HERMITIAN FORMS OVER QUATERNION ALGEBRAS

NIKITA A. KARPENKO AND ALEXANDER S. MERKURJEV

Abstract. We study a hermitian form h over a quaternion division algebra Q over a field (h is supposed to be alternating if the characteristic of the field is 2). For generic h and Q, for any integer $i \in [1,\ n/2]$, where $n := \dim_Q h$, we show that the variety of i-dimensional (over Q) totally isotropic right subspaces of h is 2-incompressible. The proof is based on a computation of the Chow ring for the classifying space of a certain parabolic subgroup in a split simple adjoint affine algebraic group of type C_n. As an application, we determine the smallest value of the J-invariant of a non-degenerate quadratic form divisible by a 2-fold Pfister form; we also determine the biggest values of the canonical dimensions of the orthogonal Grassmannians associated to such quadratic forms.

Contents

Date: May 12, 2013.

Key words and phrases. Algebraic groups, classifying spaces, hermitian and quadratic forms, projective homogeneous varieties, Chow groups and motives. *Mathematical Subject Classification (2010):* 14L17; 14C25.

The first author acknowledges the support of the French Agence Nationale de la Recherche (ANR) under reference ANR-12-BL01-0005.

The work of the second author has been supported by the NSF grant DMS #1160206.

1. Introduction

We study a hermitian form h over a quaternion division algebra Q over a field. If the characteristic of the field is 2, we additionally assume that h is *alternating* (as defined in [12, §4.A]). For generic h and Q (defined as in the beginning of Section 6), for any integer $i \in [1,\ n/2]$, where $n := \dim_Q h$, we show (Theorem 10.1) that the variety of i-dimensional (over Q) totally isotropic right subspaces of h is 2-*incompressible* (see [6] for definition). The proof is based on a computation of the Chow ring for the classifying space of a certain parabolic subgroup in a split simple adjoint affine algebraic group of type C_n made in Section 5e. As an application, we determine the smallest value of the J-invariant of a non-degenerate quadratic form divisible by a 2-fold Pfister form (Corollary 11.3); we also determine the biggest values of the canonical dimensions of the orthogonal Grassmannians associated to such quadratic forms (see Corollary 11.2).

The general outline of the paper follow the pattern of [10], where hermitian forms over quadratic extension fields (in place of quaternion algebras) and quadratic forms divisible by 1-fold Pfister forms have been treated.

We write Ch for the Chow group with coefficient in the field $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$. So, $\mathrm{Ch} = \mathrm{CH}/2\,\mathrm{CH}$, where CH is the Chow group with integer coefficients.

Variety here is just a separated scheme of finite type over a field.

We are using the Grothendieck Ch-motive and write $M(X)$ for the motive of a smooth complete variety X.

Acknowledgements. We thank Burt Totaro for useful advices. The first author gratefully acknowledges hospitality of the Fields Institute for Research in Mathematical Sciences (Toronto, Ontario), the Department of Mathematics of University of Toronto, and Thematic Program on Torsors, Nonassociative Algebras and Cohomological Invariants (January–June 2013).

2. The norm subgroup of a quaternion algebra

A *quaternion* algebra Q over a variety X is a rank 4 Azumaya algebra over X. Let $s : C \to X$ be the corresponding conic over X, i.e., the Severi-Brauer scheme of Q (the scheme of right ideals in Q of reduced rank 1).

The *norm subgroup* N_Q of Q is the image of the push-forward homomorphism

$$s_* : \mathrm{Ch}(C) \to \mathrm{Ch}(X).$$

By the projection formula, N_Q is an ideal in $\mathrm{Ch}(X)$ if X is smooth. The fiber C_x of s over a point $x \in X$ is the conic curve over the residue field $F(x)$ corresponding to the quaternion $F(x)$-algebra Q_x. If Q_x is split, C_x has a rational point (in fact $C_x \simeq \mathbb{P}^1$), hence $[x] \in N_Q$. If Q_x is not split, the subgroup of $\mathrm{CH}(X)$ generated by $s_*([y])$ for all y in C_x is equal to the subgroup generated by $2[x] = 0$. It follows that N_Q is generated by the classes $[x]$ of the points $x \in X$ such that Q_x is split.

Lemma 2.1. *Let Q and Q' be two quaternion algebras over a variety X. If $[Q] = [Q']$ in* $\mathrm{Br}(X)$*, then $N_Q = N_{Q'}$.*

Proof. For any $x \in X$, we have $[Q_x] = [Q'_x]$ in $\mathrm{Br}\,F(x)$ and hence $Q_x \simeq Q'_x$. It follows from the description of N_Q and $N_{Q'}$ before the lemma that $N_Q = N_{Q'}$. □

Note that if X is irreducible and smooth, $[Q] = [Q']$ in $\mathrm{Br}(X)$ if and only if the classes of the generic fibers of Q and Q' are equal in $\mathrm{Br}\, F(X)$, see [14, Corollary IV.2.6]).

We will need the following functorial property of N_Q:

Lemma 2.2. *Let $g : X' \to X$ be a morphism of smooth schemes and let Q be a quaternion algebra over a scheme X. Let Q' be the pull-back of Q with respect to g. Then the inverse image homomorphism $g^* : \mathrm{Ch}(X) \to \mathrm{Ch}(X')$ takes N_Q to $N_{Q'}$. In particular, g^* yields a homomorphism*

$$\mathrm{Ch}(X)/N_Q \longrightarrow \mathrm{Ch}(X')/N_{Q'}.$$

Proof. Let $s : C \to X$ be the conic associated to Q. Then the conic curve $s' : C' \to X'$ associated with Q' is the pull-back of s and we have a commutative diagram (see [16, Proposition 12.5])

$$\begin{tikzcd} \mathrm{Ch}(C) \arrow[r, "h^*"] \arrow[d, "s'_*"'] & \mathrm{Ch}(C') \arrow[d, "s_*"] \\ \mathrm{Ch}(X) \arrow[r, "g^*"] & \mathrm{Ch}(X'), \end{tikzcd}$$

where $h : C' \to C$ is the induced morphism. □

3. Projective Q-spaces

Let Q be a quaternion division algebra over a field F. For any finite-dimensional right Q-vector space V, the *projective Q-space* $Q\mathbb{P}(V)$ is defined as the F-variety of 1-dimensional subspaces in V. The dimension of subspaces here is taken over Q; the dimension over F is 4; the *reduced dimension* (defined as the dimension over F divided by $\deg Q = 2$) is therefore 2. For any $n \geq 0$, we define $Q\mathbb{P}^n$ as $Q\mathbb{P}(Q^{n+1})$. In particular, $Q\mathbb{P}^0$ is the point $\mathbf{pt} = \mathrm{Spec}\, F$. For any $n \geq 0$, $Q\mathbb{P}^n$ is a smooth projective F-variety of dimension $4n$.

Of course, the variety $Q\mathbb{P}(V)$ (and therefore $Q\mathbb{P}^n$) can be defined for an arbitrary (non-split or split) quaternion algebra Q as the variety of Q-submodules in V (resp., in Q^{n+1}) of reduced dimension 2 (where V is a right Q-module of even reduced dimension). Then $Q\mathbb{P}(V)_L = Q_L\mathbb{P}(V_L)$ and $Q\mathbb{P}^n_L = Q_L\mathbb{P}^n$ for any extension field L/F. Once Q is split, an identification of Q with the matrix algebra $M_2(F)$ identifies $Q\mathbb{P}^n$ with the Grassmannian of 2-planes in the $2(n+1)$-dimensional F-vector space $F^{2(n+1)}$. Our main case of interest, however, is the case where Q is a division algebra.

The following statement is true for Q non-split or split but only needs to be proved in the split case:

Lemma 3.1. *Let $a \in \mathrm{Ch}^4(Q\mathbb{P}^n)$ be the 4-th Chern class of the (rank 4) tautological vector bundle on $Q\mathbb{P}^n$. Then $\deg(a^n) = 1$.*

Proof. Replacing F by a field extension splitting Q, we may assume that Q itself is split. In this case there is an isomorphism of $Q\mathbb{P}^n$ with the Grassmannian X of 2-planes in $F^{2(n+1)}$ such that the tautological bundle on $Q\mathbb{P}^n$ corresponds to a vector bundle on X isomorphic to $\mathcal{T} \oplus \mathcal{T}$, where $\mathcal{T}$ is the tautological bundle on X. It follows that the element a corresponds to $c_2^2(\mathcal{T})$. By [4, Proposition 14.6.5], $c_2^{2n}(\mathcal{T})$ is a generator of the group $\mathrm{Ch}_0(X)$. Therefore $\deg(a^n) = \deg(c_2^{2n}(\mathcal{T})) = 1$. □

Lemma 3.2. *The* Ch*-motive* $M(Q\mathbb{P}^n)$ *decomposes in a direct sum of three summands:* $M(\mathbf{pt})$, $\bigoplus_{i=1}^{2n} M(C)(i)$, *and* $M(Q\mathbb{P}^{n-1})(4)$.

Proof. This is a particular case of [5, Theorem 10.9 and Corollary 10.19], where the characteristic of the base field is assumed to be different from 2. We refer to [1] for a characteristic-free treatment. □

Corollary 3.3. *The motive of* $Q\mathbb{P}^n$ *decomposes in a direct sum of two summands: the first one is* $\bigoplus_{i=0}^{n} M(\mathbf{pt})(4i)$ *and the second one is a direct sum of shifts of* $M(C)$. □

Since the motive of the product $C \times C$ is isomorphic to the sum $M(C) \oplus M(C)(1)$, we get

Corollary 3.4. *For any integers* $n_1, \dots, n_r \geq 0$*, the motive of the product*

$$P := Q\mathbb{P}^{n_1} \times \dots \times Q\mathbb{P}^{n_r}$$

decomposes in a direct sum of $(n_1+1)\dots(n_r+1)$ *shifts of* $M(\mathbf{pt})$ *and several shifts of* $M(C)$.

For an arbitrary smooth F-variety X, let us consider the (constant) Azumaya algebra Q_X over X given by the pull-back of Q. The norm subgroup N_{Q_X} is defined in Section 2 as the image of the push-forward homomorphism $\mathrm{Ch}(C_X) \to \mathrm{Ch}(X)$ with respect to the projection $C_X := X \times C \to X$. (Note that C_X is the conic over X associated with the quaternion algebra Q_X.)

If Q is split, the quotient $\mathrm{Ch}(X)/N_{Q_X}$ is 0. For a non-split Q we, for instance, have $\mathrm{Ch}(\mathbf{pt})/N_Q = \mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Ch}(C)/N_{Q_C} = 0$.

Proposition 3.5. *Assume that* Q *is division algebra. For* P *as in Corollary 3.4, let*

$$a_1, \dots, a_r \in \mathrm{Ch}(P)/N_{Q_P}$$

be the elements given by the 4*-th Chern classes of the tautological vector bundles of the factors of* P*. The* $\mathbb{F}_2$*-algebra* $\mathrm{Ch}(P)/N_{Q_P}$ *is generated by these elements subject to the relations* $a_1^{n_1+1} = 0, \dots, a_r^{n_r+1} = 0$.

Proof. Since the degree of any closed point on C is divisible by 2, the degree of any closed point on $P \times C$ is also divisible by 2 so that the degree homomorphism $\deg : \mathrm{Ch}(P) \to \mathbb{F}_2$ is defined on the quotient $\mathrm{Ch}(P)/N_{Q_P}$.

The monomials $a_1^{i_1} \dots a_r^{i_r}$ with $0 \leq i_1 \leq n_1$, …, $0 \leq i_r \leq n_r$, are linearly independent. Indeed, if a linear combination α of the monomials with coefficients $\alpha_{i_1 \dots i_r} \in \mathbb{F}_2$ is 0, then

$$0 = \deg(\alpha \cdot a_1^{n_1 - i_1} \dots a_r^{n_r - i_r}) = \alpha_{i_1 \dots i_r}$$

for every $i_1, \dots, i_r$.

It follows that the dimension (over $\mathbb{F}_2$) of the $\mathbb{F}_2$-subalgebra in $\mathrm{Ch}(P)/N_{Q_P}$ generated by $a_1, \dots, a_r$ is at least the product $(n_1+1)\dots(n_r+1)$. Since this product is equal to $\dim_{\mathbb{F}_2} \mathrm{Ch}(P)/N_{Q_P}$ (see Corollary 3.4), the $\mathbb{F}_2$-algebra $\mathrm{Ch}(P)/N_{Q_P}$ is generated by $a_1, \dots, a_r$.

The elements $a_1, \dots, a_r$ satisfy the indicated relations simply by dimension reason: a_i is given by the pull-back of an element in $\mathrm{Ch}^4(Q\mathbb{P}^{n_i})$ and $\dim Q\mathbb{P}^{n_i} = 4n_i$. The resulting

$\mathbb{F}_2$-algebra epimorphism

$$\mathbb{F}_2[t_1,\dots,t_r]/(t_1^{n_1+1},\dots,t_r^{n_r+1}),\ \ t_i\mapsto a_i$$

is an isomorphism by dimension reason once again. □

4. Smooth Q-fibrations

The following lemma is a distant descendant of [21, Statement 2.13]:

Lemma 4.1. *Let $f: X'\to X$ be a smooth morphism of smooth F-varieties, Q a quaternion algebra on X, $Q'=f^*Q$ and r an integer. Let $B\subset \mathrm{Ch}(X')$ be a homogeneous $\mathrm{Ch}(X)$-submodule containing $N_{Q'}$. Suppose that for any integer i and any point $x\in X$ of codimension i such that the restriction of Q on x is not split, the composition*

$$B^{r-i}\hookrightarrow \mathrm{Ch}^{r-i}(X')\to \mathrm{Ch}^{r-i}(X'_x)\to \mathrm{Ch}^{r-i}(X'_x)/N^{r-i}_{Q'_x}, \tag{4.2}$$

where Q'_x is the restriction of Q' on the fiber X'_x of f over x, is surjective. Then $B^r=\mathrm{Ch}^r(X')$.

Proof. First of all we notice that $\mathrm{Ch}(X'_x)/N_{Q'_x}=0$ if the restriction of Q on x is split. Therefore the composition (4.2) is surjective for any point $x\in X$ of codimension i.

For any integer i, we write $\mathcal{F}^i\,\mathrm{Ch}^r(X')$ for the subgroup in $\mathrm{Ch}^r(X')$ generated by the classes of cycles on X' whose images in X have codimension $\geq i$; these are terms of a descending ring filtration on $\mathrm{Ch}^r(X')$. For any point $x\in X$ of codimension i, we have a homomorphism $\mathrm{Ch}^{r-i}(X'_x)\to \mathcal{F}^i\,\mathrm{Ch}^r(X')/\mathcal{F}^{i+1}\,\mathrm{Ch}^r(X')$ mapping the class of a point $x'\in X'_x$ to the the class modulo $\mathcal{F}^{i+1}\,\mathrm{Ch}^r(X')$ of the class of x' considered as a point of X'. The composition

$$\mathrm{Ch}^{r-i}(X')\to \mathrm{Ch}^{r-i}(X'_x)\to \mathcal{F}^i\,\mathrm{Ch}^r(X')/\mathcal{F}^{i+1}\,\mathrm{Ch}^r(X')$$

is the multiplication by $[x]\in \mathrm{Ch}^i(X)$. The sum $\bigoplus_x \mathrm{Ch}^{r-i}(X'_x)$ over all points $x\in X$ of codimension i surjects onto the quotient $\mathcal{F}^i\,\mathrm{Ch}^r(X')/\mathcal{F}^{i+1}\,\mathrm{Ch}^r(X')$.

Let $\alpha\in \mathrm{Ch}^r(X')$. Inducting on i, we will show that $\alpha\in B^r+\mathcal{F}^i\,\mathrm{Ch}^r(X')$ for any i. With a sufficiently large i this will give the required statement.

The case of $i\leq 0$ being trivial, we assume that $\alpha\in B^r+\mathcal{F}^i\,\mathrm{Ch}^r(X')$ for some $i\geq 0$, and we show that $\alpha\in B^r+\mathcal{F}^{i+1}\,\mathrm{Ch}^r(X')$. We write α as a sum of an element of B^r and some $\beta\in\mathcal{F}^i\,\mathrm{Ch}^r(X')$

The class of β modulo $\mathcal{F}^{i+1}\,\mathrm{Ch}^r(X')$ decomposes into a sum of elements of two kinds. An element of the first kind is in the image of the composition

$$B^{r-i}\hookrightarrow \mathrm{Ch}^{r-i}(X')\to \mathrm{Ch}^{r-i}(X'_x)\to \mathcal{F}^i\,\mathrm{Ch}^r(X')/\mathcal{F}^{i+1}\,\mathrm{Ch}^r(X')$$

for some x, and therefore is represented by an element of $B^{r-i}\cdot[x]\subset B^r$.

An element of the second kind is in the image of the composition

$$N^{r-i}_{Q'_x}\hookrightarrow \mathrm{Ch}^{r-i}(X'_x)\to \mathcal{F}^i\,\mathrm{Ch}^r(X')/\mathcal{F}^{i+1}\,\mathrm{Ch}^r(X').$$

We claim that any element of this image is represented by an element of $N^r_{Q'}$. Since $N^r_{Q'}\subset B^r$, Lemma 4.1 is proved with this claim.

To prove the claim, we recall that $N^{r-i}_{Q'_x}$ is generated by $[x']$ with $x'\in X'_x$ of codimension $r-i$ such that the restriction of Q'_x on x' is split. The image of such a generator in

$\mathcal{F}^i \operatorname{Ch}^r(X')/\mathcal{F}^{i+1} \operatorname{Ch}^r(X')$ is represented by the class $[x'] \in \operatorname{Ch}(X')$ of x' viewed as a point of X'. Since the restriction of Q'_x on $x' \in X'_x$ coincides with the restriction of Q' on $x' \in X'$, the class $[x'] \in \operatorname{Ch}^r(X')$ is in $N^r_{Q'} \subset \operatorname{Ch}^r(X')$. □

Example 4.3. Let V be a right Q-module of even reduced rank $2(n+1)$. The corresponding *projective Q-bundle* $Q\mathbb{P}(V)$ is then defined as the X-scheme of Q-submodules in V of reduced rank 2. The structure morphism $Q\mathbb{P}(V) \to X$ is smooth and proper; its fiber over a point $x \in X$ is the projective Q_x-space $Q_x\mathbb{P}(V_x)$ defined in the previous section. It follows by Propositions 3.5 and Lemma 4.1 that the $\operatorname{Ch}(X)/N_Q$-algebra $\operatorname{Ch}(Q\mathbb{P}(V))/N_Q$ is generated by the 4-th Chern class a of the tautological vector bundle on $Q\mathbb{P}(V)$. More precisely, the $\operatorname{Ch}(X)/N_Q$-module $\operatorname{Ch}(Q\mathbb{P}(V))/N_Q$ is generated by the powers a^i with $i = 0, \dots, n$.

5. Chow rings of some classifying spaces

5a. **Chow rings of classifying spaces.** Let G be an algebraic group over a field F. Write BG for the *classifying space* of G viewed as the stack of G-torsors over the category of F-varieties (see [22]).

In [19], Totaro defined the Chow ring $\operatorname{CH}(BG)$ as follows. Let V be a generically free linear representation of G over F and $U \subset V$ a G-invariant open subscheme admitting a G-torsor $U \to U/G$ for a variety U/G. For any integer $i \geq 0$, the Chow group $\operatorname{CH}^i(U/G)$ does not depend (up to canonical isomorphism) on the choice of V and U provided that $\operatorname{codim}_V(V \setminus U) > i$. We write $\operatorname{CH}^i(BG)$ for $\operatorname{CH}^i(U/G)$ and refer to U/G as to an *i-th approximation* of the classifying space BG of G.

By [19, Theorem 1.3], the group $\operatorname{CH}^i(BG)$ is naturally identified with the set of functorial assignments α to every smooth variety X over F with a G-torsor E over X of an element $\alpha(E) \in \operatorname{CH}^i(X)$.

Let $E \to \mathbf{pt}$ be a G-torsor and ${}^E G := \operatorname{Aut}_G(E)$ the twist of G by E. The correspondence $I \mapsto \operatorname{Iso}_G(I, E)$ gives rise to an equivalence between BG and $B({}^E G)$. In particular, there is a natural ring isomorphism between $\operatorname{CH}(BG)$ and $\operatorname{CH}(B{}^E G)$.

Example 5.1. The ring $\operatorname{CH}(B\,\mathbf{GL}_n)$ is the polynomial ring on the Chern classes

$$c_1, c_2, \dots, c_n$$

of the tautological vector bundle on $B\,\mathbf{GL}_n$ (see [19]). A representation $\rho : G \to \mathbf{GL}_n$ yields the pull-back homomorphism $\rho^* : \operatorname{CH}(B\,\mathbf{GL}_n) \to \operatorname{CH}(BG)$. The elements $\rho^*(c_i)$ are the Chern classes of the pull-back of the tautological vector bundle under

$$B\rho : BG \to B\,\mathbf{GL}_n\,.$$

Example 5.2. Let A be a central simple algebra of degree n over F and $G = \mathbf{GL}_1(A)$. For every $N > 0$, the group G acts on the open subvariety U_N of nondegenerate elements in the free A-module A^N. Then BG is approximated by the Grassmannian varieties $A\mathbb{P}^{N-1} := U_N/G$ of A-submodules in A^N of reduced dimension n. We write $BG = A\mathbb{P}^\infty$ (an infinite "projective space" over A). The left multiplication action of A on itself yields a representation $\mathbf{GL}_1(A) \to \mathbf{GL}(A) = \mathbf{GL}_{n^2}$. The pull-back of the tautological vector bundle on $B\,\mathbf{GL}_{n^2}$ is the tautological vector bundle on $BG = A\mathbb{P}^\infty$.

Example 5.3. The projective linear group $\mathbf{PGL}_2$ is the automorphism group of the matrix algebra M_2. The twisted forms of M_2 are the quaternion algebras. It follows that every $\mathbf{PGL}_2$-torsor over a scheme X is isomorphic to the torsor of isomorphisms between a unique (up to canonical isomorphism) quaternion algebra Q over X and the matrix algebra M_2. The algebra Q carries a canonical (symplectic) involution τ. The kernel Q^0 of the endomorphism $1+\tau$ on Q is a sub-bundle of Q of rank 3. The natural homomorphism

$$\mathbf{PGL}_2 = \mathrm{Aut}(M_2) \to \mathrm{Aut}((M_2)^0) = \mathbf{O}_3^+$$

is an isomorphism between the group $\mathbf{PGL}_2$ of Dynkin type A_1 and the split special orthogonal group $\mathbf{O}_3^+$ of type C_1 (see [12, §15]). It is proved in [19, §16] (see also [15]) that if $\mathrm{char}(F) \neq 2$, then the ring $\mathrm{CH}(B\,\mathbf{PGL}_2) = \mathrm{CH}(B\,\mathbf{O}_3^+)$ is generated by the Chern classes $c_2(Q^0)$ and $c_3(Q^0)$ with the relation $2c_3(Q^0) = 0$. If $\mathrm{char}(F) = 2$, the result (and the proof) still holds. Moreover, in this case, Q^0 contains the trivial line sub-bundle generated by 1, hence $c_3(Q^0) = 0$.

Let V be a generically free representation of G and let U/G be an approximation of BG as above. The morphism $U/G \to BG$ induced by the *versal* G-torsor $U \to U/G$ yields a ring homomorphism $\mathrm{CH}(BG) \to \mathrm{CH}(U/G)$.

Lemma 5.4. *The ring homomorphism* $\mathrm{CH}(BG) \to \mathrm{CH}(U/G)$ *is surjective.*

Proof. Suppose that an algebraic group G acts on a variety X over F. The G-*equivariant* Chow group $\mathrm{CH}^G(X)$ was defined in [2]. In particular, $\mathrm{CH}^G(\mathbf{pt}) = \mathrm{CH}(BG)$. The homomorphism $\mathrm{CH}(BG) \to \mathrm{CH}(U/G)$ coincides with the composition

$$\mathrm{CH}(BG) = \mathrm{CH}^G(\mathbf{pt}) \xrightarrow{\alpha} \mathrm{CH}^G(V) \xrightarrow{\beta} \mathrm{CH}^G(U) = \mathrm{CH}(U/G),$$

where the pull-back homomorphism α is an isomorphism by the homotopy invariance property and the restriction β is surjective by localization. $\square$

5b. **Semidirect products.** Let an algebraic group K over F act on another algebraic group H by group automorphisms (so that we can form a semidirect product $H \rtimes K$). For a K-torsor E over $\mathbf{pt}$ we can twist H by E. The resulting group is denoted by ${}^E\!H$.

Proposition 5.5. *Let $S = H \rtimes K$ be a semidirect product of algebraic groups over F, $f : S \to K$ the corresponding split surjective homomorphism. Let U/S and W/K be i-th approximations of BS and BK respectively, and let $E \to \mathrm{Spec}\,F$ be the fiber of $W \to W/K$ over a point $x \in (W/K)(F)$. Then $(U \times W)/S$ is an i-th approximation of BS and the fiber of the natural morphism $(U \times W)/S \to W/K$ over x is an i-th approximation of $B({}^E\!H)$, where ${}^E\!H$ is the twist of H by E.*

Proof. Write $E_{\mathrm{sep}} = K_{\mathrm{sep}}w$ for w in W_{sep}. Then the isomorphism

$$U_{\mathrm{sep}}/H_{\mathrm{sep}} \longrightarrow p^{-1}(x)_{\mathrm{sep}}, \quad H_{\mathrm{sep}}v \mapsto K_{\mathrm{sep}}(v,w)$$

over F_{sep} descents to an isomorphism between ${}^E\!U/{}^E\!H$ and $p^{-1}(x)$ over F. Note that the variety ${}^E\!U/{}^E\!H$ is an i-th approximation of $B({}^E\!H)$. $\square$

5c. **The parabolic subgroup P.** Let $\widetilde{G} = \mathbf{Sp}_{2n}$ be the symplectic group of the alternating form on the $2n$-dimensional vector space V with a symplectic basis $\{v_i, w_i\}$, $i = 1, 2, \dots, n$. Write T for the maximal torus $T = (\mathbf{G}_m)^n$ acting on the basis vector by $tv_i = t_i v_i$ and $tw_i = t_i^{-1} w_i$. Write $\{e_i\}$ for the standard basis of the character group $T^* = \mathbb{Z}^n$. The simple roots of $\widetilde{G}$ are $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n$ (see [12, §24]).

Consider the subset $\Lambda = \{\alpha_1, \alpha_3, \dots\}$ of all odd simple roots. Let $\widetilde{S}$ be a (reductive) subgroup of $\widetilde{G}$ generated by the torus T and the root subgroups U_α with α in the root system $\pm\Lambda \simeq A_1 + \cdots + A_1$. The lattice T^* splits accordingly into a direct sum of rank 2 lattices $\mathbb{Z}e_{2i-1} \oplus \mathbb{Z}e_{2i}$ for $i = 1, 2, \dots, m := [n/2]$ (and $\mathbb{Z}e_n$ if n is odd). It follows that

$$\widetilde{S} \simeq \begin{cases} (\mathbf{GL}_2)^m, & \text{if } n \text{ is even } (n = 2m); \\ (\mathbf{GL}_2)^m \times \mathbf{SL}_2, & \text{if } n \text{ is odd } (n = 2m+1). \end{cases} \tag{5.6}$$

The group $\widetilde{S}$ is the Levi subgroup of the parabolic subgroup $\widetilde{P}$ of $\widetilde{G}$ corresponding to the set of simple roots Λ. The group $\widetilde{P}$ is the stabilizer of the flag

$$\{0\} = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_m \subset V$$

of totally isotropic subspaces $W_i := \operatorname{span}(v_1, v_2, \dots, v_{2i})$. The projection of $\widetilde{S}$ onto the i-th component $\mathbf{GL}_2$ in (5.6), $i = 1, \dots, m$, is given by the action on the factor space W_i/W_{i-1}, i.e., coincides with $\widetilde{S} \to \mathbf{GL}(W_i/W_{i-1}) = \mathbf{GL}_2$.

The projective symplectic group $G = \mathbf{PGSp}_{2n}$ is the factor group of $\widetilde{G}$ by $\boldsymbol{\mu}_2$. Write P for the parabolic subgroup $\widetilde{P}/\boldsymbol{\mu}_2$ in G. The Levi subgroup S of P is $\widetilde{S}/\boldsymbol{\mu}_2$ with the $\boldsymbol{\mu}_2$ embedded diagonally into the product of $\mathbf{GL}_2$ and $\mathbf{SL}_2$ with respect to the decomposition (5.6). By the above, we have

$$S = \begin{cases} (\mathbf{GL}_2)^m/\boldsymbol{\mu}_2, & \text{if } n \text{ is even;} \\ \big((\mathbf{GL}_2)^m \times \mathbf{SL}_2\big)/\boldsymbol{\mu}_2, & \text{if } n \text{ is odd.} \end{cases}$$

Let

$$H := \begin{cases} (\mathbf{GL}_2)^{m-1} \times \mathbf{G}_m, & \text{if } n \text{ is even;} \\ (\mathbf{GL}_2)^m, & \text{if } n \text{ is odd.} \end{cases}$$

We view H as a subgroup of S via the map

$$(s_1, s_2, \dots, s_{m-1}, \lambda^2) \mapsto (s_1\lambda, s_2\lambda, \dots, s_{m-1}\lambda, \lambda)\boldsymbol{\mu}_2$$

if n is even and

$$(s_1, s_2, \dots, s_m) \mapsto (s_1, s_2, \dots, s_m, 1)\boldsymbol{\mu}_2$$

if n is odd.

We also view the group $K := \mathbf{PGL}_2$ as a subgroup of P embedded diagonally. Note that S coincides with the the semidirect product $H \rtimes K$ and K acts on H by component-wise conjugation.

Consider the representations

$$\rho_i : S \to \mathbf{GL}_4$$

defined by

$$\rho_i((a_1, a_2, \dots a_m)\boldsymbol{\mu}_2) = a_i \otimes a_m, \quad i = 1, 2, \dots, m-1,$$

if n is even and by

$$\rho_i((a_1, a_2, \dots a_{m+1})\boldsymbol{\mu}_2) = a_i \otimes a_{m+1}, \quad i = 1, 2, \dots, m,$$

is n is odd.

A quaternion algebra Q over F can be viewed as a K-torsor. Twisting by Q the composition of the embedding of the i-th component $\mathbf{GL}_2 \hookrightarrow H$ and ρ_i restricted to H, we get a natural representation

$$\mathbf{GL}_1(Q) \to \mathbf{GL}_4 .$$

If n is even, write $\tau : S \to \mathbf{G}_m$ for the homomorphism

$$\tau((a_1, a_2, \dots a_m)\boldsymbol{\mu}_2) = \det(a_m).$$

5d. **Quaternion Azumaya algebra associated to a P-torsor.** Let $G = \mathbf{PGSp}_{2n}$ be a split adjoint group G of type C_n. There is a natural embedding of G into $\mathbf{PGL}_{2n}$. Therefore, for every G-torsor $h : E \to X$ we have associated a $\mathbf{PGL}_{2n}$-torsor, i.e., an Azumaya algebra $A(h)$ over X of degree $2n$.

Let P be the parabolic subgroup of G introduced in Section 5c. We have the composition $P \to S \to K = \mathbf{PGL}_2$. Therefore, for every P-torsor $f : I \to X$ we have associated a $\mathbf{PGL}_2$-torsor, i.e., a quaternion Azumaya algebra $Q = Q(f)$ over X.

Every P-torsor $f : I \to X$ yields a G-torsor $\operatorname{res}_{G/P}(f) := G \times_P I \to X$.

Lemma 5.7. *For a P-torsor $f : I \to X$, we have $[A(\operatorname{res}_{G/P}(f))] = [Q(f)]$ in $\operatorname{Br}(X)$.*

Proof. Consider the group $G' = \mathbf{GSp}_{2n}$ of symplectic similitudes (see [12, §12]). The group G is the factor group of G' by the center $\mathbf{G}_m$ of scalar matrices. Let P' the inverse image of P in G'. By Section 5c, the diagram

$$\begin{tikzcd}
\mathbf{G}_m \arrow[r, equal] \arrow[d] & \mathbf{G}_m \arrow[r, equal] \arrow[d] & \mathbf{G}_m \arrow[r, equal] \arrow[d] & \mathbf{G}_m \arrow[d] \\
\mathbf{GL}_2 \arrow[d] & P' \arrow[l, "\alpha"'] \arrow[r, hook] \arrow[d] & G' \arrow[r, hook] \arrow[d] & \mathbf{GL}_{2n} \arrow[d] \\
\mathbf{PGL}_2 & P \arrow[l, "\beta"'] \arrow[r, hook] & G \arrow[r, hook] & \mathbf{PGL}_{2n},
\end{tikzcd}$$

where α is the composition $P' \to S' \to \mathbf{GL}(W_1) = \mathbf{GL}_2$ and β is the composition $P \to S = H \rtimes K \to K = \mathbf{PGL}_2$, is commutative. It follows that the diagram

$$\begin{tikzcd}
H^1_{\text{ét}}(X, P) \arrow[r] \arrow[d] & H^1_{\text{ét}}(X, \mathbf{PGL}_2) \arrow[r, hook] & \operatorname{Br}(X) \arrow[d, equal] \\
H^1_{\text{ét}}(X, G) \arrow[r] & H^1_{\text{ét}}(X, \mathbf{PGL}_{2n}) \arrow[r, hook] & \operatorname{Br}(X)
\end{tikzcd}$$

is also commutative, whence the result. $\square$

5e. **Chow ring of BP.**

Lemma 5.8. *Let P be a parabolic subgroup in a semisimple group G defined over F, R the unipotent radical of P and S a Levi subgroup of P. Then*

(1) *the map $H^1(F,S)\to H^1(F,P)$ is a bijection;*
(2) *the map $A^p(\operatorname{Spec} F, K_q)\to A^p(R,K_q)$ of K-cohomology groups is an isomorphism for all p and q.*

Proof. (1) We have $P=R\rtimes S$, hence the map $s: H^1(F,P)\to H^1(F,S)$ is split surjective. By [12, Proposition 28.11], for any cocycle $\xi\in Z^1(F,S)$ there is a surjection from $H^1(F,{}^{\xi}R)$ to the fiber of s over ξ. The group ${}^{\xi}R$ is the unipotent radical of the parabolic subgroup ${}^{\xi}P$ of the semisimple group ${}^{\xi}G$. Hence by [18, Proposition. 16.1.1], ${}^{\xi}R$ is split over F, therefore, $H^1(F,{}^{\xi}R)=1$. It follows that s is a bijection.

(2) As R is split, there is a sequence of normal subgroups $1=R_0\subset R_1\subset\cdots\subset R_n=U$ such that R_{i+1}/R_i is isomorphism to the additive group $\mathbf{G}_a$ for all i. Hence every fiber of the natural morphism $R/R_i\to R/R_{i+1}$ is isomorphic to an affine line. By homotopy invariance, the map $A^p(R/R_{i+1},K_q)\to A^p(R/R_i,K_q)$ is an isomorphism for all p and q. □

Proposition 5.9. *Let P be a parabolic subgroup in a semisimple group G defined over F and S a Levi subgroup of P. Then the natural map $\operatorname{CH}(BS)\to\operatorname{CH}(BP)$ is an isomorphism.*

Proof. By Lemma 5.8(1), every fiber of (an approximation of) the natural morphism $BP\to BS$ over a point ξ has a rational point and hence is isomorphic to ${}^{\xi}(P/S)\simeq{}^{\xi}R$, where R is the unipotent radical of P. The result follows from Lemma 5.8(2) applied to the parabolic subgroup ${}^{\xi}P$ of the semisimple group ${}^{\xi}G$ and [3, Proposition 52.10]. □

Let P be the parabolic subgroup in the split adjoint group of type C_n introduced in Section 5c, S the Levi subgroup of P, K and H as in Section 5c.

A point x of BK over a field L is a K-torsor over L, i.e., a quaternion algebra Q_x over L. Let H_x be the twist of H by x, i.e.,

$$H_x=\begin{cases}\mathbf{GL}_1(Q_x)^{m-1}\times\mathbf{G}_m, & \text{if } n \text{ is even};\\ \mathbf{GL}_1(Q_x)^m, & \text{if } n \text{ is odd},\end{cases}$$

Determine the space BH_x. By Example 5.2, $B\,\mathbf{GL}_1(Q_x)=Q_x\mathbb{P}^\infty$. In follows that

$$BH_x=\begin{cases}B\,\mathbf{GL}_1(Q_x)^{m-1}\times B\,\mathbf{G}_m=\mathbb{P}^\infty\times(Q_x\mathbb{P}^\infty)^{m-1}, & \text{if } n \text{ is even};\\ B\,\mathbf{GL}_1(Q_x)^m=(Q_x\mathbb{P}^\infty)^m, & \text{if } n \text{ is odd}.\end{cases}\tag{5.10}$$

Let Q' be the pull-back to BS of the tautological quaternion algebra over BK. Let J be the following subset of $\operatorname{Ch}(BS)$ (see Section 5c):

$$J=\begin{cases}\{\rho_1^*(c_4),\rho_2^*(c_4),\dots,\rho_{m-1}^*(c_4),\tau^*(c_1)\}, & \text{if } n \text{ is even};\\ \{\rho_1^*(c_4),\rho_2^*(c_4),\dots,\rho_m^*(c_4)\}, & \text{if } n \text{ is odd}.\end{cases}$$

Theorem 5.11. *The $\operatorname{Ch}(BK)$-algebra $\operatorname{Ch}(BS)$ is generated by $J\cup N_{Q'}$.*

Proof. Let $r>0$ be an integer and choose the r-th approximations

$$f: Y:=(U\times W)/S\to W/K=:X$$

of the morphism $BS \to BK$ as in Proposition 5.5.

We apply Lemma 4.1 to the $\mathrm{Ch}(X)$-submodule B of $\mathrm{Ch}(Y)$ generated by $J \cup N_{Q'}$. Let x be a point in X of codimension i such that the restriction Q_x of Q on x is not split. By Proposition 5.5, the fiber of f over x is an r-th approximation of BH_x as in (5.10).

By Proposition 3.5, the ring $\mathrm{Ch}(BH_x)/N_{Q_x}$ is generated by the first Chern class of the (line) tautological bundle on $\mathbb{P}^\infty$ and by the 4-th Chern classes of the (rank 4) tautological bundles on the $m-1$ factors $Q\mathbb{P}^\infty$ if n is even. If n is odd, the ring $\mathrm{Ch}(BH_x)/N_{Q_x}$ is generated by the 4-th Chern classes of the tautological bundles on the m factors $Q\mathbb{P}^\infty$. By Example 5.2 and the end of Section 5c, these Chern classes are the restrictions of the Chern classes in J. □

By Example 5.3, the ring $\mathrm{CH}(BK)$ is generated by two elements: one in codimension 2 and the other one is 2-torsion in codimension 3.

Let Q be the quaternion algebra over BP associated to a P-torsor $U \to U/P$ approximating BP. By definition, Q is the pull-back of Q' under $BP \to BS$. By Proposition 5.9, the natural map $\mathrm{CH}(BS) \to \mathrm{CH}(BP)$ is an isomorphism. We have proved

Corollary 5.12. *The ring* $\mathrm{Ch}(BP)/N_Q$ *is generated by a set of elements of degree at most* 4 *(with all elements of degree* 3 *in the set being represented by* 2*-torsion elements in* $\mathrm{CH}^3(BP)$*).* □

Let $U \to U/P$ be a P-torsor approximating BP with the associated quaternion algebra $Q(U)$. By Lemma 5.4, the natural ring homomorphism $\mathrm{CH}(BP) \to \mathrm{CH}(U/P)$ is surjective. The induced surjection $\mathrm{Ch}(BP) \to \mathrm{Ch}(U/P)$ takes N_Q to $N_{Q(U)}$ (see Section 2). We have proved

Corollary 5.13. *Let $U \to U/P$ be a P-torsor with U/P approximating BP with the associated quaternion algebra $Q(U)$. Then the ring* $\mathrm{Ch}(U/P)/N_{Q(U)}$ *is generated by a set of elements of degree at most* 4 *(with all elements of degree* 3 *in the set being represented by* 2*-torsion elements in* $\mathrm{CH}^3(U/P)$*).* □

Consider a G-torsor $U \to U/G$ with U/G approximating BG. The generic fiber $f : E \to \mathrm{Spec}(K)$, where $K = F(U/G)$ is the associated *generic* G-torsor. Let L/K be a field extension. The fiber of $g : E \to E/P =: X$ over an L-point of X is a P-torsor $h : I \to \mathrm{Spec}(L)$ over L. Clearly, $\mathrm{res}_{G/P}(h) = f_L$. Therefore, $[Q(h)] = [A(f)_L]$ in $\mathrm{Br}(L)$ by Lemma 5.7.

Now take $L = F(X)$. We have the following two quaternion Azumaya algebras over X_L. One is $Q(g_L)$ for the P-torsor $g_L : E_L \to X_L$. Another one is the constant algebra coming from the L-algebra $Q(h)$.

We claim that the pull-backs to $L(X) = L(X_L)$ of both algebras are Brauer equivalent (and hence isomorphic). We see that the two pull-backs are obtained from each other by the automorphism of the field $L(X)$ induced by the exchange automorphism of $X \times X$. Since the Brauer class of $Q(h)$ comes from the field F, the exchange automorphisms acts identically on $Q(h)$ proving the claim.

As a consequence of the claim, the norm subgroups N in $\mathrm{Ch}(X_L)$ for both quaternion algebras are the same by Lemma 2.1.

Corollary 5.14. *Let $f : E \to \operatorname{Spec} K$ be a generic G-torsor and let L be the function field of $X := E/P$. Then the image of the natural ring homomorphism $\operatorname{Ch}(X) \to \operatorname{Ch}(X_L)$ is generated by elements of codimensions 1, 2, 4 and the norm subgroup $N_Q \subset \operatorname{Ch}(X_L)$ of the constant quaternion algebra Q on X_L Brauer equivalent to $A(f)$ lifted to X_L.*

Proof. Recall that E is the generic fiber of $U \to U/G$ for an appropriate U, hence X is a localization of U/P. By the localization property for Chow groups, the pull-back ring homomorphism $\operatorname{Ch}(U/P) \to \operatorname{Ch}(X)$ is surjective. By Lemma 2.2 and the discussion before the theorem, it takes the standard norm group in $\operatorname{Ch}(U/P)$ to the norm group N_Q. The result follows from Corollary 5.13 and from the fact that the (integral) Chow group $\operatorname{CH}(X_L)$ is torsion free as the (integral) CH-motive of the projective homogeneous variety X_L is a sum of shifts of several copies of the motives of the point and of a conic (see [5] and [1]). $\square$

6. Generic flag variety

Let k be a field, n an integer ≥ 2, F the field of rational functions in $n+2$ variables $t, t', t_1, \dots, t_n$, Q the quaternion (division) F-algebra given by the elements t, t', and h the hermitian form $\langle t_1, \dots, t_n \rangle$ on the right Q-module Q^n. Note that in the characteristic 2 case, the hermitian form h is *alternating* (as defined in [12, §4.A]). By [12, Theorem 4.2], this means that the adjoint to h involution on the matrix algebra $M_n(Q)$ is symplectic (in any characteristic).

The pair Q, h is generic in the following sense: there exist a smooth connected k-variety X (namely, the affine space of dimension $n+2$), a quaternion algebra $\tilde{Q}$ over X and a hermitian form $\tilde{h}$ on $\tilde{Q}^n$ such that:
(1) the pair Q, h is $\tilde{Q}, \tilde{h}$ restricted to the generic point of X and
(2) for any quaternion algebra Q' over an extension field F'/k with a non-degenerate hermitian form h' on Q' (alternating in characteristic 2), there exists an F'-point of X such that Q', h' is isomorphic to the restriction of $\tilde{Q}, \tilde{h}$ to the point.

Indeed, h' can be diagonalized and the diagonal entries are elements of Q which are symmetric (alternating in characteristic 2) with respect to the canonical symplectic involution on Q. It follows by [12, Proposition 2.6] that the diagonal entries are elements of F'.

A different construction of such a generic pair occurs in the proof of the following

Proposition 6.1. *Let Y be the F-variety of flags of totally isotropic subspaces in Q^n of Q-dimensions $1, 2, \dots, [n/2]$ (i.e., of reduced dimensions $2, 4, \dots, 2[n/2]$). Then the ring $\overline{\operatorname{Ch}}(Y)/\overline{N}_{Q_Y}$ is generated by codimension 1, 2, and 4.*

Proof. Let P be the parabolic subgroup of the split simple adjoint affine algebraic group G of type C_n over the field k considered (P and G) in Section 5c. Let E be a generic G-torsor over an extension field K/k, and let us consider the projective homogeneous K-variety $Y' := E/P$. The torsor E is given by a central simple K-algebra A of degree $2n$ with a symplectic involution σ. The variety Y' is isomorphic to the variety of flags of σ-isotropic right ideals in A of reduced dimensions $2, 4, 6, \dots, 2[n/2]$. There is a quaternion algebra Q' over Y' Brauer-equivalent to $A_{Y'}$.

Let X be the generalized Severi-Brauer variety $\mathrm{SB}_2(A)$. By Example 4.3, the $\mathrm{Ch}(Y')$-algebra $\mathrm{Ch}(Y' \times_K X)/N_{Q'_{Y'\times X}}$ is generated by an element of codimension 4. It follows by Corollary 5.14 that the ring $\overline{\mathrm{Ch}}(Y'_{K(X)})/\overline{N}_{Q'_{Y'_{K(X)}}}$ is generated by codimensions $1, 2, 4$. By a specialization argument similar to that used in [10, Proof of Corollary 4.8], the ring $\overline{\mathrm{Ch}}(Y'_{K(X)})/\overline{N}_{Q'_{Y'_{K(X)}}}$ is isomorphic to the ring $\overline{\mathrm{Ch}}(Y)/\overline{N}_{Q_Y}$. □

7. Projective Q-bundles for constant Q

In this section, Q is a quaternion algebra over F, X is a smooth F-variety, V is a right Q_X-module of reduced dimension $2(n+1)$.

The following statement generalizes Lemma 3.1:

Lemma 7.1. *As usual, let a be the 4-th Chern class of the tautological vector bundle on $Q\mathbb{P}(V)$. Let π be the structure morphism $Q\mathbb{P}(V) \to X$. Then $\pi_{X*}(a^i) = 0$ for any $i = 0, \dots, n-1$ and $\pi_{X*}(a^n) = [X]$.*

Proof. For $i < n$, we have $\pi_{X*}(a^i) = 0$ by dimension reason. The remaining formula $\pi_{X*}(a^n) = [X]$ may be checked over an extension of F so that we may assume Q is split. In this case $Q\mathbb{P}(V)$ is identified with the Grassmannian of 2-planes in a rank $2(n+1)$ vector bundle over X the way that $a = c_2^2$, where c_2 is the 2-nd Chern class of the tautological vector bundle on the Grassmannian. The desired formula becomes a particular case of Duality Theorem [4, 14.6.3]. □

Proposition 7.2. *For a Q-module V of even reduced rank $2(n+1)$, the $\mathrm{Ch}(X)/N_{Q_X}$-module $\mathrm{Ch}(Q\mathbb{P}(V))/N_{Q_{Q\mathbb{P}(V)}}$ is free with the basis $\{a^i\}_{i=0}^n$.*

Proof. We know already by Example 4.3 that the system $\{a^i\}_{i=0}^n$ generates the module. It remains to check that it is free.

Assuming that $\alpha := \sum_{i=0}^n \alpha_i a^i \in N_{Q_{Q\mathbb{P}(V)}}$ for some $\alpha_0, \dots, \alpha_n \in \mathrm{Ch}(X)$, we show that all α_i are in N_{Q_X} using a descending induction on i. Let i be the biggest index for which we didn't prove $\alpha_i \in N_{Q_X}$ yet. Calculating $\pi_{X*}(\alpha \cdot a^{n-i}) \in N_{Q_X}$ using Lemma 7.1, we get that $\alpha_i \in N_{Q_X}$. □

Now we are going to look at the *reduced* Chow ring $\overline{\mathrm{Ch}}(X)$ defined as the quotient of $\mathrm{Ch}(X)$ by the ideal of the elements vanishing over an extension field of F. Note that for any extension field L/F, the change of field homomorphism $\overline{\mathrm{Ch}}(X) \to \overline{\mathrm{Ch}}(X_L)$ is injective. We write $\overline{N}_{Q_X}$ for the image of N_{Q_X} in $\overline{\mathrm{Ch}}(X)$.

Proposition 7.3. *The element a (more precisely, its class) in the $\overline{\mathrm{Ch}}(X)/\overline{N}_{Q_X}$-algebra $\overline{\mathrm{Ch}}(Q\mathbb{P}(V))/\overline{N}_{Q_{Q\mathbb{P}(V)}}$ satisfies the relation*

$$\sum_{i=0}^{n+1} c_{4i} a^{n+1-i} = 0,$$

where $c_i := c_i(V)$ is the i-th Chern class of the vector bundle V.

Proof. The tautological vector bundle $\mathcal{T}$ on $Y := Q\mathbb{P}(V)$ is a subbundle of the vector bundle V_Y. The exact sequence of vector bundles

$$0 \longrightarrow \mathcal{T} \longrightarrow V_Y \longrightarrow V_Y/\mathcal{T} \longrightarrow 0$$

gives the relation $c(\mathcal{T})c(V_Y/\mathcal{T}) = c(V)$ of the total Chern classes. In particular,

$$
\begin{aligned}
a_4 + a_3b_1 + a_2b_2 + a_1b_3 + b_4 &= c_4(V) \\
a_4b_4 + a_3b_5 + a_2b_6 + a_1b_7 + b_8 &= c_8(V) \\
&\vdots \\
a_4b_{4n-4} + a_3b_{4n-3} + a_2b_{4n-2} + a_1b_{4n-1} + b_{4n} &= c_{4n}(V) \\
a_4b_{4n} &= c_{4n+4}(V),
\end{aligned} \tag{7.4}
$$

where $a_i := c_i(\mathcal{T})$ and $b_i := c_i(V_Y/\mathcal{T})$.

We claim that the classes of a_1, a_2, a_3 in $\overline{\mathrm{Ch}}(Y)/\overline{N}_{Q_Y}$ are zero. Indeed, let L/F be an extension field splitting Q. Choosing a simple left module M over the split quaternion L-algebra Q_L and defining $\mathcal{T}'$ as the tensor product of $\mathcal{T}_L$ and M over Q_L, we get an isomorphism $\mathcal{T}_L \simeq \mathcal{T}' \oplus \mathcal{T}'$ of vector bundles over Y_L. It follows that the images of a_1 and a_3 in $\mathrm{Ch}(Y_L)$ are 0. Therefore a_1 and a_3 are zero already in $\overline{\mathrm{Ch}}(Y)$ – before factorization by $\overline{N}_{Q_Y}$.

It remains to deal with the image of a_2 in $\mathrm{Ch}(Y_L)$ which is equal to $c_1^2(\mathcal{T}')$. It suffices to prove the following

Lemma 7.5. *The element $c_1^2(\mathcal{T}')$ is in the image of the composition $\mathrm{Ch}(Y \times C) \to \mathrm{Ch}(Y) \to \mathrm{Ch}(Y_L)$.*

Proof. The following square commutes:

$$
\begin{array}{ccc}
\mathrm{Ch}(Y \times C) & \longrightarrow & \mathrm{Ch}(Y \times C)_L \\
\downarrow & & \downarrow \\
\mathrm{Ch}(Y) & \longrightarrow & \mathrm{Ch}(Y_L).
\end{array}
$$

Let $\mathcal{C}$ be the tautological vector bundle on C. This is a right Q-module, but we may view it as a left Q-module via the canonical involution on Q in order to take the tensor product $\mathcal{F}$ of $\mathcal{T}_{Y\times C}$ and $\mathcal{C}_{Y\times C}$ over $Q_{Y\times C}$. Pulling-back the $(Y \times C)$-vector bundle $\mathcal{F}$ to $(Y \times C)_L = Y_L \times C_L$, we get a vector bundle isomorphic to the tensor product of $\mathcal{T}'$ and $\mathcal{C}'$, where $\mathcal{C}'$ is the (rank 1) vector bundle on C_L defined as the tensor product over Q_L of $\mathcal{C}$ and M.

Since $\mathcal{C}'$ is a line bundle and $\mathcal{T}'$ is a vector bundle of rank 2, we have

$$c_2(\mathcal{T}' \otimes \mathcal{C}') = c_2(\mathcal{T}') \times [C_L] + c_1(\mathcal{T}') \times c_1(\mathcal{C}') + [Y_L] \times c_1^2(\mathcal{C}').$$

Note that the last summand is 0 by dimension reason and that $\deg(\mathcal{C}') = 1$. Therefore the image of $c_2(\mathcal{T}' \otimes \mathcal{C}')$ under the push-forward to $\mathrm{Ch}(Y_L)$ is $c_1(\mathcal{T}')$ showing that $c_1(\mathcal{T}')$ is in the image of the composition $\mathrm{Ch}(Y \times C) \to \mathrm{Ch}(Y) \to \mathrm{Ch}(Y_L)$. Therefore $c_1^2(\mathcal{T}')$ is also in the image of the composition. $\square$

Turning back to the proof of Proposition 7.3 and passing from $\mathrm{Ch}(Y)$ to $\overline{\mathrm{Ch}}(Y)/\overline{N}_{Q_Y}$ in the relations (7.4), we get the following simpler relations

$$\begin{aligned} a_4 + b_4 &= c_4(V) \\ a_4 b_4 + b_8 &= c_8(V) \\ &\vdots \\ a_4 b_{4n-4} + b_{4n} &= c_{4n}(V) \\ a_4 b_{4n} &= c_{4n+4}(V). \end{aligned}$$

Starting with the last relation and sequentially excluding b_{4i} for $i = n, n-1, \dots$ with the help of the previous relations, we get the relation desired. □

Corollary 7.6. *The $\overline{\mathrm{Ch}}(X)/\overline{N}_{Q_X}$-algebra $\overline{\mathrm{Ch}}(Q\mathbb{P}(V))/\overline{N}_{Q_{Q\mathbb{P}(V)}}$ is generated by the element a subject to one relation*

$$\sum_{i=0}^{n+1} c_{4i} a^{n+1-i} = 0,$$

where $c_i := c_i(V)$ is the i-th Chern class of the vector bundle V.

Proof. Let A be an $\overline{\mathrm{Ch}}(X)/\overline{N}_{Q_X}$-algebra generated by one element t subject to one relation

$$\sum_{i=0}^{n+1} c_{4i} t^{n+1-i} = 0.$$

The $\overline{\mathrm{Ch}}(X)/\overline{N}_{Q_X}$-algebra homomorphism $A \to \overline{\mathrm{Ch}}(Q\mathbb{P}(V))/\overline{N}_{Q_{Q\mathbb{P}(V)}}$, $t \mapsto a$ is well defined by Proposition 7.3 and surjective by Proposition 7.2. Since both algebras, considered as $\overline{\mathrm{Ch}}(X)/\overline{N}_{Q_X}$-modules, are free of rank $n+1$ (the right one is so by Proposition 7.2 once again), the epimorphism is an isomorphism. □

8. Generic maximal Grassmannian

We use the settings of the beginning of Section 6. Let $m := [n/2]$. We consider the F-variety X of m-dimensional totally isotropic subspaces in h.

Proposition 8.1. *The components of positive codimension of the ring $\overline{\mathrm{Ch}}(X)/\overline{N}_{Q_X}$ are trivial.*

Proof. Let I be the set of integers $[1,\ m] := \{1, 2, \dots, m\}$. For any subset $J \subset I$ we consider the variety X_J of flags of totally isotropic subspaces in h of Q-dimensions given by J.

By induction on $l \in I$, we prove the following statement: the ring $\overline{\mathrm{Ch}}(X_{[l,\ m]})/\overline{N}_{Q_{X_{[l,\ m]}}}$ is generated by codimensions $1, 2, 4$ and the Chern classes of the tautological rank $4l$ vector bundle $\mathcal{T}_l$ on $X_{[l,\ m]}$ (which is the pull-back to $X_{[l,\ m]}$ of the tautological vector bundle on X_l). This statement with $l = m$ gives the required statement of Proposition 8.1 due to the following

Lemma 8.2. *Let F be a field, Q a quaternion division F-algebra, h a hermitian form on Q^n which is hyperbolic for even n and almost hyperbolic for odd n, X the corresponding maximal Grassmannian. Then all the elements of codimensions $1, 2, 4$ in the ring*

$\mathrm{Ch}(X)/N_{Q_X}$ *as well as the elements given by the Chern classes of positive codimensions of the tautological vector bundle on* X *are* 0.

Proof. It follows by [5, Theorem 15.8 and Corollary 15.14] and [1] that the motive of X decomposes in a direct sum with one summand $M(\mathbf{pt})$, one summand $M(\mathbf{pt})(3)$, and every of the remaining summands being either $M(\mathbf{pt})(i)$ with $i > 4$ or a shift of $M(C)$. Therefore all elements of codimensions $1, 2, 4$ in the ring $\mathrm{Ch}(X)/N_{Q_X}$ are 0, and it remains to prove the statement about the Chern classes of the tautological bundle.

The ring $\mathrm{Ch}(X)$ imbeds into $\mathrm{Ch}(\bar{X})$ (see, e.g., Corollary 9.2). The variety $\bar{X}$ is identified with the Grassmannian of $2m$-planes in a $2n$-dimensional vector space V which are totally isotropic with respect to a fixed non-degenerate alternating form on V. The tautological bundle on X gives rise to a vector bundle on $\bar{X}$ isomorphic to $\mathcal{T} \oplus \mathcal{T}$, where $\mathcal{T}$ is the tautological bundle on $\bar{X}$.

Let us consider the case of even n first. We claim that in this case the Chern classes of the tautological bundle on X are trivial already in $\mathrm{Ch}(\bar{X})$. Indeed, there is an exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow V_{\bar{X}} \longrightarrow \mathcal{T}^* \longrightarrow 0$$

relating the bundle $\mathcal{T}$ with its dual $\mathcal{T}^*$ and the trivial vector bundle $V_{\bar{X}}$ (where the epimorphism $V_{\bar{X}} \to \mathcal{T}^*$ is induced by the alternating form). Since $c(\mathcal{T}) = c(\mathcal{T}^*) \in \mathrm{Ch}(\bar{X})$, it follows that $c(\mathcal{T} \oplus \mathcal{T}) = 1$.

In the case of odd n, there is an exact sequence

$$0 \longrightarrow \mathcal{T}^{\perp} \longrightarrow V_{\bar{X}} \longrightarrow \mathcal{T}^* \longrightarrow 0,$$

where $\mathcal{T}^{\perp}$ is the orthogonal complement of $\mathcal{T}$ in $V_{\bar{X}}$. The bundle $\mathcal{T}^{\perp}$ contains $\mathcal{T}$ as a subbundle, the quotient $\mathcal{T}^{\perp}/\mathcal{T}$ is of rank 2. We have

$$1 = c(\mathcal{T}^{\perp})c(\mathcal{T}^*) = c(\mathcal{T})^2 c(\mathcal{T}^{\perp}/\mathcal{T}) \in \mathrm{Ch}(\bar{X}).$$

Multiplying by $c(\mathcal{T}^{\perp}/\mathcal{T})$, we get that $c(\mathcal{T}^{\perp}/\mathcal{T}) \in \mathrm{Ch}(X)$. Passing to the quotient by N_{Q_X}, we see that $c(\mathcal{T}^{\perp}/\mathcal{T}) = 1 \in \mathrm{Ch}(X)/N_{Q_X}$ (because $\mathcal{T}^{\perp}/\mathcal{T}$ is of rank 2 and the ring $\mathrm{Ch}(X)/N_{Q_X}$ has no non-zero elements in codimensions 1 and 2). Therefore $c(\mathcal{T})^2 = 1 \in \mathrm{Ch}(X)/N_{Q_X}$. □

We turn back to the inductive proof of the statement formulated in the beginning of the proof of Proposition 8.1. The induction base $l = 1$ follows from Proposition 6.1. Now, assuming that $l \geq 2$, let us do the passage from $l - 1$ to l.

The projection $X_{[l-1,\, m]} \to X_{[l,\, m]}$ is the projective Q-bundle given by the dual of the vector Q-bundle $\mathcal{T}_l$. Therefore, by Corollary 7.6, the $\overline{\mathrm{Ch}}(X_{[l,\, m]})/\overline{N}_{Q_{X_{[l,\, m]}}}$-algebra $\overline{\mathrm{Ch}}(X_{[l-1,\, m]})/\overline{N}_{Q_{X_{[l-1,\, m]}}}$ is generated by certain codimension 4 element a subject to one relation $\sum_{i=0}^{l} c_{4i}a^{l-i} = 0$, where $c_i := c_i(\mathcal{T}_l)$. In particular, the $\overline{\mathrm{Ch}}(X_{[l,\, m]})/\overline{N}_{Q_{X_{[l,\, m]}}}$-module $\overline{\mathrm{Ch}}(X_{[l-1,\, m]})/\overline{N}_{Q_{X_{[l-1,\, m]}}}$ is free of rank l.

Now let $C \subset \overline{\mathrm{Ch}}(X_{[l,\, m]})/\overline{N}_{Q_{X_{[l,\, m]}}}$ be the subring generated by all c_i together with the elements of codimensions $1, 2, 4$. The coefficients of the above relation are then in C. Therefore the subring of $\overline{\mathrm{Ch}}(X_{[l-1,\, m]})/\overline{N}_{Q_{X_{[l-1,\, m]}}}$ generated by C and a is also free (now as a C-module) of rank l. On the other hand, this subring coincides with the total ring by the induction hypothesis. Indeed, it contains all the elements of codimension $1, 2, 4$ in $\overline{\mathrm{Ch}}(X_{[l-1,\, m]})/\overline{N}_{Q_{X_{[l-1,\, m]}}}$ because any such element is either equal to a or lies in the image of $\overline{\mathrm{Ch}}(X_{[l,\, m]})/\overline{N}_{Q_{X_{[l,\, m]}}}$. It also contains the Chern classes of the vector bundle $\mathcal{T}_{l-1}$ on $X_{[l-1,\, m]}$ because these Chern classes are polynomials in $c_i := c_i(\mathcal{T}_l)$ and $a := c_4(\mathcal{T}_l/\mathcal{T}_{l-1})$ (we recall that the 1-st, 2-nd and 3-d Chern classes of the quotient $\mathcal{T}_l/\mathcal{T}_{l-1}$ are trivial in $\overline{\mathrm{Ch}}(X_{[l-1,\, m]})/\overline{N}_{Q_{X_{[l-1,\, m]}}}$ as shown in the proof of Proposition 7.3).

It follows that $C = \overline{\mathrm{Ch}}(X_{[l,\, m]})/\overline{N}_{Q_{X_{[l,\, m]}}}$. □

9. Essential motives of Q-Grassmannians

We fix the following notation. Let F be a field (of arbitrary characteristic). Let Q be a quaternion division F-algebra and C the corresponding conic. Let n be an integer ≥ 0. Let V be a right vector space over Q of dimension n. Let h be a non-degenerate hermitian (with respect to the canonical involution of Q) form on V. If $char F = 2$, we additionally assume that h is alternating. For any integer r, let X_r be the F-variety of totally isotropic subspaces in V of Q-dimension r (so that $X_0 = \operatorname{Spec} F$ and $X_r = \emptyset$ for r outside of the interval $[0,\ n/2]$).

Lemma 9.1 ([5, Theorem 15.8 and Corollary 15.14] and [1])**.** *Assume that the hermitian form h is isotropic: n is ≥ 2 and $h \simeq \mathbb{H} \perp h'$, where $\mathbb{H}$ is the hyperbolic plane, h' a hermitian form of dimension $n-2$. For any integer r one has*

$$M(X_r) \simeq M(X'_{r-1}) \oplus M(X'_r)(i) \oplus M(X'_{r-1})(j) \oplus M,$$

where X'_{r-1} and X'_r are the varieties of h', $i = (\dim X_r - \dim X'_r)/2$, $j = \dim X_r - \dim X'_{r-1}$, and M is a sum of shifts of the motive of C.

The following Corollary is also a consequence of a general result of [7] or of [1]:

Corollary 9.2. *If h is* split *(meaning* hyperbolic *for even n or* "almost hyperbolic" *for odd n), then $M(X_r)$ is a sum of shifts of $M(\mathbf{pt})$ and of $M(C)$.* □

Corollary 9.3. *There is a decomposition $M(X_r) \simeq M_r \oplus M$ such that the motive M is a sum of shifts of $M(C)$ and for any field extension L/F with split h_L the motive M_r is* split *(meaning* is a sum of Tate motives*).*

Proof. Apply [8, Proposition 4.1] inductively to $E := F(X_{[n/2]})$ and $S := C$. Note that the variety S_E is still irreducible and has indecomposable motive because the quaternion E-algebra $Q \otimes_F E$ is non-split (see, e.g., [13]). Therefore Condition (1) of [8, Proposition 4.1] is satisfied.

Since $X_{[n/2]}(F(C)) \neq \emptyset$, the field extension $E(S)/F(S)$ is purely transcendental, that is, Condition (2) is satisfied as well.

Clearly, the hermitian form h_E is split so that the motive of X_r *over E* is a sum of shifts of $M(\mathbf{pt})$ and of $M(C)$ (Corollary 9.2). The inductive application of [8, Proposition 4.1] shows that the sum M of all copies of shifts of $M(C)$ present in the complete decomposition of $M(X_r)$ over E, can be extracted from $M(X_r)$ over F. The complementary summand M_r of the motive of X_r has the desired property. □

Remark 9.4. The reduced Chow group (homological or cohomological one) of the motive M (as a subgroup of $\overline{\mathrm{Ch}}(X_r)$) is equal to $\overline{N}_{Q_{X_r}}$ (it is evidently contained in $\overline{N}_{Q_{X_r}}$ and coincides in fact with $\overline{N}_{Q_{X_r}}$ because $\overline{N}_{Q_{X_r}}$ intersects the reduced Chow group of M_r trivially). Therefore the reduced Chow group of M_r is identified with the quotient $\overline{\mathrm{Ch}}(X_r)/\overline{N}_{Q_{X_r}}$. In the case where h is hyperbolic or almost hyperbolic, the reduced Chow group in the above statements can be replaced by the usual Chow group. We refer to [3, §64] for the definition of homological and cohomological Chow group of a motive. The coincidence of descriptions of homological and cohomological Chow groups for the motives M and M_r is explained by their symmetry: $M \simeq M^*(\dim X_r)$ (and the same for M_r), where M^* is the *dual* motive, [3, §65].

Definition 9.5. The motive M_r (defined by X_r uniquely up to an isomorphism) will be called the *essential motive* of X_r (or the *essential part* of the motive of X_r).

It follows that the decomposition of the essential motive in the isotropic case has precisely the same shape as the decomposition of the motive of an isotropic orthogonal Grassmannian [9, Decomposition 2.6]:

Corollary 9.6. *Under the hypotheses of Lemma 9.1, one has*

$$M_r \simeq M'_{r-1} \oplus M'_r(i) \oplus M'_{r-1}(j),$$

where M'_{r-1} and M'_r are the essential motives of X'_{r-1} and X'_r, $i = (\dim X_r - \dim X'_r)/2$, $j = \dim X_r - \dim X'_{r-1}$. □

According to the general result of [7], any summand of the complete motivic decomposition of the variety X_r is a shift of the *upper motive* $U(X_s)$ for some $s \geq r$ or a shift of $M(C)$. Therefore we get

Corollary 9.7. *Any summand of the complete decomposition of the essential motive M_r is a shift of the upper motive $U(X_s)$ for some $s \geq r$.* □

Remark 9.8. A motive is *split* if it is isomorphic to a finite direct sum of Tate motives. A motive is *geometrically split* if it becomes split over an extension of the base field. *Dimension* $\dim P$ of a geometrically split motive P is the maximum of the distance $|i-j|$ between i and j running over the integers such that the Tate motives $M(\mathbf{pt})(i)$ and $M(\mathbf{pt})(j)$ are direct summands of P_L, where L/F is a field extension splitting N.

Since the quaternion F-algebra Q remains non-split over the function field $L := F(X_{[n/2]})$ and the motive of $(X_r)_L$ contains the Tate summands $\mathbb{F}_2$ and $\mathbb{F}_2(\dim X_r)$, these Tate motives are summands of $(M_r)_L$. It follows that $\dim M_r = \dim X_r$.

10. Generic Grassmannians

In the statements below we use the notion of the *essential motive* M_r of the variety X_r, introduced in the previous section. It turns out that in the generic case, this motive is indecomposable:

Theorem 10.1. *For F, Q, n, and h as in the beginning of Section 6, for any $r = 0, 1, \ldots, [n/2]$, the essential motive M_r of the variety X_r is indecomposable, the variety X_r is 2-incompressible.*

Proof. We induct on n in the proof of the first statement. The induction base is the trivial case of $n < 2$. Now we assume that $n \geq 2$.

We do a descending induction on r. The case of the maximal $r = [n/2]$ is an immediate consequence of Proposition 8.1. Indeed, one summand of the complete decomposition of the motive M_r for such r is the upper motive $U(X_r)$ of X_r. The remaining summands (if any) are positive shifts $U(X_r)(i)$ $(i > 0)$ of the upper motive, see Corollary 9.7. But if we have a summand $U(X_r)(i)$, then the reduced Chow group $\overline{\mathrm{Ch}}^i(M_r)$ is non-zero. However by Remark 9.4, $\overline{\mathrm{Ch}}(M_r)$ is isomorphic to $\overline{\mathrm{Ch}}(X_r)/\overline{N}_{QX_r}$ which is 0 in positive codimensions by Proposition 8.1.

Now we assume that $r < [n/2]$. Since the case of $r = 0$ is trivial, we may assume that $r \geq 1$ (and therefore $n \geq 4$).

Let $L := F(X_1)$. We have $h_L \simeq \mathbb{H} \perp h'$, where h' is a hermitian form of dimension $n - 2$ and $\mathbb{H}$ is the hyperbolic plane.

For any integer s, we write X'_s for the variety X_s of the hermitian form h', and we write M'_s for the essential motive of the variety X'_s. By Corollary 9.6, the motive $(M_r)_L$ decomposes in a sum of three summands:

$$(M_r)_L \simeq M'_{r-1} \oplus M'_r(i) \oplus M'_{r-1}(j), \tag{10.2}$$

where $i := (\dim X_r - \dim X'_r)/2$ and $j := \dim X_r - \dim X'_{r-1}$. Let us check that each of three summands of decomposition (10.2) is indecomposable.

Let F' be the function field of the variety of totally isotropic subspaces of Q-dimension 1 of the hermitian form $\langle t_{n-1}, t_n \rangle$. Then $h_{F'} \simeq \mathbb{H} \perp \langle t_1, \dots, t_{n-2} \rangle$ so that we have a motivic decomposition similar to (10.2) where each of the three summands is indecomposable by the induction hypothesis. Since the field extension $F'(X_1)/F'$ is purely transcendental, the complete decomposition of $(M_r)_{F'(X_1)}$ has only three summands. Since $L = F(X_1) \subset F'(X_1)$, the complete decomposition of $(M_r)_L$ has at most three summands so that the summands of decomposition (10.2) are indecomposable.

It follows by [9, Proposition 2.4]) that if the motive M_r is decomposable (over F), then it has a summand P with $P_L \simeq M'_r(i) = U(X'_r)(i)$. Note that $U(X'_r) \simeq U((X_{r+1})_L)$. Again by [9, Proposition 2.4], $P \simeq U(X_{r+1})(i)$, showing that $U(X_{r+1})_L \simeq M'_r$. By the induction hypothesis, the motive M_{r+1} is indecomposable, that is, $U(X_{r+1}) = M_{r+1}$. Therefore we have an isomorphism $(M_{r+1})_L \simeq M'_r$ and, in particular, $\dim X_{r+1} = \dim X'_r$ (see Remark 9.8). However $\dim X_{r+1} = (r+1)\big(4n - 6(r+1) + 1\big)$, $\dim X'_r = r\big(4(n-2) - 6r + 1\big)$, and the difference is $4n - 4r - 5 > 2n - 5 > 0$ (recall that $n \geq 4$ now).

To show that X_r is 2-incompressible, we show that its canonical 2-dimension $\mathrm{cd}_2 X_r$ equals $\dim X_r$. By [6, Theorem 5.1], $\mathrm{cd}_2 X_r = \dim U(X_r)$. By the first part of Theorem 10.1, $U(X_r) = M_r$. Finally, $\dim M_r = \dim X_r$ (see Remark 9.8). $\square$

11. Connection with quadratic forms

Let F, Q, V, and h be as in the beginning of Section 9. For any $v \in V$ the value $h(v, v)$ is in F and the map $q : V \to F$, $v \mapsto h(v, v)$ is a non-degenerate quadratic form on V considered this time as a vector space over F. Note that the dimension of q, that is, the dimension of V over F is the dimension n of V over Q multiplied by 4. Moreover, q is isomorphic to the tensor product of the 2-fold Pfister quadratic form Nrd_Q given by the reduced norm of Q by an n-dimensional non-degenerate symmetric bilinear form. Note that an arbitrary anisotropic 2-fold Pfister form over F is isomorphic to Nrd_Q for a unique up to an isomorphism quaternion division F-algebra Q ([3, Corollary 2.15]). Any non-degenerate quadratic form divisible by Nrd_Q arises the way described above from an appropriate hermitian form over Q (unique up to an isomorphism). For the case of characteristic not 2, we may refer to [17, §1 of Chapter 10].

The Witt indexes $i(h)$ and $i(q)$ of h and q are related as follows (and this relationship implies the above uniqueness statement, c.f. [10, Corollary 9.2], known as Jacobson Theorem):

Lemma 11.1. $i(q) = 4i(h)$.

Proof. For any integer $r \geq 0$, the inequality $i(h) \geq r$ implies $i(q) \geq 4r$. Indeed, if $i(h) \geq r$, V contains a totally h-isotropic Q-subspace W of dimension r. This W is also totally q-isotropic and has dimension $4r$ over F. Therefore $i(q) \geq 4r$.

To finish, we prove by induction on $r \geq 0$ that $i(q) \geq 4r - 3$ implies $i(h) \geq r$. This is trivial for $r = 0$. If $r > 0$ and $i(q) \geq 4r - 3$, then q is isotropic. But any q-isotropic vector is also h isotropic, therefore the Q-vector space V decomposes in a direct sum of h-orthogonal subspaces $V = U \oplus V'$ such that $h|_U$ is a hyperbolic plane. The subspaces U and V' are also q-orthogonal and $q|_U$ is hyperbolic (of dimension 4). For $h' := h|_{V'}$ and $q' := q|_{V'}$ it follows that $i(h') = i(h) - 1$ and $i(q') = i(q) - 4$, and we are done by the induction hypothesis applied to h' (of course, q' is the quadratic form given by h'). □

For any integer r, let X_r be the variety of totally h-isotropic r-dimensional Q-subspaces in V and let Y_r be the variety of totally q-isotropic r-dimensional F-subspaces in V. The variety Y_{2n}, where $n := \dim_Q V$, is not connected and has two isomorphic connected components; changing notation, we let Y_{2n} be one of its connected component in this case.

Corollary 11.2. *For any r, the upper motives of the varieties X_r, Y_{4r}, Y_{4r-1}, Y_{4r-2}, and Y_{4r-3} are isomorphic. In particular, these varieties have the same canonical 2-dimension. This canonical dimension is maximal and equal to*

$$\dim X_r = r(4n - 6r + 1)$$

in the case of generic Q and h.

Proof. By Lemma 11.1, each of the three varieties possesses a rational map to each other. Therefore the upper motives are isomorphic by [11, Corollary 2.15]. For the first statement on canonical dimension see [6, Theorem 5.1]. The statement on the maximal canonical dimension follows from Theorem 10.1. □

Let us recall that according to the original definition [20, Definition 5.11(2)] due to A. Vishik of the J-invariant $J(q)$ of a non-degenerate quadratic form q over F of dimension $4n$, $J(q)$ is a certain subset of the set of integers $\{0, 1, \ldots, 2n-1\}$. Note that in [3, §88], the name *J-invariant* and the notation $J(q)$ stand for the complement of the above subset (with the "excuse" that this choice simplifies several formulas involving the J-invariant). In the present paper we are using the original definition and notation.

Theorem 10.1 with Corollary 11.2 make it possible to compute for any n the smallest (in the sense of inclusion) value of the J-invariant of a quadratic form given by tensor product an n-dimensional non-degenerate symmetric bilinear form by a 2-fold quadratic Pfister form. Only the case of even n is of interest, because the J-invariant is $\{2, 3, \dots, 2n-1\}$ (everything but 1) for odd n, if the 2-fold Pfister form is anisotropic.

Corollary 11.3. *For any even $n \geq 2$, the smallest value of J-invariant discussed right above is the set of the integers in the interval $[0,\ 2n-1]$ which are not congruent to 3 modulo 4.*

Proof. If a quadratic form q over a field F is given by tensor product of an n-dimensional non-degenerate bilinear form by a 2-fold quadratic Pfister form, the Witt index of q over any extension field of F is divisible by 4. It follows by [3, Proposition 88.8] that the J-invariant of q contains the set indicated.

Let now Q and h be as in the beginning of Section 6. We calculate the J-invariant of the quadratic form q associated with h. Note that q is divisible by the 2-fold Pfister form Nrd_Q. We are using the above notation for the varieties associated to h and to q.

By [3, Theorem 90.3], the canonical 2-dimension of Y_{2n} is $\dim Y_{2n} = n(2n-1)$ minus the sum of the elements of the J-invariant. On the other hand, by Corollary 11.2 and Theorem 10.1, the canonical 2-dimension of Y_{2n} is equal to the dimension of $X_{n/2}$ which is

$$\dim X_{n/2} = n(n+1)/2 = n(2n-1) - \sum_{j\in[0,\ 2n-1],\ j\not\equiv 3 \pmod 4} j.$$

Therefore the J-invariant is equal to the set indicated. □

Lemma 11.1 and Corollaries 11.2 and 11.3 are analogues of [10, Lemma 9.1 and Corollaries 9.3 and 9.4]. The reader may discover on his own the analogues of the remaining statements of [10, §9].

References

[1] Chernousov, V., Gille, S., and Merkurjev, A. Motivic decomposition of isotropic projective homogeneous varieties. *Duke Math. J. 126*, 1 (2005), 137–159.

[2] Edidin, D., and Graham, W. Equivariant intersection theory. *Invent. Math. 131*, 3 (1998), 595–634.

[3] Elman, R., Karpenko, N., and Merkurjev, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[4] Fulton, W. *Intersection theory*, second ed., vol. 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1998.

[5] Karpenko, N. A. Cohomology of relative cellular spaces and of isotropic flag varieties. *Algebra i Analiz 12*, 1 (2000), 3–69.

[6] Karpenko, N. A. Canonical dimension. In *Proceedings of the International Congress of Mathematicians. Volume II* (New Delhi, 2010), Hindustan Book Agency, pp. 146–161.

[7] Karpenko, N. A. Upper motives of outer algebraic groups. In *Quadratic forms, linear algebraic groups, and cohomology*, vol. 18 of *Dev. Math.* Springer, New York, 2010, pp. 249–258.

[8] Karpenko, N. A. Hyperbolicity of unitary involutions. *Sci. China Math. 55*, 5 (2012), 937–945.

[9] Karpenko, N. A. Sufficiently generic orthogonal Grassmannians. *J. Algebra 372* (2012), 365–375.

[10] Karpenko, N. A. Unitary Grassmannians. *J. Pure Appl. Algebra 216*, 12 (2012), 2586–2600.

[11] Karpenko, N. A. Upper motives of algebraic groups and incompressibility of Severi-Brauer varieties. *J. Reine Angew. Math.* (Ahead of Print), doi: 10.1515/crelle.2012.011.

[12] Knus, M.-A., Merkurjev, A., Rost, M., and Tignol, J.-P. *The book of involutions*, vol. 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.

[13] Merkurjev, A. S., Panin, I. A., and Wadsworth, A. R. Index reduction formulas for twisted flag varieties. I. *K-Theory 10*, 6 (1996), 517–596.

[14] Milne, J. S. *Étale cohomology*, vol. 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.

[15] Molina Rojas, L. A., and Vistoli, A. On the Chow rings of classifying spaces for classical groups. *Rend. Sem. Mat. Univ. Padova 116* (2006), 271–298.

[16] Rost, M. Chow groups with coefficients. *Doc. Math. 1* (1996), No. 16, 319–393 (electronic).

[17] Scharlau, W. *Quadratic and Hermitian forms*, vol. 270 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985.

[18] Springer, T. A. *Linear algebraic groups*, second ed. Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, 2009.

[19] Totaro, B. The Chow ring of a classifying space. In *Algebraic K-theory (Seattle, WA, 1997)*, vol. 67 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1999, pp. 249–281.

[20] Vishik, A. On the Chow groups of quadratic Grassmannians. *Doc. Math. 10* (2005), 111–130 (electronic).

[21] Vishik, A. Fields of u-invariant $2^r + 1$. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, vol. 270 of *Progr. Math.* Birkhäuser Boston Inc., Boston, MA, 2009, pp. 661–685.

[22] Vistoli, A. Grothendieck topologies, fibered categories and descent theory. In *Fundamental algebraic geometry*, vol. 123 of *Math. Surveys Monogr.* Amer. Math. Soc., Providence, RI, 2005, pp. 1–104.

Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie, Paris, FRANCE
E-mail address: `karpenko` *at* `math.jussieu.fr`

Department of Mathematics, University of California, Los Angeles, CA, USA
E-mail address: `merkurev` *at* `math.ucla.edu`