THE GROUP SK_1 FOR SIMPLE ALGEBRAS

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Let A be central simple algebra over a field F. Denote by $K_1(A)$ the Whitehead group and by $SK_1(A)$ the reduced Whitehead group of A [2, §23]. If the Schur index $\operatorname{ind}(A)$ is squarefree, then the group $SK_1(A)$ is trivial [2, §23, Cor. 4]. In the case when $\operatorname{ind}(A)$ is not squarefree, the reduced Whitehead group can still be trivial over some classes of fields, for example, global and local fields. On the other hand, A. Suslin has conjectured in [15] that if $\operatorname{ind}(A)$ is not squarefree, then the group $SK_1(A)$ is nontrivial generically, i.e., there is a field extension L/F such that $SK_1(A_L) \neq 0$ where $A_L = A \otimes_F L$.

In the present paper we give a prove of the following case of Suslin's conjecture:

Theorem. *Let A be a central simple algebra over a field F. If $\operatorname{ind}(A)$ is divisible by 4, then $SK_1(A_L) \neq 0$ for some field extension L/F.*

Note that in [7] this theorem was proven in the case $\operatorname{char} F \neq 2$ with help of the algebraic theory of quadratic forms. The present proof does not rely on quadratic forms and could be viewed as a step towards the proof of the conjecture in general.

Let $\mathbf{GL}_1(A)$ be the algebraic group of invertible elements of A (with the group of F-points equal to $A^\times$). The kernel of the reduced norm homomorphism $\operatorname{Nrd} : \mathbf{GL}_1(A) \to \mathbf{GL}_1(F)$ is denoted by $\mathbf{SL}_1(A)$.

It is shown in [16, §18, Cor. 3] that the theorem yields

Corollary. *If A is a central simple algebra of index divisible by 4, then the variety of the algebraic group $\mathbf{SL}_1(A)$ is not stably rational.*

1. Proof of Theorem

Let X be a variety over a field F. For every integer $p \geq 0$, denote by $X^{(p)}$ the set of points of X of codimension p. Let M_* be a cycle module over X [12, §2]. We write $A^p(X, M_n)$ for the homology group of the complex

$$\coprod_{x \in X^{(p-1)}} M_{n-p+1}F(x) \xrightarrow{\partial} \coprod_{x \in X^{(p)}} M_{n-p}F(x) \xrightarrow{\partial} \coprod_{x \in X^{(p+1)}} M_{n-p-1}F(x).$$

Example 1.1. We denote by K_* the cycle module given by Quillen's K-groups [12, Remark 2.5]. The group $A^p(X, K_p)$ coincides with the Chow group $\operatorname{CH}^p(X)$ of the classes of cycles on X of codimension p.

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Let $f : X \to Y$ be a flat morphism of varieties over F. For a point $y \in Y$, denote by X_y the fiber of f over y, so that X_y is a variety over the residue field $F(y)$. For every integer n there is the *fibration spectral sequence* [12, Cor. 8.2]

$$E_1^{p,q} = \coprod_{y \in Y^{(p)}} A^q(X_y, M_{n-p}) \Rightarrow A^{p+q}(X, M_n).$$

In the rest of the section we formulate three propositions and show that they imply the theorem. The proofs of the propositions are given in the last three sections of the paper.

Let $X = \mathrm{SB}(A)$ be the Severi-Brauer variety of A of right ideals of A of dimension $\deg(A) = \dim(A)^{1/2}$. The dimension d of X is equal to $\deg(A) - 1$. We define the cycle module S_* over F by

$$S_n(E) = A^d(X_E, K_{d+n})$$

for a field extension E/F, where $X_E = X \times_{\operatorname{Spec} F} \operatorname{Spec} E$ [12, Th. 7.3]. The push-forward homomorphism

$$p_* : A^d(X_E, K_{d+n}) \to A^0(\operatorname{Spec} E, K_n) = K_n(E)$$

induced by the structure morphism $p : X_E \to \operatorname{Spec} E$, gives rise to the *norm* map $N : S_* \to K_*$ of cycle modules.

If A is a split algebra, the variety X is isomorphic to the projective space $\mathbb{P}^d_F$. The projective bundle theorem for the K-cohomology [13, Cor. 8.2.1] implies that the push-forward homomorphism p_* is an isomorphism, hence N is also an isomorphism.

Consider the algebraic group $G = \mathbf{SL}_1(A)$. The group $A^1(G, K_2)$ is infinite cyclic with a canonical generator and does not change under field extensions [3, Th. 9.3].

Proposition 1.2. *Suppose that the norm homomorphism* $A^1(G, S_2) \to A^1(G, K_2)$ *is not surjective. Then* $\mathrm{SK}_1(A_{F(G)}) \neq 0$.

Now suppose that A is a biquaternion algebra, i.e., $\deg(A) = 4$ and the exponent $\exp(A)$ divides 2. In this case $d = 3$.

Consider the fibration spectral sequence associated with the projection $G \times X \to G$:

$$E_1^{p,q} = \coprod_{g \in G^{(p)}} A^q(X_{F(g)}, K_{5-p}) \Rightarrow A^{p+q}(G \times X, K_5). \tag{1}$$

Clearly, $E_*^{p,q} = 0$ if $q > \dim X = 3$ and $E_2^{p,3} = A^p(G, S_2)$. Let

$$\delta : A^1(G, S_2) = E_2^{1,3} \to E_2^{3,2}$$

be the differential in the spectral sequence.

Proposition 1.3. *If* $\operatorname{ind} A = 2$*, then the image of* $\operatorname{Ker} \delta$ *under the norm homomorphism*

$$E_2^{1,3} = A^1(G, S_2) \xrightarrow{N} A^1(G, K_2) = \mathbb{Z}$$

is contained in $2\mathbb{Z}$.

Let $Y = \mathrm{SB}(2, A)$ be the generalized Severi-Brauer variety of right ideals of A of dimension 8. We set $\widetilde{F} = F(Y)$, $\widetilde{A} = A_{\widetilde{F}}$, $\widetilde{X} = X_{\widetilde{F}}$, $\widetilde{G} = G_{\widetilde{F}}$ and write $\widetilde{E}_*^{p,q}$ for the terms of the fibration spectral sequence associated with the projection $\widetilde{G} \times \widetilde{X} \to \widetilde{G}$. There is a natural morphism of spectral sequences $\kappa : E_*^{p,q} \to \widetilde{E}_*^{p,q}$.

Proposition 1.4. *If* $\operatorname{ind} A = 4$, *then* $\kappa : E_2^{3,2} \to \widetilde{E}_2^{3,2}$ *is the zero map.*

We deduce now the theorem from Propositions 1.2, 1.3 and 1.4. By the index reduction formula in [1], there is a field extension E/F such that the algebra A_E is similar to a biquaternion algebra of index 4. Replacing A by A_E we get A similar to a biquaternion algebra. Since the groups SK_1 of similar algebras are canonically isomorphic [2, §23], we may assume that A itself is a biquaternion algebra of index 4.

Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
\mathbb{Z} & = & A^1(G, K_2) & \xleftarrow{N} & A^1(G, S_2) & = & E_2^{1,3} & \xrightarrow{\delta} & E_2^{3,2} \\
\| & & \downarrow & & \downarrow & & \downarrow & & \downarrow\kappa \\
\mathbb{Z} & = & A^1(\widetilde{G}, K_2) & \xleftarrow{\widetilde{N}} & A^1(\widetilde{G}, S_2) & = & \widetilde{E}_2^{1,3} & \xrightarrow{\widetilde{\delta}} & \widetilde{E}_2^{3,2}.
\end{array}$$

Since $A_{\widetilde{F}}$ is of index 2, by Propositions 1.3 (applied to $\widetilde{A}$) and 1.4, the image of the composite

$$A^1(G, S_2) \to A^1(\widetilde{G}, S_2) \xrightarrow{\widetilde{N}} A^1(\widetilde{G}, K_2) = \mathbb{Z}$$

is contained in $2\mathbb{Z}$. On the other hand, this composite coincides with N. Therefore the norm homomorphism $N : A^1(G, S_2) \to A^1(G, K_2)$ is not surjective and Proposition 1.2 completes the proof of the theorem.

2. Proof of Proposition 1.2

Consider the split case first. The generic matrix determines a canonical element $\alpha \in K_1(\mathbf{SL}_n)$. Since $\det(\alpha) = 1$, the element α vanishes over the generic point of $\mathbf{SL}_n$, therefore α belongs to the first term $K_1(\mathbf{SL}_n)^{(1)}$ of the topological filtration on $K_1(\mathbf{SL}_n)$ [11, §7]. By [14, Th. 2.7], the first Chern class $c_1(\alpha)$ generates the group $A^1(\mathbf{SL}_n, K_2)$.

Lemma 2.1. *The image of* α *under the canonical homomorphism* $K_1(\mathbf{SL}_n)^{(1)} \to A^1(\mathbf{SL}_n, K_2)$ *is equal to the generator* $c_1(\alpha)$.

Proof. Let $\beta \in K_1(\mathbf{GL}_n)$ be the element given by the generic matrix. By [14, Th. 3.10], $\gamma_{p+1}(\beta) \in K_1(\mathbf{GL}_n)^{(p)}$ for all $p \geq 0$, where γ is the gamma-operation, and the image of $-\gamma_2(\beta)$ under the canonical homomorphism

$$K_1(\mathbf{GL}_n)^{(1)} \to A^1(\mathbf{GL}_n, K_2)$$

is equal to $c_1(\beta)$. On the other hand, the sum of $\gamma_{p+1}(\beta)$ for all $p \geq 1$ coincides with $\Lambda^n(\beta) = \det(\beta)$ [14, p. 65]. Hence $-\gamma_2(\beta) \equiv \beta - \det(\beta)$ modulo $K_1(\mathbf{GL}_n)^{(2)}$.

Pulling back with respect to the embedding of $\mathbf{SL}_n$ into $\mathbf{GL}_n$ we have $-\gamma_2(\alpha) \equiv \alpha$ modulo $K_1(\mathbf{SL}_n)^{(2)}$ since $\det(\alpha) = 0$ and therefore the image of α under the homomorphism $K_1(\mathbf{SL}_n)^{(1)} \to A^1(\mathbf{SL}_n, K_2)$ is equal to $c_1(\alpha)$. $\square$

Now let A be a central simple algebra over F, let $G = \mathbf{SL}_1(A)$ and set $X = \mathrm{SB}(A)$. Filtering the category of coherent $A \otimes_F \mathcal{O}_X$-modules by codimension of support as in [11, §7.5], we get the Gersten-Quillen spectral sequence

$$E_1^{p,q} = \coprod_{g \in G^{(p)}} K_{-p-q} A_{F(g)} \Rightarrow K_{-p-q}(G, A),$$

where the limit is the K-group of the category of coherent $A \otimes_F \mathcal{O}_X$-modules. Consider the cycle module R_* over F given by $R_n(E) = K_n(A_E)$. We have $E_2^{p,q} = A^p(G, R_{-q})$.

By [13, §26], there is *reduced norm homomorphism* $\mathrm{Nrd} : R_n \to K_n$ for $n \le 2$. It yields a homomorphism

$$\mathrm{Nrd}_G : A^1(G, R_2) \to A^1(G, K_2).$$

If A is split, the homomorphisms Nrd and Nrd_G are isomorphisms.

The generic element of G defines a canonical element $\alpha_A \in K_1(G, A) = K_1\big(A \otimes_F F[G]\big)$.

Suppose that $\mathrm{SK}_1(A_{F(G)}) = 0$. Then α_A vanishes over the generic point of G and therefore it belongs to the first term $K_1(G, A)^{(1)}$ of the topological filtration. Denote by ξ the image of α_A under the canonical homomorphism

$$K_1(G, A)^{(1)} \to E_2^{1,-2} = A^1(G, R_2).$$

Let E/F be a splitting field of the algebra A. In the commutative diagram

$$\begin{tikzcd}
 & K_1(G,A)^{(1)} \arrow[r] \arrow[dl] & A^1(G,R_2) \arrow[dl, "r"'] \arrow[d, "\mathrm{Nrd}_G"] \\
K_1(G_E,A_E)^{(1)} \arrow[r] \arrow[d, "\wr"] & A^1(G_E,R_2) \arrow[d, "\mathrm{Nrd}_{G_E}"', "\wr"] & A^1(G,K_2) \arrow[dl, "k"'] \\
K_1(G_E)^{(1)} \arrow[r] & A^1(G_E,K_2) &
\end{tikzcd}$$

the two lower vertical homomorphisms are canonical isomorphisms. The image of α_A in the group $K_1(G_E)^{(1)}$ is equal to the generic element α defined above. It follows from the commutativity of the diagram that the element $\mathrm{Nrd}_{G_E}(r(\xi))$ is the image of α from $K_1(G_E)^{(1)}$. By Lemma 2.1, the element $\mathrm{Nrd}_{G_E}(r(\xi))$ is a generator of $A^1(G_E, K_2)$. Since k is an isomorphism, the element $\mathrm{Nrd}_G(\xi)$ is a generator of $A^1(G, K_2)$. Therefore, the map Nrd_G is surjective.

Note that by [10], there is a natural isomorphism $R_n \simeq S_n$ for $n = 0$ and 1. Hence the images of Nrd_G and of the norm map $A^1(G, S_2) \to A^1(G, K_2)$ coincide. Therefore the latter map is surjective, a contradiction.

3. Proof of Proposition 1.3

We assume that A is a biquaternion algebra of index 2, so that $A = M_2(Q)$, where Q is a quaternion division algebra. We have $G = \mathbf{SL}_2(Q)$ and denote by C the conic curve $\mathrm{SB}(Q)$.

Consider the fibration spectral sequence associated with the projection $G \times C \to G$:

$$\hat{E}_1^{p,q} = \coprod_{g\in G^{(p)}} A^q(C_{F(g)}, K_{3-p}) \Rightarrow A^{p+q}(G \times C, K_3). \tag{2}$$

We have for $p = 2, 3$:

$$\hat{E}_1^{p,0} = \coprod_{g\in G^{(p)}} A^0(C_{F(g)}, K_{3-p}) = \coprod_{g\in G^{(p)}} K_{3-p}(F(g)),$$

hence $\hat{E}_2^{3,0} = \mathrm{CH}^3(G)$. The spectral sequence (2) yields an exact sequence

$$A^2(G \times C, K_3) \to \hat{E}_2^{1,1} \xrightarrow{\hat{\delta}} \hat{E}_2^{3,0}. \tag{3}$$

By [4, Cor. 1.3.2], the motive $M(X)$ in the category of Chow motives over F is isomorphic canonically to $M(C) \oplus M(C)(2)$. Hence,

$$A^{q+2}(X, K_{n+2}) \simeq A^{q+2}(C, K_{n+2}) \oplus A^q(C, K_n).$$

If $q \geq 0$, then $A^{q+2}(C, K_{n+2}) = 0$, hence there is an isomorphism

$$A^q(C, K_n) \xrightarrow{\sim} A^{q+2}(X, K_{n+2}),$$

which in fact is induced by a closed embedding $C \to X$ [8, p. 326].

We compare the spectral sequences (1) and (2). The push forward homomorphism with respect to the embedding $C \to X$ yields a morphism of spectral sequences $\hat{E}_*^{p,q} \to E_*^{p,q+2}$ which is an isomorphism for $q = 0$ and 1. In particular,

$$E_2^{3,2} = \hat{E}_2^{3,0} = \mathrm{CH}^3(G). \tag{4}$$

We also identify the differential $\delta : E_2^{1,3} \to E_2^{3,2}$ with the differential $\hat{\delta} : \hat{E}_2^{1,1} \to \hat{E}_2^{3,0}$. The exactness of (3) implies that it is sufficient to show that the image of the composite

$$A^2(G \times C, K_3) \to \hat{E}_2^{1,1} = E_2^{1,3} = A^1(G, S_2) \xrightarrow{N} A^1(G, K_2) = \mathbb{Z}$$

is contained in $2\mathbb{Z}$. Since the norm map N is induced by the push-forward homomorphism for the projection $C \to \operatorname{Spec} F$, this composite is the push-forward homomorphism

$$r_*^G : A^2(G \times C, K_3) \to A^1(G, K_2) = \mathbb{Z}$$

with respect to the projection $r^G : G \times C \to G$.

Consider the group $H = \mathbf{SL}_1(Q)$ as a subgroup of G. Let $i : H \to G$ be the embedding morphism. Choose a field extension E/F splitting the algebra Q and consider the commutative diagram

$$\begin{array}{ccccc}
A^2(G\times C,K_3) & \xrightarrow{(i\times 1_C)^*} & A^2(H\times C,K_3) & \xrightarrow{\text{res}} & A^2(H_E\times C_E,K_3)\\
{\scriptstyle r_*^G}\downarrow & & {\scriptstyle r_*^H}\downarrow & & {\scriptstyle r_*^{H_E}}\downarrow\\
A^1(G,K_2) & \xrightarrow{i^*} & A^1(H,K_2) & \xrightarrow{\text{res}'} & A^1(H_E,K_2).
\end{array}$$

The algebra Q is isomorphic over C to $\mathrm{End}_{\mathcal{O}_C}(J)$, where J is the canonical locally free sheaf on C of rank 2. By [14, Th. 4.2],

$$A^2(H\times C,K_3)=\mathrm{CH}^1(C)\cdot c_1(\alpha),$$
$$A^2(H_E\times C_E,K_3)=\mathrm{CH}^1(C_E)\cdot c_1(\alpha_E)$$

where $\alpha\in K_1(G\times C)$ is the generic element and $c_1(\alpha)\in A^1(H\times C,K_2)$ is the first Chern class of α. Thus the homomorphism res in the diagram is isomorphic to the restriction homomorphism $\mathrm{CH}^1(C)\to\mathrm{CH}^1(C_E)$. Since $C_E\simeq\mathbb{P}^1_E$ and all closed points of C are of even degree, the latter homomorphism is isomorphic to the inclusion of $2\mathbb{Z}$ into $\mathbb{Z}$, hence the image of res is divisible by 2. By [3, Ex. 7.10, Th. 9.3], the homomorphisms i^* and res' are isomorphisms. It follows that the image of r_*^G is also divisible by 2.

4. Proof of Proposition 1.4

Let A be a central simple algebra over F and let $G=\mathbf{SL}_1(A)$. We first collect some information about the Chow groups of G.

Proposition 4.1. *The groups* $\mathrm{CH}^1(G)$ *and* $\mathrm{CH}^2(G)$ *are trivial and there is a natural surjective homomorphism* $A^1(G,K_2)\to\mathrm{CH}^3(G)$*. In particular,* $\mathrm{CH}^3(G)$ *is a cyclic group. Moreover,* $2\cdot\mathrm{CH}^3(G)=0$.

Proof. Consider the Gersten-Quillen spectral sequence [11, §7, Th. 5.4]

$$E_2^{p,q}=A^p(G,K_{-q})\Rightarrow K_{-p-q}(G).$$

By [14, Th. 6.1], $K_0(G)=\mathbb{Z}$, hence for $p=1$ and 2 we have

$$\mathrm{CH}^p(G)=E_2^{p,-p}=E_\infty^{p,-p}=0$$

and the differential

$$A^1(G,K_2)=E_2^{1,-2}\to E_2^{3,-3}=\mathrm{CH}^3(G)$$

is surjective. The last assertion is proven in [9, Prop. 4.3]. □

Corollary 4.2. *Suppose that* $\mathrm{ind}(A)\le 2$*. Then* $\mathrm{CH}^p(G)=0$ *if* $p\ne 0$ *and* 3.

Proof. If A is split, then $\mathrm{CH}^p(G)=0$ for every $p>0$ [14, Th. 2.7]. We may therefore assume that $G=\mathbf{SL}_n(Q)$, where Q is a quaternion algebra. We proceed by induction on n. In the case $n=1$ we have $\dim(G)=3$ and the statement follows from Proposition 4.1.

Let $H=\mathbf{SL}_{n-1}(Q)$. We view H as a subgroup of G with respect to the embedding $a\mapsto diag(1,a)$. Consider the closed subvariety V of the affine space Q^{2n} consisting of tuples $(b_1, \dots, b_n, c_1, \dots, c_n)$ such that $\sum b_i c_i = 1$. Define the morphism

$$f : G \to V, \qquad a = (a_{ij}) \mapsto (a_{11}, \dots, a_{1n}, a'_{11}, \dots, a'_{n1}),$$

where $(a'_{ij}) = a^{-1}$. Clearly, f is an H-torsor over V. For any field extension E/F, in the exact sequence of Galois cohomology

$$G(E) \xrightarrow{f(E)} V(E) \to H^1(E, H) \xrightarrow{r} H^1(E, G)$$

the map r is an isomorphism (both sets are identified with $E^\times / \operatorname{Nrd}(A_E^\times)$ and r is the identity map [6, Cor. 2.9.4]). Hence f is surjective on E-points.

Let W be the open subset of the affine space Q^n consisting of all tuples $(b_1, \dots, b_n)$ such that $\sum b_i Q = Q$. Clearly $\mathrm{CH}^p(W) = 0$ for $p > 0$. The obvious projection $V \to W$ is an affine bundle, hence by the homotopy invariance property,

$$\mathrm{CH}^p(V) \simeq \mathrm{CH}^p(W) = 0 \tag{5}$$

for every $p > 0$.

For every m, consider the fibration spectral sequence associated with the morphism f:

$$E_1^{p,q} = \coprod_{v \in V^{(p)}} A^q(G_v, K_{m-p}) \Rightarrow A^{p+q}(G, K_m).$$

Since f is surjective on points, we have $G_v \simeq H_{F(v)}$ for every $v \in V$. By the induction hypothesis and Proposition 4.1, $E_1^{m-q,q} = 0$ if $q \neq 0, 3$ and there are surjections $\mathrm{CH}^m(V) \to E_2^{m,0}$, $\mathrm{CH}^{m-3}(V) \to E_2^{m-3,3}$. By (5), if $m \neq 0$ and 3, $E_2^{m,0}$ and $E_2^{m-3,3}$ are trivial, hence $\mathrm{CH}^m(G) = A^m(G, K_m) = 0$. □

Suppose now that A is a biquaternion division algebra. Since $\operatorname{ind} A_{\widetilde{F}} = 2$, it follows from (4) that

$$\widetilde{E}_2^{3,2} = \mathrm{CH}^3(\widetilde{G}).$$

The group $E_2^{3,2}$ is generated by the images of the groups $A^2(X_{F(g)}, K_2) = \mathrm{CH}^2(X_{F(g)})$ for all $g \in G^{(3)}$. Since over a splitting field the variety X is isomorphic to a projective space, and the group CH^2 of a projective space is canonically isomorphic to $\mathbb{Z}$, we have a canonical homomorphism $\mathrm{CH}^2(X_{F(g)}) \to \mathbb{Z}$.

It is proven in [5, Cor. 7] that

$$\operatorname{Im}\bigl(\mathrm{CH}^2(X_{F(g)}) \to \mathbb{Z}\bigr) = \begin{cases} 2\mathbb{Z}, & \text{if } \operatorname{ind} A_{F(g)} = 4; \\ \mathbb{Z}, & \text{if } \operatorname{ind} A_{F(g)} \leq 2. \end{cases} \tag{6}$$

Recall that $Y = \mathrm{SB}(2, A)$ so that $\dim(Y) = 4$. The variety Y has a rational point if and only if $\operatorname{ind}(A) \leq 2$. Therefore we have the following computation of the image of the degree homomorphism for every $g \in G$:

$$\operatorname{Im}\bigl(\mathrm{CH}^4(Y_{F(g)}) \xrightarrow{\deg} \mathbb{Z}\bigr) = \begin{cases} 2\mathbb{Z}, & \text{if } \operatorname{ind} A_{F(g)} = 4; \\ \mathbb{Z}, & \text{if } \operatorname{ind} A_{F(g)} \leq 2. \end{cases} \tag{7}$$

Consider the cycle module M_* over F defined by

$$M_n(E) = A^4(Y_E, K_{n+4}).$$

It follows from (6) and (7) that the image of $\kappa : E_2^{3,2} \to \widetilde{E}_2^{3,2}$ coincides with the image of the composite

$$A^3(G, M_3) \to A^3(G, K_3) = \mathrm{CH}^3(G) \to \mathrm{CH}^3(\widetilde{G}) = \widetilde{E}_2^{3,2},$$

where the first homomorphism is induced by the norm map $M_* \to K_*$. It is sufficient to prove that the first homomorphism in the composite is trivial.

Consider the fibration spectral sequence associated with the projection $G \times Y \to G$:

$$E_1^{p,q} = \coprod_{g \in G^{(p)}} A^q(Y_{F(g)}, K_{7-p}) \Rightarrow A^{p+q}(G \times Y, K_7).$$

Since by dimension consideration $E_1^{p,q} = 0$ if $q > 4$, the spectral sequence yields a surjective homomorphism $\mathrm{CH}^7(G \times Y) \to A^3(G, M_3)$. The composite

$$\mathrm{CH}^7(G \times Y) \to A^3(G, M_3) \to A^3(G, K_3) = \mathrm{CH}^3(G)$$

is the push-forward homomorphism p_* with respect to the projection $p : G \times Y \to G$. Thus, it is sufficient to show that $p_* = 0$.

Lemma 4.3. *The Chow group* $\mathrm{CH}^7(G \times Y)$ *is generated by images of the push-forward homomorphisms* $\mathrm{CH}^3(G_{F(y)}) \to \mathrm{CH}^7(G \times Y)$ *for all closed points* $y \in Y$.

Proof. Consider the fibration spectral sequence associated with the projection $G \times Y \to Y$:

$$E_1^{p,q} = \coprod_{y \in Y^{(p)}} A^q(G_{F(y)}, K_{7-p}) \Rightarrow A^{p+q}(G \times Y, K_7).$$

We have $\mathrm{ind}(A_{F(y)}) \le 2$ for every $y \in Y$, hence by Corollary 4.2, $\mathrm{CH}^q(G_{F(y)}) = 0$ if $q > 3$. This implies that $E_1^{7-q,q} = 0$ if $q > 3$, whence the result. □

By Lemma 4.3, it is sufficient to prove that for every closed point $y \in Y$, the push-forward homomorphism

$$\mathrm{CH}^3(G_{F(y)}) \to \mathrm{CH}^3(G)$$

is trivial. Note that $\deg(y)$ is even since $\mathrm{ind}(A) = 4$. It follows from Proposition 4.1 that the horizontal morphisms in the commutative diagram

$$\begin{array}{ccccc}
\mathbb{Z} & = & A^1(G_{F(y)}, K_2) & \longrightarrow & \mathrm{CH}^3(G_{F(y)}) \\
{\scriptstyle \deg(y)}\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z} & = & A^1(G, K_2) & \longrightarrow & \mathrm{CH}^3(G)
\end{array}$$

are surjective. The statement follows from the equality $2 \cdot \mathrm{CH}^3(G) = 0$ (Proposition 4.1).

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