskresenskiĭ. *Algebraic tori.* Nauka, Moscow, 1977 (Russian).
[28] S. Wang. On the commutator group of a simple algebra. *Amer. J. Math.* **72** (1950), 323–334.
[29] A. Weil. Algebras with involutions and the classical groups. *J. Indian Math. Soc.* **24** (1960), 589–623.

Alexander Merkurjev, Department of Mathematics, University of California, Los Angeles, CA 90095-1555, USA
E-mail address: merkurev@math.ucla.edu