# MOTIVIC DECOMPOSITION OF ISOTROPIC PROJECTIVE HOMOGENEOUS VARIETIES

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Abstract. We give a decomposition of the Chow motive of an isotropic projective homogeneous variety of a semisimple algebraic group in terms of twisted motives of simpler projective homogeneous varieties. As an application we prove a generalization of Rost nilpotence theorem.

## 1. Introduction

Let $k$ be a field. The idea of the category of Chow motives over $k$ is due to Grothendieck (see [10]). This category contains a full subcategory $\mathbf{Corr}(k)$ of correspondences (Section 7). There is a natural functor from the category of smooth complete varieties over $k$ to $\mathbf{Corr}(k)$. For a smooth complete variety $X$ we denote by $\mathcal{M}(X)$ the corresponding *motive* in $\mathbf{Corr}(k)$.

It turns out that for certain varieties $X$ the motive $\mathcal{M}(X)$ is isomorphic to a direct sum of simple motives. For example, it was noticed in [10] that $\mathcal{M}(\mathbb{P}^n_k) \simeq \mathbb{Z} \oplus \mathbb{Z}(1) \oplus \cdots \oplus \mathbb{Z}(n)$, where $\mathbb{Z}(i)$ is the Tate motive. The projective space $\mathbb{P}^n_k$ is a projective homogeneous variety of the split algebraic group $\mathrm{PGL}_{n+1}$. Köck proved in [9] that the motive of every projective homogeneous variety of a split algebraic group is isomorphic to a direct sum of Tate motives.

Another example of a projective homogeneous variety is a smooth projective quadric $X$ given by a quadratic form $f$. If $X$ has a rational point, i.e., if $f$ is isotropic, there is a quadratic form $g$ such that $f \simeq g \perp \mathbb{H}$, where $\mathbb{H}$ is a hyperbolic plane. Let $Y$ be the quadric given by $g$. Rost has shown in [12] that

$$\mathcal{M}(X) \simeq \mathbb{Z} \oplus \mathcal{M}(Y)(1) \oplus \mathbb{Z}(d),$$

where $\mathcal{M}(Y)(1) = \mathcal{M}(Y) \otimes \mathbb{Z}(1)$ and $d = \dim X$. This result has been generalized by Karpenko in [6] to the case of arbitrary isotropic flag varieties, i.e., to the class of projective homogeneous varieties of semisimple groups of classical types.

In the present paper we consider isotropic projective homogeneous varieties of arbitrary semisimple groups. The main result (Theorem 7.5) asserts that the motive of a such variety is isomorphic to a direct sum of twisted motives of simpler projective homogeneous varieties. As one should expect all the information to calculate such a decomposition is contained in the root system of the group. We illustrate this in the last section, where we work out the (essentially combinatorial) details for projective spaces, isotropic quadrics (where we get another proof of Rost's decomposition stated above), and a homogeneous variety for a group of type $E_6$.

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As an application we generalize Rost's nilpotence theorem to the case of arbitrary projective homogeneous varieties (Section 8).

## 2. Projective homogeneous varieties

Let $G$ be a semisimple algebraic group defined over a field $k$. Suppose $G$ acts on an algebraic variety $X$ over $k$. We say that $X$ is a *projective homogeneous variety* of $G$ if $G$ acts transitively and the stabilizer of every point of $X$ is a parabolic subgroup of $G$. Since the center $C$ of $G$ is contained in every parabolic subgroup, it acts trivially on $X$. Hence, replacing $G$ by $G/C$ we may assume that $G$ is an adjoint semisimple group. We need this assumption in Section 6 where we consider the $k$-structure of $X$.

A projective homogeneous variety $X$ is called *isotropic* if $X$ has a $k$-point. In other words, $X$ is $k$-isomorphic to a factor variety $G/P$, where $P$ is a parabolic subgroup of $G$ defined over $k$.

Let $P$ be a parabolic subgroup of $G$. Choose a maximal torus $T$ of $G$ and a Borel subgroup $B$ such that $T \subset B \subset P$. Let $\Sigma$ be the root system of $G$ with respect to $T$ and $\Pi \subset \Sigma$ the system of simple roots corresponding to $B$. Let $\Sigma^+$ (respectively $\Sigma^-$) be the set of positive (respectively negative) roots with respect to the basis $\Pi$. We write $U$ for the unipotent radical of $B$, $U^-$ for the unipotent radical of the opposite Borel subgroup $B^-$ and $U_\alpha$ for the root subgroups of $\alpha \in \Sigma$ (cf. [1, p. 176]).

Denote by $W$ the Weyl group of $\Sigma$. The group $W$ is generated by the set $V$ of reflections with respect to all roots in $\Pi$.

Let $S$ be the subset of $\Pi$ consisting of all $\alpha$ such that $U_{-\alpha} \subset P$. The parabolic subgroup $P$ is generated by $B$ and the root subgroups $U_{-\alpha}$ for all $\alpha \in S$. Let $G_P$ be the subgroup of $G$ generated by $U_{\pm\alpha}$, $\alpha \in S$. The group $G_P$ is the semisimple part of a Levi subgroup of $P$. The root system $\Sigma_P$ of $G_P$ with respect to the torus $T_P = T \cap G_P$ is the root subsystem in $\Sigma$ generated by $S$. The Weyl group $W_P$ of $G_P$ is the subgroup of $W$ generated by the set $V_S$ of reflections with respect to the roots in $S$.

## 3. Preliminary results

If $w \in W$, we let $l(w)$ denote the length of $w$ with respect to the set of the generators $V$ of $W$. We say that a decomposition $w = v_1 \cdots v_m$, where $v_i \in V$, is *reduced*, if $l(w) = m$. A proof of the following result can be found in [3, Chap. IV, §1, no. 5 and §1, no. 7].

**Proposition 3.1.** *Let $w = v_1 v_2 \dots v_m$ be a reduced decomposition and $v \in V$.*

(i) *If $l(vw) < l(w)$ then there exists $j$ such that $1 \le j \le m$ and*
$$vw = v_1 \dots v_{j-1} v_{j+1} \dots v_m \,.$$

(ii) *If $w = v_1' v_2' \dots v_m'$ is another reduced decomposition then*
$$\{ v_1, v_2, \dots, v_m \} = \{ v_1', v_2', \dots, v_m' \} \,.$$

The following two propositions are taken from [3, Chap. IV, §1, Ex. 1 and 3].

**Proposition 3.2.** *Let $w = v_1 v_2 \dots v_l$ be a decomposition with $v_i \in V$. Then there exists a sequence of indices $1 \le s_1 < s_2 < \dots < s_j \le l$ such that $w = v_{s_1} v_{s_2} \dots v_{s_j}$ is a reduced decomposition of $w$.*

*Proof.* If the decomposition $w = v_1 v_2 \ldots v_l$ is not reduced, there is an index $i$ such that $2 \le i \le l$ and $w' = v_i \ldots v_l$ is a reduced decomposition with $l(v_{i-1}w') < l(w')$. By Proposition 3.1(i), there exists $j$ such that $i \le j \le l$ with $v_{i-1}w' = v_i \ldots v_{j-1}v_{j+1} \ldots v_l$. The result follows by induction on $l$. □

**Corollary 3.3.** *All terms of any reduced decomposition of an element of $W_P$ are reflections in $V_S$.*

**Proposition 3.4.** *Let $w \in W$. Assume that $w$ is an element of minimal length in the double coset $W_P w W_P$. Then any element $w_1 \in W_P w W_P$ can be written in the form $w_1 = awb$ with $a, b \in W_P$ such that $l(w_1) = l(a) + l(w) + l(b)$. In particular $W_P w W_P$ contains a unique element of minimal length.*

*Proof.* Write $w_1 = cwd$ with $c, d \in W_P$. By Proposition 3.2, to get a reduced decomposition of $w_1$, one needs to take the product of reduced decompositions of $c, w, d$ and to delete some generators from that product. Let $a, b$ be the words left from $c$ and $d$ respectively after deleting some generators in the product $cwd$. Note that $a, b \in W_P$ by Corollary 3.3. Since $w$ has minimal length, we delete nothing from the reduced decomposition of $w$. Then $w_1 = awb$ and the words $a, w, b$ are reduced, being subwords of a reduced word, so the result follows. □

**Proposition 3.5.** *Let $w \in W$ be an element of minimal length in the double coset in $W_P w W_P$. Then for every $a \in W_P$ we have $l(wa) = l(aw) = l(a) + l(w)$.*

*Proof.* This easily follows from Proposition 3.2 and from the property that $w$ has minimal length in $W_P\, w\, W_P$. □

For $\alpha \in \Pi$, let $w_\alpha \in W$ denote the reflection with respect to $\alpha$.

**Proposition 3.6.** *Let $w \in W$ and $\alpha \in \Pi$.*

(i) *If $w^{-1}(\alpha) \in \Sigma^+$, then $l(w_\alpha w) = l(w) + 1$.*
(ii) *If $w^{-1}(\alpha) \in \Sigma^-$, then $l(w_\alpha w) = l(w) - 1$.*
(iii) *If $w(\alpha) \in \Sigma^+$, then $l(w\, w_\alpha) = l(w) + 1$.*
(iv) *If $w(\alpha) \in \Sigma^-$, then $l(w\, w_\alpha) = l(w) - 1$.*

*Proof.* See [13, Appendix, Lemma 19]. □

**Proposition 3.7.** *Let $w = v_1 v_2 \ldots v_m$ be a reduced decomposition and let $v_m = w_\alpha$, where $\alpha \in \Pi$. Then $w(\alpha) \in \Sigma^-$.*

*Proof.* This follows from Proposition 3.6(iii). □

Consider projective homogeneous varieties $Y = G/B$ and $X = G/P$. Let us recall some standard facts on the Bruhat decomposition in $G$, $Y$ and $X$. For proofs we refer to [1, §14, 21]. For every $w \in W$, we fix a representative $\bar{w}$ of $w$ in $N_G(T)$. The subset $U\bar{w}B \subset G$ (respectively $U\bar{w}B/B \subset Y$, $U\bar{w}P/P \subset X$) is called a *Bruhat cell in $G$* (respectively *in $Y$* and *in $X$*) and we denote it by $C(w)_G$ (respectively $C(w)_Y$ and $C(w)$).

**Proposition 3.8.** (i) *Given $w \in W$, there exists a unique element $w_P$ of smallest length in the coset $wW_P$. For such an element one has $C(w) = C(w_P)$.*

(ii) *$\{w_P | w \in W\}$ is a set of representatives for the left cosets $wW_P$.*

(iii) *Let $w_1, w_2 \in W$. Then $C(w_1) = C(w_2)$ if and only if $w_1 W_P = w_2 W_P$.*

*Proof.* For the proofs of (i) and (ii) see [1, p. 241]. The third statement follows immediately from the first two. □

Let $w = v_1 \dots v_l$ be a reduced decomposition. Consider the set

$$A_w = \{v_{i_1} \dots v_{i_m},\ 1 \le i_1 < \dots < i_m \le l\}.$$

**Proposition 3.9.** *One has*

$$\overline{C(w)} = \bigcup_{u \in A_w} C(u).$$

*Proof.* Consider the natural map $\varphi_1 : Y \to X$. It is a proper map and takes cells onto cells, so we may assume that $X = Y$, i.e., $P = B$. Let $\varphi_2 : G \to Y$ be the natural morphism. Obviously, $\varphi_2(C(w)_G) = C(w)_Y$, hence

$$C(w)_Y \subset \varphi_2(\overline{C(w)_G}) \subset \overline{C(w)_Y}.$$

By [1, Th. 21.26], one has

$$\overline{C(w)_G} = \bigcup_{u \in A_w} C(u)_G.$$

Let $A = G \setminus \overline{C(w)_G}$. Since $\varphi_2$ is open, $\varphi_2(A)$ is open in $Y$. Since $A$ and $\overline{C(w)_G}$ consist of cells, we have

$$\varphi_2(A) \cap \varphi_2(\overline{C(w)_G}) = \emptyset \text{ and } \ \varphi_2(A) \cup \varphi_2(\overline{C(w)_G}) = Y.$$

This implies $\varphi_2(\overline{C(w)_G})$ is closed and the result follows. □

If $w \in W$, we set

$$\Sigma_w = \{\gamma \in \Sigma^+ \mid w^{-1}(\gamma) \in \Sigma^+\}$$

and

$$\Sigma'_w = \{\gamma \in \Sigma^+ \mid w^{-1}(\gamma) \in \Sigma^-\}.$$

Consider the closed subgroups of $U$:

$$U_w = U \cap \bar{w}\, U\, \bar{w}^{-1}, \qquad U'_w = U \cap \bar{w}\, U^-\, \bar{w}^{-1}.$$

**Theorem 3.10.** (i) *$G$ is the disjoint union of cells $UwB$, $w \in W$. For every $w \in W$, the morphism*

$$U'_w \times B \longrightarrow C(w)_G = U\bar{w}B, \quad (u, b) \mapsto u\bar{w}b$$

*is an isomorphism of varieties.*

(ii) (*Cellular decomposition of $G/B$*) *The variety of Borel subgroups is the disjoint union of cells $UwB/B$. If $w \in W$, then the morphism*

$$U'_w \longrightarrow C(w)_Y = U\bar{w}B/B, \quad u \mapsto u\bar{w}B$$

*is an isomorphism of varieties.*

*Proof.* See [1, Th. 14.12]. □

Let $H_i$, $i \in I$, be a finite family of closed connected subgroups of a connected group $H$. We say that $H$ is directly spanned by the $H_i$ in order $i_1, \dots, i_n$ if the product morphism

$$H_{i_1} \times \dots \times H_{i_n} \longrightarrow H$$

is an isomorphism of varieties.

**Proposition 3.11.** *Let $w \in W$. Then $U$, $U_w$, $U'_w$ are directly spanned in any order by the $U_\gamma$ for $\gamma \in \Sigma^+$, $\Sigma_w$, $\Sigma'_w$ respectively and* $\dim U'_w = |\Sigma'_w| = l(w)$.

*Proof.* The unipotent groups $U$, $U_w$, $U'_w$ satisfy all the conditions of [1, Prop. 14.4] (see also the discussion in [1, p. 193]), so the result follows from that proposition. $\square$

## 4. A filtration on $X$

Recall that $X = G/P$. Let $D \in W_P\backslash W/W_P$ be a double coset. By Proposition 3.4, $D$ contains a unique element $w$ of minimal length. We set $l(D) = l(w)$ and call this number the *length* of $D$.

Let

$$X_D = \bigcup_{w \in D} C(w) \subset X.$$

By Proposition 3.8(iii), $X_D \cap X_{D'} = \emptyset$ if $D \neq D'$. For every $i \geq 0$ we set

$$X_i = \bigcup_{l(D) \leq i} X_D.$$

Since $X$ is a finite union of Bruhat cells, we have a finite filtration

$$X_0 \subset X_1 \subset \cdots \subset X_m = X$$

for some integer $m$. Clearly, $X_0 = \operatorname{Spec} k$.

**Proposition 4.1.** *Let $D \in W_P\backslash W/W_P$ be a double coset of length $i$. Then $\overline{X}_D \subset X_i$ and $\overline{X}_D \setminus X_{i-1} = X_D$.*

*Proof.* Let $C(w_1) \subset \overline{C(w_2)}$ and $C(w_2) \subset X_D$ for some $w_1, w_2 \in W$. It is sufficient to prove that $C(w_1)$ is contained in either $X_D$ or $X_{i-1}$. By Proposition 3.4, $w_2 = awb$ for some $a, b \in W_P$ and $w \in W$ such that $l(w_2) = l(a) + l(w) + l(b)$ and $l(w) = i$. By Proposition 3.9, $w_1 = a'w'b'$, where $a'$, $w'$ and $b'$ are obtained from $a$, $w$ and $b$ respectively by deleting some generators. If $w' = w$, then $w_1 \in D$ and hence $C(w_1) \subset X_D$. If $w' \neq w$, then $l(w') < l(w)$ and hence the double coset of $w_1$ is of length at most $i - 1$ and therefore, $C(w_1) \subset X_{i-1}$. $\square$

**Corollary 4.2.** *Every $X_i$ is closed in Zariski topology. The set $X_i \setminus X_{i-1}$ is the disjoint union of (relatively closed and open) subsets $X_D$ with $l(D) = i$.*

## 5. A structure of cells $X_D$

We fix a double coset $D \in W_P\backslash W/W_P$ of length $i$. Let $w \in D$ be the element of (minimal) length $i$. We are going to describe the structure of the variety $X_D$. For later use we need several lemmas.

**Lemma 5.1.** *We have $w(\Sigma_P^+) \subset \Sigma^+$ and $w^{-1}(\Sigma_P^+) \subset \Sigma^+$.*

*Proof.* Let $\alpha \in S$. It is sufficient to prove that $w(\alpha) \in \Sigma^+$ and $w^{-1}(\alpha) \in \Sigma^+$. By Proposition 3.5, $l(w_\alpha w) = l(w w_\alpha) = l(w) + 1$ and the statement follows from Proposition 3.6. $\square$

Consider the subgroup $G_P \subset G$ (see Section 2). Let $B_P$ be the Borel subgroup $B \cap G_P$ and $U_P$ its unipotent part. The root system of $G_P$ coincides with $\Sigma_P$ and $S$ is its basis. We set

$$R_D = \{\alpha \in S \mid w^{-1}(\alpha) \in \Sigma_P^+\}.$$

Denote by $\Sigma_D$ the root subsystem of $\Sigma_P$ generated by $R_D$. Let $W_D$ be the corresponding Weyl group. One has $W_D \subset W_P \subset W$.

**Lemma 5.2.** *Let $v \in W_P$. Then $v(\Sigma^+ \setminus \Sigma_P^+) = \Sigma^+ \setminus \Sigma_P^+$.*

*Proof.* It suffices to consider the case $v = w_\beta$ for some $\beta \in S$. Let $\alpha \in \Sigma^+ \setminus \Sigma_P^+$. For every $\gamma \in \Pi \setminus S$, the roots $\alpha$ and $w_\beta(\alpha)$ have the same $\gamma$-coordinates. Some of them (and hence all) are positive. Hence $w_\beta(\alpha) \in \Sigma^+ \setminus \Sigma_P^+$. □

**Lemma 5.3.** *$W_D = W_P \cap wW_Pw^{-1}$.*

*Proof.* The group $W_D$ is generated by the reflections $w_\alpha$ with respect to roots $\alpha \in R_D$. For every such reflection $w_\alpha$ we have $w^{-1}w_\alpha w = w_{w^{-1}(\alpha)}$. Since $w^{-1}(\alpha) \in \Sigma_P^+$, we obtain $w^{-1}w_\alpha w \in W_P$, hence $W_D \subset W_P \cap wW_Pw^{-1}$.

Conversely, let $v \in W_P \cap wW_Pw^{-1}$. Then $v = wv'w^{-1}$ for $v' \in W_P$. Let $v = v_1 \dots v_m$ be a reduced decomposition in the group $W_P$. Here all the $v_i$ are reflections with respect to roots from $S$. Let $v_m = w_\alpha$, $\alpha \in S$. The induction argument shows that it suffices to see that $\alpha \in R_D$.

Assume the contrary. Since $\alpha \notin R_D$, we have

$$w^{-1}(\alpha) \in \Sigma^+ \setminus \Sigma_P^+,$$

and hence, in view of Lemma 5.2,

$$w^{-1}v(\alpha) = v'w^{-1}(\alpha) \in \Sigma^+,$$

since $v' \in W_P$. On the other hand, by Proposition 3.7, we have $v(\alpha) \in \Sigma_P^-$, hence

$$w^{-1}v(\alpha) \in \Sigma^-,$$

by Lemma 5.1, a contradiction. □

**Lemma 5.4.** *One has $U\bar{w}P = U'_w\bar{w}P$.*

*Proof.* By Proposition 3.11, we have $U = U'_w \cdot U_w$. So the result follows since $U_w\bar{w}P = \bar{w}P$. □

**Lemma 5.5.** *The group $W_D$ normalizes $U'_w$.*

*Proof.* It is sufficient to prove that for every $\alpha \in R_D$, the reflection $v = w_\alpha$ normalizes $U'_w$. By Proposition 3.11, the group $U'_w$ is generated by the root subgroups $U_\gamma$ such that $\gamma \in \Sigma^+$ and $w^{-1}(\gamma) \in \Sigma^-$. We need to prove that for such $\gamma$, $v(\gamma) \in \Sigma^+$ and $w^{-1}\big(v(\gamma)\big) \in \Sigma^-$.

By Lemma 5.1, $\gamma \in \Sigma^+ \setminus \Sigma_P^+$ and in view of Lemma 5.2, $v(\gamma) \in \Sigma^+$. Since $\gamma \in \Sigma^+$, by Lemma 5.1, $w^{-1}(\gamma) \notin \Sigma_P^-$. It follows from the definition of $R_D$ that $w^{-1}(\alpha) \in \Sigma_P^+$. Hence, for every $\delta \in \Pi \setminus S$, the roots $w^{-1}\big(v(\gamma)\big)$ and $w^{-1}(\gamma)$ have the same $\delta$-coordinates, some of them are negative since $w^{-1}(\gamma) \in \Sigma^-$. Therefore, $w^{-1}\big(v(\gamma)\big) \in \Sigma^-$. □

**Corollary 5.6.** *Let $v \in W_D$. Then $\bar{v}U'_w\bar{w}P = U'_w\bar{w}P$.*

*Proof.* We may assume that $v = w_\alpha$, where $\alpha \in R_D$. By Lemma 5.5,

$$\bar{v}U'_w\bar{w}P = (\bar{v}U'_w\bar{v}^{-1})\bar{v}\bar{w}P = U'_w\bar{v}\bar{w}P = U'_w\bar{w}(\bar{w}^{-1}\bar{v}\bar{w})P = U'_w\bar{w}P,$$

since $w^{-1}(\alpha) \in \Sigma_P^+$ by the definition of $R_D$ and

$$\bar{w}^{-1}\bar{v}\bar{w} = \bar{w}^{-1}\bar{w}_\alpha\bar{w} = \bar{w}_{w^{-1}(\alpha)} \in N_{G_P}(T_P) \subset P.$$

□

**Proposition 5.7.** *Let $v_1, v_2 \in W$ be such that $l(v_1v_2) = l(v_1) + l(v_2)$. Then the map*

$$U'_{v_1} \times U'_{v_2} \longrightarrow C(v_1v_2)_Y, \ (x, y) \mapsto x\bar{v}_1y\bar{v}_2\, B$$

*is an isomorphism of varieties.*

*Proof.* We obviously have $\Sigma'_{v_1v_2} \subset \Sigma'_{v_1} \cup v_1(\Sigma'_{v_2})$. By assumption,

$$|\Sigma'_{v_1v_2}| = |\Sigma'_{v_1}| + |\Sigma'_{v_2}|,$$

hence $\Sigma'_{v_1v_2}$ is the disjoint union of $\Sigma'_{v_1}$ and $v_1(\Sigma'_{v_2})$. By Proposition 3.11, the morphism

$$\varphi : U'_{v_1} \times U'_{v_2} \longrightarrow U'_{v_1v_2}, \quad (x, y) \mapsto x\bar{v}_1y\bar{v}_1^{-1},$$

is an isomorphism of varieties. It remains to notice that the morphism in question is the composite of $\varphi$ and the isomorphism

$$U'_{v_1v_2} \longrightarrow C(v_1v_2)_Y, \quad c \mapsto c\bar{v}_1\bar{v}_2B$$

from Theorem 3.10(ii). □

We now want to generalize this result to the case of the variety $X$.

**Proposition 5.8.** *Let $v$ be the element of minimal length in the coset $v\,W_P$ (such an element is unique). Then the map*

$$U'_v \longrightarrow C(v), \quad x \mapsto x\bar{v}P$$

*is an isomorphism of varieties.*

*Proof.* Our map factors as $U'_v \to C(v)_Y \to C(v)$, where the first one is the isomorphism from Theorem 3.10(ii) and the second one is a canonical isomorphism by [1, Proposition 21.29(ii)]. □

**Lemma 5.9.** *Let $a' \in W_P$ and let $a$ be the element of minimal length in the coset $a'\,W_D$. Then $C(a'w) = C(aw)$.*

*Proof.* Let $a = a'b$ where $b \in W_D$. Then we have

$$\bar{a}\bar{w}\,P = \bar{a}'\bar{b}\bar{w}\,P = \bar{a}'\bar{w}(\bar{w}^{-1}\bar{b}\bar{w})\,P = \bar{a}'\bar{w}\,P,$$

since $\bar{w}^{-1}\bar{b}\bar{w} \in N_{G_P}(T_P) \subset P$ by Lemma 5.3. □

**Proposition 5.10.** *Let $a \in W_P$ be the element of minimal length in the left coset $aW_D$. Then the product morphism*

$$U'_a \times U'_w \longrightarrow C(aw), \ (x, y) \mapsto x\bar{a}y\bar{w}\,P$$

*is an isomorphism of varieties.*

*Proof.* As above, our map factors as

$$U'_a \times U'_w \to C(aw)_Y \to C(aw).$$

The first map is an isomorphism, by Proposition 5.7 (note that $l(aw) = l(a) + l(w)$, by Proposition 3.5). To see that the second one is an isomorphism, by [1, Proposition 21.29(ii)], one needs to check that $aw$ is the element of minimal length in the left coset $awW_P$.

If this is not the case then there is $b \in W_P$ such that $w' = awb$ has length lower than $l(aw) = l(a) + l(w)$. To get a reduced decomposition of $w'$ we have to delete some generators from reduced decompositions of $a, w$ and $b$. Let $a', w', b'$ be the words left from $a, w, b$ after deleting. If $w' \neq w$, then the double coset

$$W_P\, w\, W_P = W_P\, awb\, W_P = W_P\, a'w'b'\, W_P$$

contains the element $w'$ which has length lower than $l(w)$ which is impossible.

Thus $w' = w$ and $awb = a'wb'$. Let $a'' = (a')^{-1}a$. Then

$$w^{-1}a''w = b'b^{-1} \in W_P.$$

By Lemma 5.3, $a'' \in W_D$, hence $a' \in a\, W_D$. Since $a$ is the element of minimal length in the coset $a\, W_D$, we have $a = a'$. Then $b = b'$ and hence

$$l(awb) = l(a') + l(w') + l(b') = l(a) + l(w) + l(b),$$

implying that $aw$ is the element of minimal length. □

**Remark 5.11.** Note that for $a \in W_P$ one has $U \cap \bar{a}\, U^- \,\bar{a}^{-1} = U_P \cap \bar{a}\, U_P^- \,\bar{a}^{-1}$, where $U_P = G_P \cap U$ and $U_P^- = G_P \cap U^-$, so that there is no confusion in writing $U'_a$. Indeed, if $U_\gamma \subset U \cap \bar{a}\, U^- \,\bar{a}^{-1}$, i.e., $\gamma \in \Sigma^+$ and $a^{-1}(\gamma) \in \Sigma^-$, then by Lemma 5.2, $\gamma \in \Sigma_P^+$ and hence $U_\gamma \subset U_P \cap \bar{a}\, U_P^- \,\bar{a}^{-1}$.

Let $Q_D$ be the parabolic subgroup of $G_P$ corresponding to $R_D$, i.e., the subgroup generated by the Borel subgroup $B_P \subset G_P$ and the root subgroups $U_{-\alpha}$ for all $\alpha \in R_D$. Set $Z_D = G_P/Q_D$. By definition, $Z_D$ is a projective homogeneous variety of the group $G_P$. Our next aim is to construct a surjective morphism $\lambda_D : X_D \to Z_D$ with fibers isomorphic to affine spaces of dimension $l(D)$. First we define $\lambda_D$ on the level of sets and then verify that it is a morphism.

By Lemma 5.9 and Proposition 5.10, any element $s \in X_D$ can be written uniquely in the form $s = x\bar{a}y\bar{w}\, P$ where $a \in W_P$, $x \in U'_a$, $y \in U'_w$ and $a$ has minimal length in the coset $aW_D$. Then we define $\lambda_D(s) = x\bar{a}\, Q_D$.

**Proposition 5.12.** *Let $a \in W_P$. Then*
(i) $\lambda_D^{-1}\big(C(a)_{Z_D}\big) = C(aw)$.
(ii) *There exists an isomorphism $\mu : C(a)_{Z_D} \times \mathbb{A}_k^{l(D)} \xrightarrow{\sim} C(aw)$ such that the composite $\lambda_D \circ \mu$ is the projection.*

*Proof.* (i) Let $s \in C(aw)$. By Lemma 5.9 and Proposition 5.10, we can write $s$ in the form $s = x\bar{a}'y\bar{w}\, P$ where $a' \in W_P$, $x \in U'_{a'}$, $y \in U'_w$ and $a'$ has minimal length in the coset $aW_D$. Then $\lambda_D(s) = x\bar{a}'Q_D \in C(a')_{Z_D} = C(a)_{Z_D}$.

Conversely, let $s = x\bar{a}'y\bar{w}\, P \in C(a'w)$ where $a' \in W_P$, $x \in U'_{a'}$, $y \in U'_w$ and $a'$ has minimal length in the coset $a'W_D$. Suppose that $\lambda_D(s) = x\bar{a}'Q_D \in C(a)_{Z_D}$. Hence, $C(a)_{Z_D} = C(a')_{Z_D}$ and therefore, $aW_D = a'W_D$ by Proposition 3.8(iii). It follows from Lemma 5.3, that $a^{-1}a' \in W_D \subset wW_Pw^{-1}$ and $awW_P = a'wW_P$. Again by Proposition 3.8(iii), $s \in C(a'w) = C(aw)$.

(ii) Consider the commutative diagram

$$\begin{array}{ccc} U'_a \times U'_w & \longrightarrow & C(aw) \\ {\scriptstyle pr}\big\downarrow & & \big\downarrow{\scriptstyle \lambda_D} \\ U'_a & \longrightarrow & C(a)_{Z_D}, \end{array}$$

where the horizontal isomorphisms are defined in Propositions 5.8 and 5.10. Note that $U'_w \simeq \mathbb{A}^{l(D)}$. The result follows. □

**Lemma 5.13.** *For every $g \in G_P$ and $s \in X_D$, we have $gX_D = X_D$ and $\lambda_D(gs) = g\lambda_D(s)$.*

*Proof.* Write $s$ in the form $s = x\bar{a}y\bar{w}\,P$ where $a \in W_P$, $x \in U'_a$, $y \in U'_w$ and $a$ has minimal length in the coset $aW_D$. We have $\lambda_D(s) = x\bar{a}\,Q_D$. By the Bruhat decomposition, the element $gx\bar{a}$ of $G_P$ can be written in the form $u\bar{b}q$, where $b \in W_P, u \in U'_b, q \in Q_D$ and $b$ has minimal length in the coset $bW_D$. In view of the Bruhat decomposition, $Q_D$ is generated by the maximal torus $T_P$, $U_P$ and $W_D$. The application of Lemma 5.4 and Corollary 5.6 shows that $gs$ can be written in the form $gs = u\bar{b}z\bar{w}\,P$ for some $z \in U'_w$. Hence, $gs \in X_D$ and

$$\lambda_D(gs) = u\bar{b}\,Q_D = gx\bar{a}\,Q_D = g\lambda_D(s).$$

□

**Theorem 5.14.** *$\lambda_D$ is a flat morphism.*

*Proof.* Let $a \in W_P$ be the longest element. The set $C(a)_{Z_D}$ is the big (open) cell in $Z_D$. By Proposition 5.12, the restriction of $\lambda_D$ on the open set $\lambda_D^{-1}\big(C(a)_{Z_D}\big) = C(aw)$ is a flat morphism. The variety $Z_D$ is covered by the $G_P$-translations of $C(a)_{Z_D}$, and the statement follows from the fact that by Lemma 5.13, $\lambda_D$ commutes with the $G_P$-translations. □

## 6. The $k$-structure of $X_D$

In this section we assume that $P$ is defined over $k$, in particular $X(k) \neq \emptyset$. We also assume that the maximal torus $T \subset G$ contains a maximal $k$-split torus $L$ of $G$. Recall first some basic facts on the structure of isotropic groups (see [14]).

Let $k_{sep}$ be a separable closure of $k$ and let $\Gamma = Gal(k_{sep}/k)$. There is a quasisplit group $G^{qs}$ defined over $k$ that is a twisted inner form of $G$. In particular, the groups $G$ and $G^{qs}$ are isomorphic over $k_{sep}$. The isomorphism between $G$ and $G^{qs}$ can be chosen in such a way that it takes $T$ and $B$ into a maximal $k$-torus $T^{qs}$ containing a maximal $k$-split torus of $G^{qs}$ and a Borel subgroup $B^{qs}$ containing $T^{qs}$. Thus, we can identify $\Sigma^{qs}$, $\Pi^{qs}$ and $W^{qs}$ with $\Sigma$, $\Pi$ and $W$ respectively. The corresponding $\Gamma$-action on $G(k_{sep})$, $\Sigma$, $\Pi$ and $W$ is called the *$*$-action.* For $\sigma \in \Gamma$ let $\sigma^*$ denote the $*$-action. Since $G$ is an adjoint group, there is a cocycle $g_\sigma \in G(k_{sep})$ such that the $\Gamma$-action on $G(k_{sep})$ is given by the formula

$$\sigma(g) = g_\sigma(\sigma^*(g))g_\sigma^{-1}$$

for all $g \in G(k_{sep})$ and $\sigma \in \Gamma$. Since $T$ is $k$-defined in $G$, the cocycle normalizes $T$, i.e., $g_\sigma \in N_G(T)$.

Given a group $G$, one can associate a geometric picture called the *Tits index*. It consists of the corresponding Dynkin diagram of $G^{qs}$ with some vertices circled. The set $S_0$ of uncircled vertices corresponds to the $k$-subgroup $G_0$ of $G$ called a *semisimple anisotropic kernel*. This subgroup is the semisimple part of the centralizer $C_G(L)$. Let $T_0 = T \cap G_0$. Since $L$ is $k$-split, one has

$$g_\sigma \in C_G(L) \cap N_G(T) = N_{G_0}(T_0) \cdot T\,.$$

Lastly, we note that a variety $X$ of parabolic subgroups of type $S$ is defined over $k$ if and only if $S$ is stable with respect to $*$-action. Moreover $X$ contains a $k$-point, i.e. $G$ contains a $k$-defined parabolic subgroup of type $S$, if and only if $S$ is $*$-stable and contains $S_0$.

Thus our $S$ is $*$-stable, hence so is $W_P$. Hence, there is a $*$-action of $\Gamma$ on the set $W_P\backslash W/W_P$ of double cosets.

**Proposition 6.1.** *Let $D \in W_P\backslash W/W_P$ be a double coset such that $\sigma^*(D) = D$. Then the varieties $X_D$ and $Z_D$ are defined over $k$. The morphism $\lambda_D : X_D \to Z_D$ is defined over $k$.*

*Proof.* Since $S_0 \subset S$, we have $g_\sigma \in N_{G_P}(T_P)\cdot T \subset G_P \cdot T$. Let us describe the action of $\Gamma$ on $X$. A point $s = gP \in X$ corresponds to the parabolic subgroup $gPg^{-1}$ in $G$. Then for $\sigma \in \Gamma$ we have

$$\sigma(gPg^{-1}) = g_\sigma(\sigma^*(gPg^{-1}))g_\sigma^{-1} = g_\sigma\sigma^*(g)P\sigma^*(g)^{-1}g_\sigma^{-1}.$$

Thus we obtain

$$\sigma(s) = \sigma(gP) = g_\sigma\sigma^*(g)P.$$

Using Proposition 5.10, we can write $s$ in the form $s = x\bar{a}y\bar{w}P$, where $w \in D$ is the element of minimal length, $a \in W_P$, $x \in U'_a$, $y \in U'_w$. We have $\sigma^*(w) = w$, then $\sigma^*(U'_w) = U'_w$ and

$$\sigma(s) = g_\sigma\sigma^*(x\bar{a})\sigma^*(y)\bar{w}\,P.$$

The group $G_P$ is $k$-defined, hence $g_\sigma\sigma^*(x\bar{a}) \in G_P \cdot T$. Since $\sigma^*(y) \in U'_w$, we have $\sigma^*(y)\bar{w}\,P \in X_D$ and therefore, $\sigma(s) \in X_D$ by Lemma 5.13, so that $X_D$ is defined over $k$.

By construction, $Z_D$ is the variety of parabolic subgroups in the $k$-defined group $G_P$ of type $R_D$. To prove that $Z_D$ is defined over $k$ one needs to show that $\sigma^*(R_D) = R_D$ for every $\sigma \in \Gamma$. Let $\alpha \in R_D$. Then $w^{-1}(\alpha) \in \Sigma_P^+$. Applying the $*$-action we have

$$w^{-1}(\sigma^*(\alpha)) = \sigma^*(w^{-1})(\sigma^*(\alpha)) = \sigma^*\big(w^{-1}(\alpha)\big) \in \sigma^*(\Sigma_P^+) = \Sigma_P^+,$$

hence $\sigma^*(\alpha) \in R_D$.

It remains to show that $\lambda_D$ is $k$-defined. Let $g_\sigma = g'_\sigma g''_\sigma$ where $g'_\sigma \in G_P$ and $g''_\sigma$ lies in the center of the group $C_G(L)$. By Lemma 5.13,

$$\lambda_D(\sigma s) = \lambda_D(g_\sigma\sigma^*(x\bar{a})\sigma^*(y)\bar{w}\,P) = \\ g'_\sigma\sigma^*(x\bar{a})\lambda_D(g''_\sigma\sigma^*(y)\bar{w}\,P) = g'_\sigma\sigma^*(x\bar{a})\,Q_D = \sigma(x\bar{a}\,Q_D) = \sigma\lambda_D(s),$$

hence $\lambda_D$ is defined over $k$. □

Let $D \in W_P\backslash W/W_P$ be an arbitrary double coset. Let $k(D)$ be the finite separable field extension of $k$ corresponding to the stabilizer of $D$ in $\Gamma$ (with respect to the $*$-action) by Galois theory. By Proposition 6.1, the varieties $X_D$ and $Z_D$ are defined over $k(D)$. Let $X_D \to \operatorname{Spec} k(D)$ and $Z_D \to \operatorname{Spec} k(D)$ be the corresponding structure morphisms. We consider $X_D$ and $Z_D$ as schemes over $k$ with respect to the composites $X_D \to \operatorname{Spec} k(D) \to \operatorname{Spec} k$ and $Z_D \to \operatorname{Spec} k(D) \to \operatorname{Spec} k$. By Proposition 6.1, we can view $\lambda_D : X_D \to Z_D$ as a morphism of schemes defined over $k$.

**Remark 6.2.** The variety $Z_D$ may be defined over some subfield of $k(D)$.

**Proposition 6.3.** *Let $z \in Z_D$ be a point (not necessary closed). Then $\lambda_D^{-1}(z) \simeq \mathbb{A}^m_{k(z)}$, where $m = l(D)$.*

*Proof.* Replacing $k$ by $k(z)$ we may assume that $Z_D$ contains a $k$-point. But the group $Q_D$ may not be defined over $k$. The reason is that the torus $T$ may not contain a maximal split torus anymore. Let $G_0$ be the semisimple part of the centralizer $C_G(L)$. Choose a maximal split torus $L'$ containing $L$. Since $L' \subset C_G(L)$, there is a maximal $k$-defined torus $T'$ of $C_G(L)$ containing $L'$. The maximal tori $T$ and $T'$ of $C_G(L)$ are conjugate by an element of $C_G(L)$ and hence by an element of $G_0$ since $C_G(L)$ is a product of $G_0$ and the center of $C_G(L)$. Since $G_0 \subset G_P$, we have $T' = gTg^{-1}$ for some $g \in G_P$.

Now we replace $T$ by $T'$ so that we may assume that $T$ contains a maximal split torus. We also replace $U$, $B$, $P$, $\Sigma$, $\Pi$,... by the corresponding objects $U'$, $B'$, $P'$, $\Sigma'$, $\Pi'$,... using conjugation by $g$. Since $g \in P$, we have $P' = P$, $X' = X$. The conjugation also provides identification of $W'_P$ with $W_P$ and the double cosets of $W'_P\backslash W'/W'_P$ and $W_P\backslash W/W_P$. By Lemma 5.13, $gX_D = X_D$, hence $X'_D = X_D$.

Since $Z_D$ contains a rational $k$-point, the set of uncircled vertices in the Tits index of $G_P$ (and hence of $G$) over $k$ lies in $R_D$. Therefore, the group $Q'_D = gQ_Dg^{-1}$ is defined over $k$. Under the canonical identification of $Z'_D = G_P/Q'_D$ with $Z_D$ the morphisms $\lambda'_D : X'_D \to Z'_D$ and $\lambda_D : X_D \to Z_D$ coincide. Thus, replacing $Q_D$ by $Q'_D$ we may assume that the group $Q_D$ is defined over $k$ and therefore, $S_0 \subset R_D$.

Let $z \in Z_D$ be a rational point. Since the natural map $G_P(k) \to Z_D(k)$ is surjective [2, Th. 4.13(a)], there is $h \in G_P(k)$ such that $z = hQ_D$. By Propositions 5.8 and 5.12,

$$\lambda_D^{-1}(z) = hU'_w\bar{w}P.$$

Since $Z_D$ is defined over $k(D)$ and has a rational point, we have $k(D) = k$ and hence $\sigma^*(D) = D$ for all $\sigma \in \Gamma$. Since $S_0 \subset R_D$, the elements $g_\sigma$ belong to the inverse image of $W_D$ under the natural map $N_G(T) \to W$. The group $W_D$ normalizes $U'_w$ by Lemma 5.5, therefore, the group $U'_w$ is defined over $k$. The isomorphism

$$U'_w \to hU'_w\bar{w}P = \lambda_D^{-1}(z), \quad u \mapsto hu\bar{w}P$$

is defined over $k$. Thus, it is sufficient to prove that $U'_w$ is isomorphic to $\mathbb{A}_k^{l(D)}$ over $k$. Since $U'_w$ is normalized by $T$, $U'_w$ is a $k$-split subgroup of $G$ by [2, Cor. 3.18]. In view of [1, Remark 15.13], any $k$-split subgroup group as a variety is $k$-isomorphic to an affine space. □

## 7. Motivic decomposition

We start by recalling the definition of the category of Chow-correspondences (with twist) $\mathbf{Corr}(k)$ over a field $k$. The objects of $\mathbf{Corr}(k)$ are formal finite direct sums of pairs $(X, n)$, where $X$ is a smooth complete variety over $k$ and $n$ a non negative integer. Let $(X,n)$ and $(Y,m)$ be two objects of $\mathbf{Corr}(k)$ and let $X_1, X_2, \dots, X_k$ be irreducible (connected) components of $X$. Then

$$\operatorname{Hom}_{\mathbf{Corr}(k)}\big((X,n),(Y,m)\big) = \coprod_{i=1}^{k} \mathrm{CH}_{\dim X_i + n - m}(X_i \times Y),$$

where $\mathrm{CH}_i(Z)$ stands for the Chow group of cycles on $Z$ of dimension $i$.

The additive category $\mathbf{Corr}(k)$ has a tensor structure given by $(X,n)\otimes(Y,m) = (X\times Y, n+m)$. We write $\mathcal{M}(X)$ for the object $(X,0)$ and $\mathbb{Z}(n)$ for $(\operatorname{Spec} k, n)$. The object $\mathbb{Z}(n)$ is called the *Tate motive.* We also set

$$\mathcal{M}(X)(n) = \mathcal{M}(X)\otimes\mathbb{Z}(n) = (X,n).$$

Note that the group

$$\operatorname{Hom}_{\mathbf{Corr}(k)}\big(\mathbb{Z}(n),\mathbb{Z}(m)\big) = \begin{cases} \mathbb{Z}, & \text{if } n=m; \\ 0, & \text{otherwise} \end{cases}$$

does not depend on the field $k$.

**Remark 7.1.** The collection of all objects $(X,0)=\mathcal{M}(X)$ constitute a full subcategory of $\mathbf{Corr}(k)$, the category of 0-correspondences $\mathbf{Corr}^0(k)$. The pseudo-abelian completion of $\mathbf{Corr}^0(k)$ is the category of effective motives $\mathcal{M}_k$, see [8] and [10]. Our category of correspondences $\mathbf{Corr}(k)$ is the full additive subcategory of $\mathcal{M}_k$ generated by the pure motives $\mathcal{M}(X)$ and their (positive) Tate twists $\mathcal{M}(X)(n)$ $(n\geq 1)$, cf. the formulas in [12, Sect. 1.3].

Note that Köck [9] and Rost [12] work with $\mathcal{M}_k$. Karpenko [7] works with still another category of correspondences. He considers formal direct sums of pairs $(X,n)$, as we do, but allows $n$ to be negative, i.e. the Tate motive is formally inverted.

All these categories are full subcategories of the category of motives as e.g. introduced in [8, p. 57]. This category is constructed by adding formally negative twists of the Tate motive to $\mathcal{M}(k)$.

Let $X$ and $Y$ be smooth complete varieties over $k$. A correspondence $f\in \operatorname{End}\mathcal{M}(X)$ acts on the Chow group $\mathrm{CH}_*(Y\times X)$ by means of the homomorphism

$$f_*:\mathrm{CH}_*(Y\times X)\to \mathrm{CH}_*(Y\times X)$$

defined by $f_*(u)=f\circ u$ for $u\in \mathrm{CH}_*(Y\times X)=\operatorname{Hom}_{\mathbf{Corr}(k)}\big(\mathcal{M}(Y)(*),\mathcal{M}(X)\big)$.

Let $X$ be a smooth complete variety over $k$. Suppose there is a filtration by closed subvarieties

$$\emptyset = X_{-1}\subset X_0\subset X_1\subset\cdots\subset X_n = X$$

together with flat morphisms $f_i: X_i\setminus X_{i-1}\to Y_i$ of constant relative dimension $a_i$ for every $i=1,2,\dots,n$, where $Y_i$ are smooth complete varieties over $k$. Suppose in addition that the fiber of every $f_i$ over any point $y\in Y_i$ is isomorphic to the affine space $\mathbb{A}^{a_i}_{k(y)}$.

**Theorem 7.2.** [6, Th. 6.5, Cor. 6.11] *There is an isomorphism in* $\mathbf{Corr}(k)$*:*

$$\mathcal{M}(X)\simeq\coprod_{i=1}^{n}\mathcal{M}(Y_i)(a_i).$$

**Remark 7.3.** Karpenko proved Theorem 7.2 under assumption that every $f_i$ is a vector bundle morphism. The only property of vector bundle he needs is that for a vector bundle $f : X \to Y$ the pull-back homomorphism $f^* : H^*(Y, K_*) \to H^*(X, K_*)$ of $K$-cohomology groups exists and it is an isomorphism. The pull-back homomorphism $f^*$ exists if $f$ is flat of constant relative dimension. The localization and homotopy invariance properties of $K$-cohomology (see [11]) imply that $f^*$ is an isomorphism if every fiber of $f$ is an affine space.

Let $X$ be a projective homogeneous variety over $k$. We use the notation of the preceding sections. We assume that $X = G/P$, where $P$ is a parabolic subgroup of $G$ defined over $k$. Consider the $*$-action on the set of double cosets $W_P\backslash W/W_P$. Let $\Delta$ be the set of all $*$-orbits. For an orbit $\delta \in \Delta$, the length $l(D)$ does not depend on the choice of a representative $D \in \delta$. We denote this number by $l(\delta)$. The varieties $X_D$ and $Z_D$ considered as schemes over $k$ also do not depend on $D \in \delta$. We denote them by $X_\delta$ and $Z_\delta$ respectively.

**Remark 7.4.** There is a canonical $\Gamma$-equivariant bijection between $W_P\backslash W/W_P$ (with the $*$-action), $P\backslash G/P$ and the set of $P$-orbits on $X$. Thus, $\Delta$ can be identified with the set of $\Gamma$-orbits on the set of $P$-orbits on $X$.

**Theorem 7.5.** *Let $P$ be a parabolic $k$-subgroup of a semisimple group $G$ and let $X = G/P$. Then in the above notation there is an isomorphism in* $\mathbf{Corr}(k)$*:*

$$\mathcal{M}(X) \simeq \coprod_{\delta\in\Delta} \mathcal{M}(Z_\delta)\big(l(\delta)\big).$$

*Proof.* Take the filtration

$$X_0 \subset X_1 \subset \cdots \subset X_m = X$$

constructed in Section 4. By Corollary 4.2, we can refine it

$$X'_0 \subset X'_1 \subset \cdots \subset X'_t = X$$

in such a way that every $X'_i \setminus X'_{i-1}$ coincides with $X_\delta$ for an appropriate $\delta \in \Delta$. By Proposition 6.1, there is a flat morphism $\lambda_\delta : X_\delta \to Z_\delta$ of constant relative dimension $l(\delta)$. In view of Proposition 6.3, every fiber of $\lambda_\delta$ is an affine space. So the result follows immediately from Theorem 7.2. □

Some of the projective homogeneous varieties $Z_\delta$ in Theorem 7.5 may be isotropic. Applying Theorem 7.5 to isotropic $Z_\delta$ again we get

**Corollary 7.6.** *Let $X$ be a projective homogeneous variety over $k$. Then there are anisotropic projective homogeneous varieties $Y_1, Y_2, \dots, Y_r$, each $Y_i$ is defined over a finite separable field extension $k_i/k$, and an isomorphism*

$$\mathcal{M}(X) \simeq \coprod_{i=1}^{r} \mathcal{M}(Y_i)(a_i)$$

*in the category* $\mathbf{Corr}(k)$ *for some non-negative integers $a_i$.*

## 8. Nilpotence Theorem

We will use the following modification due to P. Brosnan [5, Th. 3.1] of a statement proved by M. Rost [12, Prop. 1].

**Proposition 8.1.** *Let $Y$ and $X$ be smooth varieties over $k$ and let $f \in \operatorname{End} \mathcal{M}(X)$ be a correspondence. Suppose that $f_{k(y)}$ acts trivially on $\mathrm{CH}_*(X_{k(y)})$ for every point $y \in Y$. Then the correspondence $f^{\dim Y+1}$ acts trivially on $\mathrm{CH}_*(Y \times X)$.*

**Theorem 8.2.** *Let $X$ be a projective homogeneous variety over $k$. Then for every field extension $l/k$, the kernel of the natural ring homomorphism $\operatorname{End} \mathcal{M}(X) \to \operatorname{End} \mathcal{M}(X_l)$ consists of nilpotent correspondences.*

*Proof.* By Corollary 7.6, there are anisotropic projective homogeneous varieties $Y_i$, $i = 1, 2, \dots m$, such that

$$\mathcal{M}(X) \simeq \coprod_i^m \mathcal{M}(Y_i)(a_i)$$

for some integers $a_i$. We claim that there is an integer $d$, depending only on the dimension of $X$ and $m$, such that for every correspondence in the kernel of $\operatorname{End} \mathcal{M}(X) \to \operatorname{End} \mathcal{M}(X_l)$ one has $f^d = 0$.

We proceed by descending induction on $m$. If $m$ has maximum value then every $Y_i$ is equal to $\operatorname{Spec} k$, and hence $\mathcal{M}(X)$ is a direct sum of Tate motives $\mathbb{Z}(n)$. Therefore the ring $\operatorname{End} \mathcal{M}(X)$ injects into $\operatorname{End} \mathcal{M}(X_l)$ and the result follows.

In the general case, for every point $y \in Y_i$, the variety $Y_i$ is isotropic over the field $e = k(y)$. Thus, $X_e = X \otimes_k e$ is a direct sum of at least $m+1$ summands. By the induction hypothesis, $f_e^c = (f \otimes_k e)^c = 0$ for some integer $c$ not depending on $y$ and $i$. Replacing $f$ by $f^c$, we may assume that $f_e = 0$. By Proposition 8.1, the correspondence $f^{b_i}$, for $b_i = \dim Y_i + 1$, acts trivially on $\mathrm{CH}_*(Y_i \times X)$. In particular, the composite of the natural morphism $g_i : \mathcal{M}(Y_i)(a_i) \to \mathcal{M}(X)$ with $f^{b_i}$ is trivial. But the direct sum of all the $g_i$ is the identity of $X$, hence $f^b = 0$ for $b = \max(b_i)$. □

**Corollary 8.3.** *Let $X$ be a projective homogeneous variety over $k$. Suppose that for a field extension $l/k$, the image of the ring homomorphism $\operatorname{End} \mathcal{M}(X) \to \operatorname{End} \mathcal{M}(X_l)$, $f \mapsto f_l = f \otimes_k l$, contains a projector (idempotent) $q$. Then there is a projector $p \in \operatorname{End} \mathcal{M}(X)$ such that $p_l = q$.*

*Proof.* Choose a correspondence $p' \in \operatorname{End} \mathcal{M}(X)$ such that $(p')_l = q$. Let $A$ (respectively $B$) be the (commutative) subring of $\operatorname{End} \mathcal{M}(X)$ (respectively $\operatorname{End} \mathcal{M}(X_l)$) generated by $p'$ (respectively $q$). By Theorem 8.2, the kernel of the natural ring homomorphism $A \to B$ consists of nilpotent elements. By [4, Ch.II, §4, Cor.1, Prop.15], there is an idempotent $p \in A$ with $p_l = q$. □

**Corollary 8.4.** (cf., [12, Cor. 10]) *Let $X$ and $Y$ be projective homogeneous varieties over $k$ and let $f : X \to Y$ be a morphism in $\mathbf{Corr}(k)$. Suppose that for a field extension $l/k$, the correspondence $f_l$ is an isomorphism in $\mathbf{Corr}(l)$. Then $f$ is an isomorphism in $\mathbf{Corr}(k)$.*

*Proof.* Suppose first that $Y = X$. We may assume that $X$ is split over $l$, hence

$$\mathcal{M}(X_l) \simeq \coprod_i \mathbb{Z}(i)^{\oplus n_i}$$

for some $n_i$. Therefore,

$$\operatorname{Aut}\mathcal{M}(X_l) \simeq \prod_i \mathrm{GL}_{n_i}(\mathbb{Z}).$$

Let $P(t) \in \mathbb{Z}[t]$ be the characteristic polynomial of the matrix of $f_l$, so that $P(f_l) = 0$. Since $P(0) = \pm 1$ we can write $P(t) \pm 1 = tQ(t)$ for some $Q(t) \in \mathbb{Z}[t]$. We have

$$f_l Q(f_l) = Q(f_l) f_l = P(f_l) \pm 1 = \pm 1.$$

By Theorem 8.2, $fQ(f) = \pm 1 + g$ and $Q(f)f = \pm 1 + h$ for some nilpotent correspondences $g, h \in \operatorname{End}\mathcal{M}(X)$. Thus, $fQ(f)$ and $Q(f)f$ are automorphisms of $\mathcal{M}(X)$, hence so is $f$.

In the general case, note that $f$ induces an isomorphism of the Chow groups $\mathrm{CH}_i(X_l)$ and $\mathrm{CH}_i(Y_l)$. Since $\dim X$ is the largest integer $i$ such that $\mathrm{CH}_i(X_l) \neq 0$, we have $\dim X = \dim Y$. Therefore, the transpose $f^t$ of $f$ can be viewed as a morphism $Y \to X$ in $\mathbf{Corr}(k)$. Clearly, $(f^t)_l = (f_l)^t$ is an isomorphism, hence $f \circ f^t$ and $f^t \circ f$ are automorphisms of $\mathcal{M}(Y)$ and $\mathcal{M}(X)$ respectively by the first part of the proof. Thus, $f$ is an isomorphism. □

## 9. Examples of motivic decompositions

In this section we consider three examples. In all of them we number simple roots in $\Pi$ as in [3].

**Example 9.1.** *Projective spaces.* Let $G = \mathrm{PGL}_{n+1}$, it is a simple group of type $A_n$. Let $S = \Pi \setminus \{\alpha_1\}$ and let $P \subset G$ be the corresponding parabolic subgroup. The variety $X = G/P$ is isomorphic to a projective space $\mathbb{P}^n_k$. The Weyl group $W$ is the symmetric group $S_{n+1}$ and $W_P = S_n$.

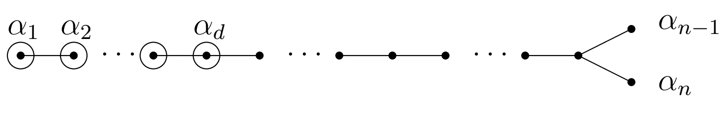
There are only two double cosets $D_1$ and $D_2$ in $W_P\backslash W/W_P$ with representatives 1 and $w_{\alpha_1}$ respectively. One easily checks that

$$R_{D_1} = S, \quad R_{D_2} = S \setminus \{\alpha_2\}.$$

It follows that $Z_{D_1} = \operatorname{Spec} k$ and $Z_{D_2} = G_P/Q_{D_2}$ is isomorphic to $\mathbb{P}^{n-1}_k$. Thus our Theorem 7.5 yields a motivic decomposition

$$\mathcal{M}(\mathbb{P}^n_k) \simeq \mathbb{Z} \oplus \mathcal{M}(\mathbb{P}^{n-1}_k)(1).$$

**Example 9.2.** *Quadrics.* Let $f = g \perp \mathbb{H}$ be a $k$-isotropic quadratic form of dimension $2n$ and let $G = \mathrm{PSO}(f)$, the projective special orthogonal group of $f$. The Tits index of $G$ is of the form




Here $d \geq 1$ is the Witt index of $f$.

The set of uncircled vertices is $S_0 = \{\alpha_{d+1}, \ldots, \alpha_n\}$. The set $S = \{\alpha_2, \ldots, \alpha_n\}$ contains $S_0$ and is $*$-stable. Let $P$ be the parabolic subgroup of $G$ corresponding to $S$. The variety $X = G/P$ is isomorphic to the isotropic projective quadric given by an equation $f = 0$ and $G_P = \mathrm{SO}(g)$.

Consider the following element in the Weyl group $W$:

$$w_0 = w_{\alpha_1} w_{\alpha_2} \cdots w_{\alpha_{n-2}} (w_{\alpha_{n-1}} w_{\alpha_n}) w_{\alpha_{n-2}} \cdots w_{\alpha_2} w_{\alpha_1}.$$

One easily checks that this is a reduced decomposition of $w_0$ and that $C(w_0)$ is a big cell in $X$. So all other cells are of the form $C(v)$ where $v$ is a word obtained by deleting some generators from the above reduced decomposition of $w_0$ (Proposition 3.9).

After deleting any generator from the above reduced decomposition of $w_0$ we obtain a word containing $w_{\alpha_1}$ once (one uses the relations

$$w_{\alpha_i}w_{\alpha_j} = w_{\alpha_j}w_{\alpha_i}, \text{ if } \ j \neq i \pm 1, \text{ and } \ w_{\alpha_{i+1}}w_{\alpha_i}w_{\alpha_{i+1}} = w_{\alpha_i}w_{\alpha_{i+1}}w_{\alpha_i}.)$$

It follows that we have only three double cosets $D_1, D_2$ and $D_3$ with representatives $1, w_{\alpha_1}$ and $w_0$ respectively. One checks that

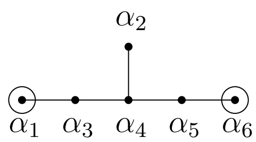
$$R_{D_1} = S, \ R_{D_2} = S \setminus \{\alpha_2\}, \ R_{D_3} = S.$$

Therefore $Z_{D_1} = Z_{D_3} = \operatorname{Spec} k$ and $Z_{D_2}$ is a projective quadric $Y$ of dimension $2n-4$ given by $g = 0$. We have $l(w_0) = 2n-2$, therefore, Theorem 7.5 yields a motivic decomposition

$$\mathcal{M}(X) \simeq \mathbb{Z} \oplus \mathcal{M}(Y)(1) \oplus \mathbb{Z}(2n-2).$$

Thus we recover the standard motivic decomposition of an isotropic quadric given in [12].

**Example 9.3.** *Type* $E_6$. Consider an adjoint group $G$ of inner type $E_6$ with the Tits index




Let $S = \{\alpha_1, \dots, \alpha_5\}$, let $P$ be the parabolic subgroup of $G$ corresponding to $S$, and set $X = G/P$. As above we conclude that $X$ has a rational $k$-point. Note also that $G_P$ is isomorphic to a spinor group of a 10-dimensional (isotropic) quadratic form $f$.

Take the following element in the Weyl group $W$:

$$w_0 = (w_{\alpha_1}w_{\alpha_3}w_{\alpha_4}w_{\alpha_5}w_{\alpha_2}w_{\alpha_4}w_{\alpha_3}w_{\alpha_1})w_{\alpha_6}(w_{\alpha_5}w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_4}w_{\alpha_5})w_{\alpha_6}$$

As above, one checks that $C(w_0)$ is a big cell. It follows that we have three double cosets $D_1, D_2$ and $D_3$ with the representatives $w_1 = 1, w_2 = w_{\alpha_6}$ and

$$w_3 = w_{\alpha_6}w_{\alpha_5}w_{\alpha_4}w_{\alpha_3}w_{\alpha_2}w_{\alpha_4}w_{\alpha_5}w_{\alpha_6}.$$

One checks that

$$R_{D_1} = S, \ R_{D_2} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \ R_{D_3} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}.$$

Hence $Z_{D_1} = \operatorname{Spec} k$, $Z_{D_2} = G_P/Q_{D_2}$ is a connected component $Z$ of the variety of totally isotropic subspaces of $f$ and $Z_{D_3} = G_P/Q_{D_3}$ is the projective quadric $Y$ given by $f$.

Thus we obtain the following motivic decomposition:

$$\mathcal{M}(X) \simeq \mathbb{Z} \oplus \mathcal{M}(Z)(1) \oplus \mathcal{M}(Y)(8).$$

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