MOTIVIC COHOMOLOGY OF THE SIMPLICIAL MOTIVE OF A ROST VARIETY

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Abstract. We compute the motivic cohomology groups of the simplicial motive $\mathcal{X}_\theta$ of a Rost variety for an n-symbol θ in Galois cohomology of a field. As an application we compute the kernel and cokernel of multiplication by θ in Galois cohomology. We also show that the reduced norm map on K_2 of a division algebra of square-free degree is injective.

1. Motivic cohomology of $\mathcal{X}_\theta$

1.1. **Introduction.** Let l be a prime integer, F a field of characteristic different from l. The Galois cohomology group $H^1_{et}(F, \mu_l)$, where μ_l is the Galois module of all lth roots of unity, is canonically isomorphic to the factor group $F^\times/F^{\times l}$. We write (a) for the class in $H^1_{et}(F, \mu_l)$ corresponding to an element $a \in F^\times$. Let $a_1, \dots, a_{n-1} \in F^\times$ for some $n \geq 1$ and $\chi \in H^1_{et}(F, \mathbb{Z}/l\mathbb{Z})$. We consider the n-tuple of 1-dimensional cohomology classes

$$\theta = \big(\chi, (a_1), \dots, (a_{n-1})\big).$$

Abusing notation we shall also write θ for the cup-product $\chi \cup (a_1) \cup \dots \cup (a_{n-1})$ in $H^n_{et}(F, \mu_l^{\otimes(n-1)})$ and call this element a *symbol*.

Note that if $\mu_l \subset F^\times$, the choice of a primitive lth root of unity identifies $\mathbb{Z}/l\mathbb{Z}$ with μ_l and, therefore, χ with (a_0) for some $a_0 \in F^\times$. Thus, θ is given by the n-tuple $(a_0, a_1, \dots, a_{n-1})$ of elements in $F^\times$.

A *Rost variety* for θ is a smooth projective variety X_θ over F satisfying the conditions given in [20, Def. 1.1] or [3, Def. 0.5].

Example 1.1. (see [20])

1) If $n = 1$, then $X_\theta = \operatorname{Spec}(L)$, where L/F is a cyclic field extension of degree l splitting θ, is a Rost variety for θ.

2) If $n = 2$, the Severi-Brauer variety $X_\theta = SB(A)$ of a central simple F-algebra A of dimension l^2 with the class θ in $H^2(F, \mu_l) \subset \operatorname{Br}(F)$ is a Rost variety for θ.

An inductive process given in [13] allows to construct a Rost variety for any θ. Denote further by $\mathcal{X}_\theta$ the Čech simplicial scheme $\check{C}(\mathcal{X}_\theta)$ of X_θ (see [17, Appendix B]) and by $M(\mathcal{X}_\theta)$ the motive of $\mathcal{X}_\theta$ in the triangulated category $\mathbf{DM}(F, \mathbb{Z})$ (see [6]). The motive of $\mathcal{X}_\theta$ in $\mathbf{DM}(F, \mathbb{Z}_{(l)})$ is independent of the

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choice of the Rost variety X_θ ([16, §5]). If $\theta = 0$, then $\mathcal{X}_\theta = \mathbb{Z}$, so in general, $\mathcal{X}_\theta$ is a "twisted form" of $\mathbb{Z}$. We write $H^{p,q}(\mathcal{X}_\theta, \mathbb{Z})$ for the motivic cohomology group $H^{p,q}\big(M(\mathcal{X}_\theta), \mathbb{Z}\big)$.

The triviality of the motivic cohomology group $H^{n+1,n}(\mathcal{X}_\theta, \mathbb{Z})$ is the essential step in the proof of Bloch-Kato Conjecture (see [16, Prop. 6.11]). In this paper we compute the motivic cohomology $H^{p,q}(\mathcal{X}_\theta, \mathbb{Z})$ for all p and q (Theorem 1.15).

In the second part of the paper some applications are given. We compute the kernel and cokernel of multiplication by θ in Galois cohomology. We also show that the reduced norm map on K_2 of a division algebra of square-free degree is injective.

We use the following notation:
$K_*(F)$ is the Milnor ring of a field F.
If X is a variety over F, we write $A_0(X, K_p)$ for the cokernel of the residue homomorphism (see [11]):

$$\coprod_{x \in X_{(1)}} K_{p+1}F(x) \to \coprod_{x \in X_{(0)}} K_pF(x),$$

where $X_{(i)}$ is the set of all points of X of dimension i.
$n \geq 2$ an integer,
$b = (l^{n-1} - 1)/(l - 1) = 1 + l + \cdots + l^{n-2}$,
$c = (l^n - 1)/(l - 1) = 1 + l + \cdots + l^{n-1} = bl + 1 = b + l^{n-1}$,
$d = l^{n-1} - 1 = b(l - 1) = c - b - 1$.

1.2. **The Bloch-Kato Conjecture and the motivic cohomology of $\mathcal{X}_\theta$.** The Bloch-Kato Conjecture asserts that the norm residue homomorphism

$$h_{n,l} : K_n(F)/lK_n(F) \to H^n_{et}(F, \mu_l^{\otimes n}),$$

taking the class of a symbol $\{a_0, a_1, \dots, a_{n-1}\}$ to the cup-product $(a_0) \cup (a_1) \cup \cdots \cup (a_{n-1})$, is an isomorphism. This conjecture was proved in [16] (see also [3], [13], [19], [20] and [21]). In view of [14], the natural maps

$$H^{p,q}(Y, \mathbb{Z}) \to H^{p,q}_{et}(Y, \mathbb{Z})$$

are isomorphisms for a smooth projective variety Y over F and $p \leq q + 1$. Moreover, the natural map

$$H^{p,q}(\mathcal{X}_\theta, \mathbb{Z}) \to H^{p,q}_{et}(\mathcal{X}_\theta, \mathbb{Z}) \tag{1}$$

is an isomorphism if $p \leq q + 1$. By [17, Lemma 7.3],

$$H^{p,q}_{et}(\mathcal{X}_\theta, \mathbb{Z}) \simeq H^{p,q}_{et}(F, \mathbb{Z}) \tag{2}$$

for all p and q.

For every $\mathcal{N} \in \mathbf{DM}(F, \mathbb{Z})$ and every $\alpha \in H^{p,q}(\mathcal{N}, \mathbb{Z})$ the *order of* α is the integer $\operatorname{ord}(\alpha) = p - q - 1$. The subgroup of $H^{*,*}(\mathcal{N}, \mathbb{Z})$ of elements of non-negative (respectively, non-positive) order will be denoted by $H^{*,*}(\mathcal{N}, \mathbb{Z})^{\geq 0}$ (respectively, $H^{*,*}(\mathcal{N}, \mathbb{Z})^{\leq 0}$).

1.3. **The motive $\widetilde{\mathcal{X}}_\theta$.** The motive $\widetilde{\mathcal{X}}_\theta$ is defined by the exact triangle

$$\widetilde{\mathcal{X}}_\theta \to M(\mathcal{X}_\theta) \to \mathbb{Z} \to \widetilde{\mathcal{X}}_\theta[1] \tag{3}$$

in $\mathbf{DM}(F, \mathbb{Z})$. Note that the motive $\widetilde{\mathcal{X}}_\theta$ differs by a shift from the one defined in [17].

It follows from (1) and (2) that

$$H^{p,q}(\mathcal{X}_\theta, \mathbb{Z}) \simeq H^{p,q}_{et}(\mathcal{X}_\theta, \mathbb{Z}) \simeq H^{p,q}_{et}(F, \mathbb{Z}) \simeq H^{p,q}(F, \mathbb{Z}) \tag{4}$$

if $p \leq q+1$. As $H^{p,q}(F, \mathbb{Z}) = 0$ when $p > q$, the exact triangle (3) yields:

Proposition 1.2. *There are canonical isomorphisms:*

$$H^{*,*}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z})^{\geq 0} \simeq H^{*,*}(\mathcal{X}_\theta, \mathbb{Z})^{\geq 0},$$
$$H^{*,*}(\mathcal{X}_\theta, \mathbb{Z})^{\leq 0} \simeq H^{*,*}(F, \mathbb{Z})^{\leq 0},$$
$$H^{*,*}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z})^{\leq 0} = 0.$$

Note that the motive $\widetilde{\mathcal{X}}_\theta$ and hence the group $H^{p,q}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z})$ vanish if $\theta = 0$. Since in general θ has a degree l splitting field extension, the group $H^{p,q}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z})$ is l-torsion.

Recall that $K_p(F) = H^{p,p}(F, \mathbb{Z})$ (see [6, §5]). Hence there is the product

$$K_s(F) \otimes H^{p,q}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z}) \to H^{p+s,q+s}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z}), \quad x \otimes \alpha \mapsto x \cdot \alpha. \tag{5}$$

Let $K^\theta_*(F)$ be the (graded) cokernel of the norm homomorphism

$$\coprod K_*(L) \to K_*(F),$$

where the coproduct is taken over all finite field extensions L/F such that θ is split over E. By projection formula, $K^\theta_*(F)$ has structure of a graded ring. Clearly, $K^\theta_*(F) = 0$ if $\theta = 0$. If $\theta \neq 0$, a transfer argument shows that the degree of a finite splitting field extension for θ is divisible by l. On the other hand, there is a splitting field extension of degree l, hence $K^\theta_0(F) = \mathbb{Z}/l\mathbb{Z}$.

It follows from Proposition 1.2 that in general the product (5) yields the structure of a left $K^\theta_*(F)$-module on $H^{*,*}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z})$ and $H^{*,*}(\mathcal{X}_\theta, \mathbb{Z})^{\geq 0}$.

1.4. **Integral elements.** We say that an element $\alpha \in H^{p,q}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z}/l\mathbb{Z})$ is *integral* if α belongs to the image of the natural homomorphism

$$H^{p,q}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z}) \to H^{p,q}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z}/l\mathbb{Z}).$$

Let

$$B : H^{*,*}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z}/l\mathbb{Z}) \to H^{*+1,*}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z}/l\mathbb{Z})$$

be the *Bockstein homomorphism*, i.e., the connecting homomorphism for the exact sequence

$$0 \to \mathbb{Z}/l\mathbb{Z} \to \mathbb{Z}/l^2\mathbb{Z} \to \mathbb{Z}/l\mathbb{Z} \to 0.$$

The following statement is a consequence of the fact that the group $H^{p,q}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z})$ is l-torsion.

Lemma 1.3. *Let $\alpha \in H^{p,q}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z}/l\mathbb{Z})$. Then the following conditions are equivalent:*

(1) *α is integral;*
(2) *$B(\alpha) = 0$;*
(3) *$\alpha \in \operatorname{Im} B$.* □

1.5. **The element** δ**.** As X_θ is a splitting variety for θ, the symbol θ belongs to the kernel of the natural homomorphism

$$\operatorname{res} : H^n_{et}\big(F, \mu_l^{\otimes(n-1)}\big) \to H^n_{et}\big(F(X_\theta), \mu_l^{\otimes(n-1)}\big).$$

Proposition 1.4. *For any $m > 0$, there is a canonical isomorphism between $H^{m+1,m-1}(\mathcal{X}_\theta, \mathbb{Z})$ and the kernel of the natural homomorphism*

$$\operatorname{res} : H^m_{et}\big(F, \mu_l^{\otimes(m-1)}\big) \to H^m_{et}\big(F(X_\theta), \mu_l^{\otimes(m-1)}\big).$$

Proof. As $H^{m+1,m-1}(\mathcal{X}_\theta, \mathbb{Z}) = H^{m+1,m-1}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z})$ is l-torsion, we have an exact sequence

$$H^{m,m-1}(\mathcal{X}_\theta, \mathbb{Z}) \to H^{m,m-1}(\mathcal{X}_\theta, \mathbb{Z}/l\mathbb{Z}) \to H^{m+1,m-1}(\mathcal{X}_\theta, \mathbb{Z}) \to 0.$$

It follows from Proposition 1.2 that the first term of the sequence is trivial. Hence the group $H^{m+1,m-1}(\mathcal{X}_\theta, \mathbb{Z})$ is canonically isomorphic to $H^{m,m-1}(\mathcal{X}_\theta, \mathbb{Z}/l\mathbb{Z})$. By the proof of [16, Lemma 6.5], the latter group is canonically isomorphic to the kernel of the homomorphism res. □

Denote by $\delta \in H^{n+1,n-1}(\mathcal{X}_\theta, \mathbb{Z})$ the element corresponding to the symbol $\theta \in \operatorname{Ker}(\operatorname{res})$ when $m = n$. Clearly, $\delta \neq 0$ if $\theta \neq 0$. We have $\operatorname{ord}(\delta) = 1$.

1.6. **Cohomological operations.** Denote by Q_i, $i = 0, 1, \ldots, n-1$, the Milnor cohomological operations of bidegree $(2l^i - 1, l^i - 1)$ on $H^{*,*}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z}/l\mathbb{Z})$ and $H^{*,*}(\mathcal{X}_\theta, \mathbb{Z}/l\mathbb{Z})$ (see [18, §13]). As $H^{p,q}(F, \mathbb{Z})$ is trivial if $p > q$, Q_i is trivial on $H^{p,p}(F, \mathbb{Z}) = K_p(F)$. It follows from the product formula (see the proof of Lemma 1.8 below), that the operations Q_i are $K_*(F)$-linear, that is $Q_i(\alpha \cdot x) = Q_i(\alpha) \cdot x$ and $Q_i(x \cdot \alpha) = (-1)^p x \cdot Q_i(\alpha)$ for all $\alpha \in H^{*,*}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z}/l\mathbb{Z})$ and $x \in K_p(F)$. The operations anti-commute: $Q_k Q_j = -Q_j Q_k$ for $j \neq k$ and $Q_i^2 = 0$ for all i. Moreover, $Q_0 = B$. Note that

$$\operatorname{ord} Q_i(\alpha) = \operatorname{ord}(\alpha) + l^i$$

for all α.

Proposition 1.5. [17, Th. 3.2], [16, Lemma 4.3] *For every $i = 1, \ldots, n-1$, the sequence*

$$H^{p-2l^i+1, q-l^i+1}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z}/l\mathbb{Z}) \xrightarrow{Q_i} H^{p,q}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z}/l\mathbb{Z}) \xrightarrow{Q_i} H^{p+2l^i-1, q+l^i-1}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z}/l\mathbb{Z})$$

is exact for all p and q. □

It follows from the equality $Q_i B = -B Q_i$ for $i \geq 1$ and Lemma 1.3 that Q_i takes integral elements to integral ones. The restriction of Q_i on the subgroup of integral elements $H^{p,q}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z})$ is still denoted by Q_i.

Proposition 1.6. *For every $i = 1, \dots, n-1$, the sequence*

$$H^{p-2l^i+1,q-l^i+1}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z}) \xrightarrow{Q_i} H^{p,q}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z}) \xrightarrow{Q_i} H^{p+2l^i-1,q+l^i-1}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z})$$

is exact for all p and q.

Proof. Suppose that $Q_i(\alpha) = 0$ for some $\alpha \in H^{p,q}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z})$. By induction on $\operatorname{ord}(\alpha)$ we prove that $\alpha = Q_i(\beta)$ for some integral β.

By Proposition 1.5, $\alpha = Q_i(\beta')$ for $\beta' \in H^{p-2l^i+1,q-l^i+1}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z}/l\mathbb{Z})$. Since α is integral, we have $(Q_iB)(\beta') = -(BQ_i)(\beta') = -B(\alpha) = 0$. Since $B(\beta')$ is integral, by the induction hypothesis, $B(\beta') = Q_i(\gamma)$ for some integral γ. By Lemma 1.3, we have $\gamma = B(\gamma')$ for some γ' and hence

$$B\big(\beta' + Q_i(\gamma')\big) = B(\beta') + (BQ_i)(\gamma') = B(\beta') - Q_i(\gamma) = 0.$$

Therefore, the element $\beta = \beta' + Q_i(\gamma')$ is integral and $Q_i(\beta) = Q_i(\beta') = \alpha$. □

Propositions 1.2 and 1.6 yield:

Corollary 1.7. *Let $Q_i(\alpha) = 0$ for some $\alpha \in H^{p,q}(\mathcal{X}_\theta, \mathbb{Z})$ and $i = 1, \dots, n-1$. Then*

1. *If $0 \le \operatorname{ord}(\alpha) < l^i$, then $\alpha = 0$.*
2. *If $\operatorname{ord}(\alpha) \ge l^i$, then $\alpha = Q_i(\beta)$ for some $\beta \in H^{p-2l^i+1,q-l^i+1}(\mathcal{X}_\theta, \mathbb{Z})$.*

1.7. **The elements γ and μ.** Set

$$\mu = (Q_1Q_2 \dots Q_{n-2})(\delta) = \pm(Q_{n-2} \dots Q_2Q_1)(\delta) \in H^{2b+1,b}(\mathcal{X}_\theta, \mathbb{Z})$$

and

$$\gamma = (Q_1Q_2 \dots Q_{n-2}Q_{n-1})(\delta) = \pm Q_{n-1}(\mu) \in H^{2c,c-1}(\mathcal{X}_\theta, \mathbb{Z}).$$

We have $\operatorname{ord}(\mu) = b$ and $\operatorname{ord}(\gamma) = c$. If $\theta \neq 0$, then $\delta \neq 0$, hence it follows from Corollary 1.7(1) by induction on $i = 1, \dots, n-1$ that $(Q_i \dots Q_2Q_1)(\delta) \neq 0$. In particular, $\mu \neq 0$ and $\gamma \neq 0$.

We write $\cup$ for the product in $H^{*,*}(\mathcal{X}_\theta, \mathbb{Z})$.

Lemma 1.8. *We have $Q_i(\gamma) = 0$ for any $i = 1, \dots, n-1$ and $Q_i(x \cup \gamma) = Q_i(x) \cup \gamma$ for every $x \in H^{p,q}(\mathcal{X}_\theta, \mathbb{Z})$.*

Proof. The first equality follows from $Q_i^2 = 0$ and the anti-commutativity of the Q_j. If l is odd, by the product formula, for every homogeneous x of degree p, we have

$$Q_i(x \cup \gamma) = Q_i(x) \cup \gamma + (-1)^p x \cup Q_i(\gamma) = Q_i(x) \cup \gamma.$$

In the case $l = 2$, the product formula looks as follows [18, Prop. 13.4]:

$$Q_i(x \cup \gamma) = x \cup Q_i(\gamma) + Q_i(x) \cup \gamma + \sum_{E,E'} u_{E,E'} Q_E(x) \cup Q_{E'}(\gamma)$$

for some $u_{E,E'} \in H^{*,*}(F, \mathbb{Z})$, where $Q_E = Q_1^{e_1}Q_2^{e_2} \dots$ and the sum is taken over all pairs of nonzero sequences $E = (e_1, e_2, \dots)$ and E' of length less than i. Note that $Q_j(\gamma) = 0$ for all $j = 1, \dots, n-1$, hence $Q_{E'}(\gamma) = 0$. □

The element μ gives rise to a morphism $M(\mathcal{X}_\theta) \to M(\mathcal{X}_\theta)(b)[2b+1]$ in $\mathbf{DM}(F,\mathbb{Z})$, still denoted by μ (see [16, 5.3]). Let $\mathcal{M}_\theta$ be the motive in $\mathbf{DM}(F,\mathbb{Z})$ defined by the exact triangle

$$M(\mathcal{X}_\theta)(b)[2b] \to \mathcal{M}_\theta \to M(\mathcal{X}_\theta) \xrightarrow{\mu} M(\mathcal{X}_\theta)(b)[2b+1].$$

For every $i = 0, 1, \dots, l-1$, let $S^i\mathcal{M}_\theta$ be the i-th symmetric power of $\mathcal{M}_\theta$ in $\mathbf{DM}(F,\mathbb{Z}_{(l)})$ (see [16, §3]). The symmetric power $\mathcal{R}_\theta := S^{l-1}\mathcal{M}_\theta$ is called the *Rost motive of* θ.

Note that in the split case,

$$\mathcal{M}_\theta = \mathbb{Z}_{(l)} \oplus \mathbb{Z}_{(l)}(b)[2b] \oplus \dots \oplus \mathbb{Z}_{(l)}((l-1)b)[2(l-1)b]. \tag{6}$$

There are exact triangles [16, (5.5) and (5.6)] in $\mathbf{DM}(F,\mathbb{Z}_{(l)})$:

$$\mathcal{R}_\theta \to S^{l-2}\mathcal{M}_\theta \to M(\mathcal{X}_\theta)(d)[2d+1] \to \mathcal{R}_\theta[1],$$

$$S^{l-2}\mathcal{M}_\theta(b)[2b] \to \mathcal{R}_\theta \to M(\mathcal{X}_\theta) \to S^{l-2}\mathcal{M}_\theta(b)[2b+1].$$

For all integers p and q we then have exact sequences

$$\begin{aligned}(7)\quad & H^{p+2d,q+d}(\mathcal{R}_\theta,\mathbb{Z}_{(l)}) \to H^{p,q}(\mathcal{X}_\theta,\mathbb{Z}_{(l)}) \xrightarrow{\partial_1} \\ & \qquad H^{p+2d+1,q+d}(S^{l-2}\mathcal{M}_\theta,\mathbb{Z}_{(l)}) \to H^{p+2d+1,q+d}(\mathcal{R}_\theta,\mathbb{Z}_{(l)})\end{aligned}$$

and

$$\begin{aligned}(8)\quad & H^{p+2c-1,q+c-1}(\mathcal{R}_\theta,\mathbb{Z}_{(l)}) \to H^{p+2d+1,q+d}(S^{l-2}\mathcal{M}_\theta,\mathbb{Z}_{(l)}) \xrightarrow{\partial_2} \\ & \qquad H^{p+2c,q+c-1}(\mathcal{X}_\theta,\mathbb{Z}_{(l)}) \to H^{p+2c,q+c-1}(\mathcal{R}_\theta,\mathbb{Z}_{(l)}).\end{aligned}$$

By [16, Lemma 5.15] and [20, Cor. 8.8], the motive $\mathcal{R}_\theta$ is a direct summand of $M(X_\theta)$ in $\mathbf{DM}(F,\mathbb{Z}_{(l)})$. It follows that $H^{p,q}(\mathcal{R}_\theta,\mathbb{Z}_{(l)}) = 0$ if $p-q > d = \dim X_\theta$. Therefore, by (7),

$$\partial_1 : H^{p,q}(\mathcal{X}_\theta,\mathbb{Z}_{(l)}) \to H^{p+2d+1,q+d}(S^{l-2}\mathcal{M}_\theta,\mathbb{Z}_{(l)}) \tag{9}$$

is an isomorphism if $p > q$ and by (8),

$$\partial_2 : H^{p+2d+1,q+d}(S^{l-2}\mathcal{M}_\theta,\mathbb{Z}_{(l)}) \to H^{p+2c,q+c-1}(\mathcal{X}_\theta,\mathbb{Z}_{(l)}) \tag{10}$$

is an isomorphism if $p + c > q + d$.

Let ∂ be the composition $\partial_2 \circ \partial_1$. Then (7) and (10) yield an exact sequence

$$\begin{aligned}(11)\quad & H^{p+2c-1,p+c-1}(\mathcal{X}_\theta,\mathbb{Z}_{(l)}) \to H^{p+2d,p+d}(\mathcal{R}_\theta,\mathbb{Z}_{(l)}) \to \\ & \qquad H^{p,p}(\mathcal{X}_\theta,\mathbb{Z}_{(l)}) \xrightarrow{\partial} H^{p+2c,p+c-1}(\mathcal{X}_\theta,\mathbb{Z}_{(l)}) \to 0\end{aligned}$$

for every p.

Proposition 1.9. *The homomorphism ∂ in* (11) *yields an isomorphism*

$$K_p^\theta(F) \xrightarrow{\sim} H^{p+2c,p+c-1}(\mathcal{X}_\theta,\mathbb{Z})$$

for every integer p.

Proof. Since both groups are l-torsion, it is sufficient to establish the isomorphism over $\mathbb{Z}_{(l)}$. By (4), we have $H^{p,p}(\mathcal{X}_\theta, \mathbb{Z}_{(l)}) = H^{p,p}(F, \mathbb{Z}_{(l)}) = K_p(F)_{(l)}$. Let L/F be a finite splitting field extension of θ. The commutativity of the diagram

$$\begin{array}{ccccc} K_p(L)_{(l)} & = & H^{p,p}(\mathcal{X}_{\theta L}, \mathbb{Z}_{(l)}) & \xrightarrow{\partial_L} & H^{p+2c,p+c-1}(\mathcal{X}_{\theta L}, \mathbb{Z}_{(l)}) \\ {\scriptstyle N_{L/F}}\downarrow & & {\scriptstyle N_{L/F}}\downarrow & & {\scriptstyle N_{L/F}}\downarrow \\ K_p(F)_{(l)} & = & H^{p,p}(\mathcal{X}_\theta, \mathbb{Z}_{(l)}) & \xrightarrow{\partial} & H^{p+2c,p+c-1}(\mathcal{X}_\theta, \mathbb{Z}_{(l)}) \end{array}$$

and the triviality of the top right corner imply that

$$\mathrm{Ker}(\partial) \supset A := \mathrm{Ker}\big(K_p(F)_{(l)} \to K_p^\theta(F)_{(l)}\big).$$

By [17, Lemma 4.11], the group $H^{p+2d,p+d}(X_\theta, \mathbb{Z})$ is canonically isomorphic to $A_0(X_\theta, K_p)$. As $\mathcal{R}_\theta$ is a direct summand of $M(X_\theta)$, the group $H^{p+2d,p+d}(X_\theta, \mathbb{Z}_{(l)})$ and hence $H^{p+2d,p+d}(\mathcal{R}_\theta, \mathbb{Z}_{(l)})$ is generated by the norms for the field extensions $F(x)/F$ over all closed points $x \in X_\theta$. Since the field $F(x)$ splits θ, we see from the exactness of (11) that $\mathrm{Ker}(\partial) \subset A$. Therefore, $\mathrm{Ker}(\partial) = A$ and ∂ yields the isomorphism in the statement of the proposition. □

Suppose that $\theta \neq 0$. We have $K_0^\theta(F) = \mathbb{Z}/l\mathbb{Z}$ and therefore by Proposition 1.9, $H^{2c,c-1}(\mathcal{X}_\theta, \mathbb{Z}) \simeq \mathbb{Z}/l\mathbb{Z}$. On the other hand, γ is a nonzero element of this group, hence $H^{2c,c-1}(\mathcal{X}_\theta, \mathbb{Z}) = (\mathbb{Z}/l\mathbb{Z})\gamma$.

Note that the motive $\mathcal{M}_\theta$ and its symmetric powers are motives over $\mathcal{X}_\theta$ (see [16]). Moreover, the morphisms in the exact triangles involving these motives are over $\mathcal{X}_\theta$. In particular, the homomorphism ∂ is $H^{*,*}(\mathcal{X}_\theta, \mathbb{Z})$-linear. Therefore, ∂ is given by multiplication by the canonical generator of $H^{2c,c-1}(\mathcal{X}_\theta, \mathbb{Z})$ that is a multiple of γ. Note that since the degree $2c$ of γ is even, γ is central in $H^{*,*}(\mathcal{X}_\theta, \mathbb{Z})$ by [6, Th. 15.9]. Then Proposition 1.9, (9) and (10) yield:

Proposition 1.10. (1) *The map $K_p^\theta(F) \to H^{p+2c,p+c-1}(\mathcal{X}_\theta, \mathbb{Z})$ given by multiplication by γ, is an isomorphism for any p.*

(2) *The map $H^{p,q}(\mathcal{X}_\theta, \mathbb{Z}) \to H^{p+2c,q+c-1}(\mathcal{X}_\theta, \mathbb{Z})$, given by multiplication by γ, is an isomorphism if $p > q$, i.e., every $\alpha \in H^{*,*}(\mathcal{X}_\theta, \mathbb{Z})$ with* $\mathrm{ord}(\alpha) > c$ *can be written in the form $\alpha = \beta \cup \gamma$ for a unique β.* □

Proposition 1.11. *The map $K_p^\theta(F) \to H^{p+n+1,p+n-1}(\mathcal{X}_\theta, \mathbb{Z})$, $x \mapsto x \cdot \delta$, is an isomorphism for all p.*

Proof. The composition

$$K_p^\theta(F) \xrightarrow{\cdot\delta} H^{p+n+1,p+n-1}(\mathcal{X}_\theta, \mathbb{Z}) \xrightarrow{Q_1 \dots Q_{n-1}} H^{p+2c,p+c-1}(\mathcal{X}_\theta, \mathbb{Z})$$

coincides with the multiplication by γ and therefore is an isomorphism by Proposition 1.10(1). The second map is injective by Corollary 1.7. □

Lemma 1.12. *If $\alpha \in H^{p,q}(\mathcal{X}_\theta, \mathbb{Z})$ such that $b <$* $\mathrm{ord}(\alpha) \leq c$, *then $Q_{n-1}(\alpha) = 0$.*

Proof. Since $\operatorname{ord} Q_{n-1}(\alpha) > b + l^{n-1} = c$, we have $Q_{n-1}(\alpha) = \beta \cup \gamma$ for some $\beta \in H^{*,*}(\mathcal{X}_\theta, \mathbb{Z})^{\geq 0}$ by Proposition 1.10(2). Hence, in view of Lemma 1.8,

$$Q_{n-1}(\beta) \cup \gamma = Q_{n-1}(\beta \cup \gamma) = Q_{n-1}^2(\alpha) = 0,$$

therefore $Q_{n-1}(\beta) = 0$ again by Proposition 1.10(2). Since

$$\operatorname{ord}(\beta) \leq c + l^{n-1} - (c+1) < l^{n-1},$$

we have $\beta = 0$ by Corollary 1.7 and therefore $Q_{n-1}(\alpha) = 0$. □

Lemma 1.13. *If $\alpha \in H^{p,q}(\mathcal{X}_\theta, \mathbb{Z})$ with $2l^i \leq \operatorname{ord}(\alpha) < l^{i+1}$ for some $i = 0, 1, \dots, n-2$, then $\alpha = 0$.*

Proof. Consider the element $\alpha' = (Q_{n-2} \dots Q_{i+1})(\alpha)$. Since

$$\operatorname{ord}(\alpha') = \operatorname{ord}(\alpha) + l^{i+1} + \dots + l^{n-2},$$

we have $b < \operatorname{ord}(\alpha') < l^{n-1} < c$. By Lemma 1.12, $Q_{n-1}(\alpha') = 0$.

Using Corollary 1.7, by descending induction on $j = n-2, \dots, i$, we deduce that $(Q_j \dots Q_{i+1})(\alpha) = 0$ since $\operatorname{ord}(Q_j \dots Q_{i+1})(\alpha) < l^{j+1}$. Therefore, $\alpha = 0$. □

Lemma 1.14. *If $\alpha \in H^{p,q}(\mathcal{X}_\theta, \mathbb{Z})$ with $\operatorname{ord}(\alpha) \leq c$ and $l^i \leq \operatorname{ord}(\alpha) < 2l^i$ for some $i = 1, 2, \dots, n-1$, then $\alpha = Q_i(\beta)$ for some $\beta \in H^{*,*}(\mathcal{X}_\theta, \mathbb{Z})^{\geq 0}$.*

Proof. By Corollary 1.7, it is sufficient to show that $Q_i(\alpha) = 0$.

Suppose first that $i = n-1$. Since $b < l^{n-1} \leq \operatorname{ord}(\alpha) \leq c$, it follows from Lemma 1.12 that $Q_{n-1}(\alpha) = 0$. In the rest of the proof we assume that $i \leq n-2$.

Case 1: $l \geq 3$. We have

$$2l^i \leq \operatorname{ord} Q_i(\alpha) < 3l^i \leq l^{i+1}.$$

By Lemma 1.13, $Q_i(\alpha) = 0$.

Case 2: $l = 2$. We prove that $Q_i(\alpha) = 0$ by descending induction on i. Since $2^{i+1} \leq \operatorname{ord} Q_i(\alpha) < 2^i + 2^{i+1}$, by the induction hypothesis, $(Q_{i+1}Q_i)(\alpha) = 0$. By Corollary 1.7(2), $Q_i(\alpha) = Q_{i+1}(\rho)$ for some $\rho \in H^{*,*}(\mathcal{X}_\theta, \mathbb{Z})^{\geq 0}$. Since $(Q_{i+1}Q_i)(\rho) = -(Q_iQ_{i+1})(\rho) = -Q_i^2(\alpha) = 0$ and $\operatorname{ord}(\rho) < 2^i$, $\operatorname{ord} Q_i(\rho) < 2^{i+1}$, it follows from Corollary 1.7 that $\rho = 0$ and therefore $Q_i(\alpha) = 0$. □

1.8. **Main theorem.** Consider the exterior algebra

$$\Lambda = K_*^\theta(F)[t][\lambda_1, \dots, \lambda_{n-1}]$$

over the polynomial algebra $K_*^\theta(F)[t]$, i.e., $\lambda_i^2 = 0$ and $\lambda_i\lambda_j = -\lambda_j\lambda_i$ for $i \neq j$. Recall (see section 1.3) that $H^{*,*}(\widetilde{\mathcal{X}}_\theta, \mathbb{Z})$ has a structure of a left $K_*^\theta(F)$-module. The operations Q_i are $K_*(F)$-linear and Q_i commute with multiplication by γ by Lemma 1.8. Therefore, we can view $H^{*,*}(\mathcal{X}_\theta, \mathbb{Z})$ as a left Λ-module with t acting by multiplication by γ and λ_i acting via the operation Q_i. The ring Λ is graded over $K_*^\theta(F)$ as follows: $\deg(t) = c+1$, $\deg(\lambda_i) = l^i$. For any homogeneous element $\lambda \in \Lambda$ and any $\alpha \in H^{p,q}(\mathcal{X}_\theta, \mathbb{Z})$, we have

$$\operatorname{ord}(\lambda\alpha) = \deg(\lambda) + \operatorname{ord}(\alpha).$$

Note that distinct monomials in Λ of the form $t^k\lambda_1^{a_1}\lambda_2^{a_2}\dots\lambda_{n-1}^{a_{n-1}}$ with $a_i = 0$ or 1 have different degree.

Theorem 1.15. *Let l be a prime integer, $n \geq 2$, F a field of characteristic different from l and $\theta \in H^n_{et}(F, \mu_l^{\otimes(n-1)})$ a nontrivial symbol. Then*

(1) *$H^{*,*}(\mathcal{X}_\theta, \mathbb{Z})^{\leq 0}$ is canonically isomorphic to $H^{*,*}(F, \mathbb{Z})^{\leq 0}$.*
(2) *$H^{*,*}(\mathcal{X}_\theta, \mathbb{Z})^{\geq 0}$ is a free left Λ-module with basis $\{\delta\}$.*

Proof. (1) follows from Proposition 1.2.

(2): We shall prove that the map

$$\Lambda \to H^{*,*}(\mathcal{X}_\theta, \mathbb{Z})^{\geq 0}, \qquad \lambda \mapsto \lambda\delta$$

is an isomorphism.

Injectivity: In view of the remark preceding the theorem, it suffices to prove that if $\lambda\delta = 0$ for a monomial $\lambda = xt^k\lambda_1^{a_1}\lambda_2^{a_2}\dots\lambda_{n-1}^{a_{n-1}}$ with $x \in K_*^\theta(F)$ and the integers $k \geq 0$ and a_i such that $a_i = 0$ or 1, i.e.,

$$\gamma^k \cup (Q_1^{a_1}Q_2^{a_2}\dots Q_{n-1}^{a_{n-1}})(x\cdot\delta) = 0,$$

then $x = 0$. By Proposition 1.10,

$$(Q_1^{a_1}Q_2^{a_2}\dots Q_{n-1}^{a_{n-1}})(x\cdot\delta) = 0.$$

It follows from Corollary 1.7 by descending induction on $i \geq 0$ that

$$(Q_1^{a_1}Q_2^{a_2}\dots Q_i^{a_i})(x\cdot\delta) = 0.$$

Therefore, $x\cdot\delta = 0$ and hence $x = 0$ by Proposition 1.11.

Surjectivity: Let $\alpha \in H^{p,q}(\mathcal{X}_\theta, \mathbb{Z})$ be an element with $\mathrm{ord}(\alpha) \geq 0$. By induction on $\mathrm{ord}(\alpha)$ we show that $\alpha = \lambda\delta$ for some $\lambda \in \Lambda$. If $\mathrm{ord}(\alpha) = 0$, then $\alpha = 0$ by Proposition 1.2. The case $\mathrm{ord}(\alpha) = 1$ is covered by Proposition 1.11. If $\mathrm{ord}(\alpha) \geq c+1$, then by Proposition 1.10(2), $\alpha = \gamma \cup \beta$ for β with $0 \leq \mathrm{ord}(\beta) < \mathrm{ord}(\alpha)$. By induction, $\beta = \lambda\delta$ for some $\lambda \in \Lambda$ and hence $\alpha = (t\lambda)\delta$.

Suppose $2 \leq \mathrm{ord}(\alpha) \leq c$. Choose an $i = 0, 1, \dots, n-1$ such that $l^i \leq \mathrm{ord}(\alpha) < l^{i+1}$. If $\mathrm{ord}(\alpha) \geq 2l^i$ (and hence $i \leq n-2$), then by Lemma 1.13, $\alpha = 0$. So we may assume that $l^i \leq \mathrm{ord}(\alpha) < 2l^i$ for $i \geq 1$. It follows from Lemma 1.14 that $\alpha = Q_i(\beta)$ for some β with $0 \leq \mathrm{ord}(\beta) < \mathrm{ord}(\alpha)$. By induction, $\beta = \lambda\delta$ for some $\lambda \in \Lambda$ and hence $\alpha = (\lambda_i\lambda)\delta$. $\square$

Remark 1.16. When $l = 2$ we have $\gamma = \mu^2$. Moreover if s is the operation of multiplication by μ, the left $K_*^\theta(F)[s][\lambda_1, \dots \lambda_{n-2}]$-module $H^{*,*}(\mathcal{X}_\theta, \mathbb{Z})^{\geq 0}$ is free with basis $\{\delta\}$. In this form the statement was proved by Orlov, Vishik, and Voevodsky (unpublished).

Remark 1.17. By Theorem 1.15, a nontrivial element $\alpha \in H^{p,q}(\mathcal{X}_\theta, \mathbb{Z})$ of positive order can be uniquely written in the form

$$\alpha = \gamma^k \cup (Q_1^{a_1}Q_2^{a_2}\dots Q_{n-1}^{a_{n-1}})(x\cdot\delta)$$

where $x \in K_*^\theta(F)$ and k, a_i are integers such that $k \geq 0$ and $a_i = 0$ or 1. Moreover,

$$\operatorname{ord}(\alpha) = 1 + k(c+1) + \sum_{i=1}^{n-1} l^{a_i}.$$

1.9. **The multiplicative structure.** The ring structure of $H^{*,*}(\mathcal{X}_\theta, \mathbb{Z})^{\geq 0}$ in the case $l = 2$ has been determined by Orlov, Vishik, and Voevodsky (unpublished).

Let l be an odd prime. The operation S of the form $\pm Q_{i_1} Q_{i_2} \dots Q_{i_k}$, where $k \geq 0$ and $i_1 < \cdots < i_k$, is called a *monomial*. For a monomial S there exists a unique monomial S' such that $S'S = Q_1 Q_2 \dots Q_{n-1}$. We shall compute the cup-product $T(\delta) \cup S(\delta)$ for two monomials T and S.

Proposition 1.18. *Let S and T be monomials. Then*

1. *If $S'T' = 0$, then $T(\delta) \cup S(\delta) = 0$.*
2. *If $S'T' \neq 0$, i.e., $S'T'$ is a monomial, let U be (the unique) monomial such that $U' = S'T'$. Then $T(\delta) \cup S(\delta) = U(\delta) \cup \gamma$.*

Proof. (1): By assumption, the monomials T and S do not contain Q_i for some i. Therefore, the i-th digit of $\operatorname{ord} T(\delta) + \operatorname{ord} S(\delta)$ written in base l is equal to 0. By Theorem 1.15, the product $T(\delta) \cup S(\delta)$, if not zero, is a $K_*^\theta(F)$-multiple of either $U(\delta) \cup \gamma$ or $U(\delta)$ for some monomial U. In the first case $\operatorname{ord} T(\delta) + \operatorname{ord} S(\delta) = \operatorname{ord} U(\delta) + c$, and this case is impossible since all digits of the right hand side written in base l are nonzero.

In the second case $\operatorname{ord} T(\delta) + \operatorname{ord} S(\delta) = \operatorname{ord} U(\delta) - 1$ and this case does not occur since $\operatorname{ord} V(\delta) \equiv 1$ modulo l for every monomial V. Therefore $T(\delta) \cup S(\delta) = 0$.

(2): If $T' = 1$, then $U = S$, $T(\delta) = \gamma$ and the equality follows.

Assume that $T' \neq 1$. By assumption $T'S = 0$. Therefore,

$$\begin{aligned} S'T'\big(T(\delta) \cup S(\delta)\big) &= S'\big(T'T(\delta) \cup S(\delta) \pm T(\delta) \cup T'S(\delta)\big) \\ &= S'\big(\gamma \cup S(\delta)\big) \\ &= \gamma \cup S'S(\delta) \\ &= \gamma^2, \end{aligned}$$

$$\begin{aligned} S'T'\big(U(\delta) \cup \gamma\big) &= U'\big(U(\delta) \cup \gamma\big) \\ &= U'U(\delta) \cup \gamma \\ &= \gamma^2. \end{aligned}$$

By Theorem 1.15, the restriction of every operation Q_i on a homogeneous component $H^{p,q}(\mathcal{X}_\theta, \mathbb{Z})$ of non-negative order is either injective or zero. Hence $T(\delta) \cup S(\delta) = U(\delta) \cup \gamma$. □

2. Applications

In this section we give some applications.

2.1. **An exact sequence.**

Theorem 2.1. *Let l be a prime integer, F a field of characteristic different from l and $\theta \in H^n_{et}(F, \mu_l^{\otimes(n-1)})$ a symbol. Then the sequence*

$$\coprod H^p_{et}(L, \mu_l^{\otimes p}) \xrightarrow{\sum N_{L/F}} H^p_{et}(F, \mu_l^{\otimes p}) \xrightarrow{\cup\theta}$$
$$H^{p+n}_{et}(F, \mu_l^{\otimes(p+n-1)}) \xrightarrow{\prod \mathrm{res}_{E/F}} \prod H^{p+n}_{et}(E, \mu_l^{\otimes(p+n-1)}),$$

where the coproduct is taken over all finite splitting field extensions L/F for θ and the product is taken over all splitting field extensions E/F, is exact.

Proof. Note that by a projection formula, the sequence is a complex. By Proposition 1.4, the kernel of the last homomorphism in the sequence is canonically isomorphic to $H^{p+n+1,p+n-1}(\mathcal{X}_\theta, \mathbb{Z})$. Under this isomorphism, the cup-product with θ corresponds to multiplication by δ. The statement follows now from Proposition 1.11, the definition of $K^\theta_p(F)$ and the bijectivity of the norm residue homomorphism. □

2.2. **Certain motivic cohomology groups of the Rost motive $\mathcal{R}_\theta$.** Let l be a prime integer, F a field of characteristic different from l, $\theta \in H^n_{et}(F, \mu_l^{\otimes(n-1)})$ a symbol, X_θ a Rost variety for θ and $\mathcal{R}_\theta$ the Rost motive of θ. Recall the exact sequence (11):

$$H^{p+2c-1,p+c-1}(\mathcal{X}_\theta, \mathbb{Z}_{(l)}) \to H^{p+2d,p+d}(\mathcal{R}_\theta, \mathbb{Z}_{(l)}) \to H^{p,p}(\mathcal{X}_\theta, \mathbb{Z}_{(l)}) = K_p(F)_{(l)}.$$

By Theorem 1.15 (see also Remark 1.17), the degree $c-2$ component of the ring Λ is zero, hence the first group in the sequence is trivial. We have proved:

Proposition 2.2. *The natural homomorphism*

$$H^{p+2d,p+d}(\mathcal{R}_\theta, \mathbb{Z}_{(l)}) \to K_p(F)_{(l)}$$

is injective.

2.3. **Injectivity of the reduced norm map for a central simple algebra.** Let D be a central simple algebra of degree m over F. An old theorem of Wang (see [2]) asserts that the classical reduced norm homomorphism

$$\mathrm{Nrd}_D : K_1(D) \to K_1(F)$$

is injective provided m is a square-free integer.

The reduced norm homomorphism Nrd_D for the K_2-groups has been defined in [12, §26]. It was proven in [10] and [7] that Nrd_D is injective in the case $m = 2$. In the following statement we generalize this result to a K_2-analog of Wang's theorem.

Theorem 2.3. *Let D be a central simple algebra of square-free degree over F. Then the reduced norm homomorphism*

$$\mathrm{Nrd}_D : K_2(D) \to K_2(F)$$

is injective.

Lemma 2.4. *Let B and C be two central simple algebras over F of relatively prime degree. Suppose the reduced norm homomorphisms for B and C are injective over any field extension of F. Then $\mathrm{Nrd}_{B\otimes C}$ is also injective.*

Proof. Let L/F be a splitting field of C of degree r dividing $\deg C$. By assumption, the bottom homomorphism of the diagram

$$\begin{tikzcd} & K_2(B\otimes C) \arrow[r, "\mathrm{Nrd}"] \arrow[d] & K_2(F) \arrow[d] \\ K_2(B\otimes L) \arrow[r, equal] & K_2(B\otimes C\otimes L) \arrow[r, "\mathrm{Nrd}"] & K_2(L) \end{tikzcd}$$

is injective. We deduce that the kernel K of $\mathrm{Nrd}_{B\otimes C}$ is r-torsion. Similarly we show that K is s-torsion for some integer s relatively prime to r, hence $K = 0$. □

Since the algebra D is a tensor product of algebras of prime degree (see [2]), Lemma 2.4 allows to assume that D is a division algebra of a prime degree l.

Let L/F be a finite extension splitting the algebra D. Then $[L : F] = kl$ for some integer k and there is an embedding of F-algebras $L \hookrightarrow M_k(D)$ (see [4]). The induced homomorphism

$$K_2(L) \to K_2\big(M_k(D)\big) = K_2(D)$$

does not depend on the choice of the embedding.

Let X be the Severi-Brauer variety of left ideals of dimension l in D. As for every closed point $x \in X$ the residue field $F(x)$ splits D, we have a canonical homomorphism $K_2F(x) \to K_2(D)$.

Lemma 2.5. *There is a homomorphism $h : A_0(X, K_2) \to K_2(D)$ satisfying the following properties:*

1. *The composition $\mathrm{Nrd}_D \circ h$ is the norm map $A_0(X, K_2) \to K_2(F)$.*
2. *For every closed point $x \in X$, the composition of h with the natural homomorphism $K_2F(x) \to A_0(X, K_2)$ coincides with the canonical map $K_2F(x) \to K_2(D)$.*

Proof. We follow the construction in [9, §8]. Let J be the canonical vector bundle of rank l over X. There is a natural right action of the opposite algebra D^{-1} on J over F. For every $i \geq 0$, the functor $M \mapsto J^i \otimes_{D^{\otimes -i}} M$ from the category of left finite $D^{\otimes -i}$-modules to the category of vector bundles over X induces a homomorphism

$$K_2(D^{\otimes -i}) \to K_2(X).$$

By Quillen's theorem [9, §8, Th. 4.1], the map

$$\coprod_{i=0}^{l-1} K_2(D^{\otimes -i}) \to K_2(X)$$

is an isomorphism. We define the map h as the composition

$$h : A_0(X, K_2) \to K_2(X) \to K_2(D),$$

where the first map is the edge homomorphism of the Gersten-Quillen spectral sequence [9, §7, Th. 5.4] and the second one is projection on the $(l-1)$-th component of the left hand side in Quillen's isomorphism. The Gersten-Quillen spectral sequence is functorial with respect to the base field change, in particular, h commutes with the norm maps for finite field extensions.

Note that the group $A_0(X, K_2)$ is generated by the norms for finite field extensions that split D. Thus, to prove the first property of h we may assume that D is split. In this case X is isomorphic to the projective space $\mathbb{P}^{l-1}$, $A_0(X, K_2) \simeq K_2(F)$ canonically and $K_2(D^{-i})$ can be identified with $K_2(F)$ via the reduced norm homomorphism. The image of an element $\alpha \in K_2(F)$ in $K_2(X)$ is equal to $\alpha \cdot [pt]$, where $[pt]$ is the class of a rational point in $K_0(X)$. Note that the Quillen's isomorphism takes $\sum a_i$ to $\sum a_i\eta^i \in K_*(X)$, where η is the class of the canonical line bundle (with the sheaf of sections $\mathcal{O}(1)$). Since $[pt] = (\eta - 1)^{l-1} = \eta^{l-1} + \dots$, the element $\alpha \cdot [pt]$ projects to $\alpha \in K_2(F) = K_2(D)$, that proves the first property of h.

To prove the second property let $x \in X$ be a closed point of degree kl and let $L = F(x)$. Choose a rational point $x' \in X_L$ over x. For every $\alpha \in K_2(L)$ the classes $\alpha x'$ and αx in the groups $A_0(X_L, K_2)$ and $A_0(X, K_2)$ respectively satisfy $N_{L/F}(\alpha x') = \alpha x$, where $N_{L/F}$ is the left vertical norm homomorphism in the commutative diagram

$$\begin{array}{ccc} A_0(X_L, K_2) & \xrightarrow{h_L} & K_2(D_L) \\ {\scriptstyle N_{L/F}}\downarrow & & \downarrow{\scriptstyle N_{L/F}} \\ A_0(X, K_2) & \xrightarrow{h} & K_2(D), \end{array}$$

where $D_L = D \otimes_F L$. Since $\mathrm{Nrd}_{D_L}\big(h_L(\alpha x')\big) = \alpha$ by the first part, it suffices to prove that the right vertical norm homomorphism coincides with the composition

$$K_2(D_L) \xrightarrow{\mathrm{Nrd}} K_2(L) \to K_2(D).$$

This follows from commutativity of the diagram

$$\begin{array}{ccccccc} K_2(D_L) & \xrightarrow{\sim} & K_2M_l(L) & \longrightarrow & K_2M_{kl}(D) & \xrightarrow{\sim} & K_2M_l(D) \\ {\scriptstyle \mathrm{Nrd}}\downarrow & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ K_2(L) & = & K_2(L) & \longrightarrow & K_2M_k(D) & \xrightarrow{\sim} & K_2(D) \end{array}$$

and the fact that the diagonal composition $K_2(D_L) \to K_2(D)$ coincides with the norm map. □

Now we can finish the proof of Theorem 2.3. As the kernel of Nrd_D is l-torsion, it suffices to prove that Nrd_D is injective after tensoring with $Z_{(l)}$. A transfer argument shows that we can replace F by a finite field extension of degree prime to l. Thus, we can assume that D is a cyclic algebra.

Let θ be the 2-symbol corresponding to D and X the Severi-Brauer variety of D. Then X is a Rost variety of θ. The Rost motive $\mathcal{R}_\theta$ is a direct sum of

$M(X)$ in $\mathbf{DM}(F,\mathbb{Z}_{(l)})$. Let

$$\mathcal{R}_\theta \xrightarrow{r} M(X) \xrightarrow{s} \mathcal{R}_\theta$$

be morphisms such that $s\circ r$ is the identity of $\mathcal{R}_\theta$. Over a splitting field extension L/F, $X_L \simeq \mathbb{P}_L^{l-1}$, therefore, the motives $(\mathcal{R}_\theta)_L$ and $M(X_L)$ are both isomorphic to $\mathbb{Z}_{(l)} \oplus \mathbb{Z}_{(l)}(1)[2] \oplus \cdots \oplus \mathbb{Z}_{(l)}(l-1)[2(l-1)]$ by (6) and hence are isomorphic. Identifying $(\mathcal{R}_\theta)_L$ and $M(X_L)$ with $M(\mathbb{P}_L^{l-1})$, we can view r_L and s_L as endomorphisms of $M(\mathbb{P}_L^{l-1})$ with s_L a left inverse of r_L. The endomorphism ring of $M(\mathbb{P}_L^{l-1})$ is $\mathrm{CH}^{l-1}(\mathbb{P}_L^{l-1}\times\mathbb{P}_L^{l-1})_{(l)}$ (see [15]) that is isomorphic to the product of l copies of $\mathbb{Z}_{(l)}$. As this ring is commutative, r_L and s_L are isomorphisms inverse to each other. By [1, Cor. 8.4.], r and s are isomorphisms, i.e., the Rost motive $\mathcal{R}_\theta$ is isomorphic to the motive of the Severi-Brauer variety X in $\mathbf{DM}(F,\mathbb{Z}_{(l)})$.

By Proposition 2.2 (see also the proof of Proposition 1.9), the norm homomorphism $N : A_0(X,K_2)\to K_2(F)$ is injective after tensoring with $Z_{(l)}$. By Lemma 2.5(1), N coincides with the composition

$$A_0(X,K_2) \xrightarrow{h} K_2(D) \xrightarrow{\mathrm{Nrd}_D} K_2(F).$$

It follows from [8, Th. 5.2] that the group $K_2(D)$ is generated by the images of natural homomorphisms $K_2F(x)\to K_2(D)$ over all closed points $x\in X$. Hence, by Lemma 2.5(2), h is surjective. It follows that Nrd_D is injective after tensoring with $Z_{(l)}$. □

Remark 2.6. Theorem 2.3 was proven independently by B. Kahn and M. Levine in [5].

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