ZERO-CYCLES ON ALGEBRAIC TORI

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1. The map φ_T

Let T be an algebraic torus over a field F and X a smooth *compactification of* T, i.e., a geometrically irreducible smooth complete variety containing T as an open set. The Chow group $\mathrm{CH}_0(X)$ of classes of zero dimensional cycles on X does not depend (up to canonical isomorphism) on the choice of X (cf. [7, 16.1.11], [4, Prop. 6.3], [8]).

Recall that two F-points $t, t' \in T(F)$ are called R-*equivalent* if there is rational morphism $f : \mathbb{A}^1 \dashrightarrow T$ defined at 0 and 1 satisfying $f(0) = t$ and $f(1) = t'$ (cf. [2, §4]). We write $T(F)/R$ for the group of R-equivalence classes in $T(F)$.

For an F-point $t \in T(F)$ let $[t]$ denote its class in $\mathrm{CH}_0(X)$. Consider the map from $T(F)$ to $\mathrm{CH}_0(X)$ taking a point t to the class $[t] - [1]$. This map does not depend on the choice of X (up to canonical isomorphism) and it factors through R-equivalence. Indeed, a map f as above extends to a morphism $g : \mathbb{P}^1 \to X$ and $[t] = g_*([0]) = g_*([1]) = [t']$, where $g_* : \mathrm{CH}_0(\mathbb{P}^1) \to \mathrm{CH}_0(X)$ is the push-forward homomorphism (cf. [7, 1.4]).

We denote the resulting map by

$$\varphi_T : T(F)/R \to \mathrm{CH}_0(X).$$

Note that there is a homomorphism $\psi_T : A_0(X) \to T(F)/R$ such that $\psi_T \circ \varphi_T$ is the identity (cf. [2, Prop. 12]). It follows that the map φ_T is injective.

One can ask whether φ_T is a homomorphism. It is known that φ_T is a homomorphism for all tori T of dimension at most 3 (cf. [10]). In this note we shall give an example of a torus T such that φ_T is not a homomorphism although it has left inverse map ψ_T that is a homomorphism. It follows that φ_T is not surjective.

The map φ_T is a homomorphism if and only if for any two points t_1 and t_2 in $T(F)$ one has

$$[t_1 t_2] - [t_1] - [t_2] + [1] = 0 \tag{1}$$

in $\mathrm{CH}_0(X)$.

Let T' be another torus with a compactification X'. Then $X \times X'$ is a compactification of $T \times T'$. Let $t \in T(F)$ and $t' \in T'(F)$. The condition (1) for the elements $t_1 = (t, 1)$ and $t_2 = (1, t')$ of $(T \times T')(F)$ amounts to

$$([t] - [1]) \times ([t'] - [1]) = 0 \tag{2}$$

in $\mathrm{CH}_0(X \times X')$, where $\times$ denotes the external product for Chow groups (cf. [7, 1.10]). In the next section we shall give examples of tori T and T' such that the condition (2) fails for some t and t'. It would follows that $\varphi_{T \times T'}$ is not a homomorphism.

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2. The tori $\mathrm{R}^1_{L/F}(\mathbf{G}_{\mathrm{m}})$

Let F be a field with $\operatorname{char} F \neq 2$. For an element $a \in F^\times$, let F_a denote the quadratic (étale) F-algebra $F[t]/(t^2-a)$.

Let $a,b \in F^\times$. Consider the biquadratic F-algebra $L = F_a \otimes F_b$ and let G be the Galois group $\operatorname{Gal}(L/F)$. Write $\sigma \in G$ for the generator of $\operatorname{Gal}(L/F_a)$ and $\tau \in G$ for the generator of $\operatorname{Gal}(L/F_b)$.

Let T be the torus $\mathrm{R}^1_{L/F}(\mathbf{G}_{\mathrm{m}})$ of norm 1 elements of the extension L/F. For a field extension K/F, a point t of $T(K)$ is an element $t \in (K\otimes L)^\times$ satisfying $N_{(K\otimes L)/K}(t) := t\cdot\sigma(t)\cdot\tau(t)\cdot\sigma\tau(t) = 1$, where $N_{(K\otimes L)/K} : (K\otimes L)^\times \to K^\times$ is the norm homomorphism. The element $N_{(K\otimes L)/(K\otimes F_a)} = t\cdot\sigma(t)$ in $K\otimes F_a$ has norm 1 in K. By Hilbert Theorem 90, applied to the quadratic extension $(K\otimes F_a)/K$, there is an element $z \in (K\otimes F_a)^\times$ with $t\cdot\sigma(t) = z\cdot\tau(z)^{-1}$. Note that z is unique up to a multiple from $K^\times$. Hence the norm $N_{(K\otimes F_a)/K}(z) = z\cdot\tau(z)$ is unique up to a multiple from $K^{\times 2}$. It follows that the class $q_K(t)$ quaternion algebra $\big(z\cdot\tau(z), b\big)_K$ in the Brauer group $\operatorname{Br}(K)$ is well defined. Thus, we get a group homomorphism

$$q_K : T(K) \to \operatorname{Br}(K), \qquad t \mapsto q_K(t).$$

The collection of the homomorphisms q_K over all field extensions K of F form a morphism q of functors T and Br from the category of all field extensions of F to the category of groups. In other words, q is an invariant of the algebraic torus T with values in the Brauer group (cf. [9]).

Remark 2.1. It is shown in [11, p. 427] that q_F induces an isomorphism between $T(F)/R$ and the subgroup of $\operatorname{Br}(F)$ consisting of classes of algebras that are split over all three quadratic subalgebras of L.

Example 2.2. Assume that F contains a square root i of -1. Then we can view i as an element of $T(F)$. We have $i\cdot\sigma(i) = -1 = z\cdot\tau(z)^{-1}$ with $z = \sqrt{a}$ in F_a. Hence $q_F(i)$ is the class of the quaternion algebra $\big(z\cdot\tau(z), b\big)_F \simeq \big(-a, b\big)_F \simeq \big(a,b\big)_F$.

Let $F(T)$ be the function field of T over F and let v be a discrete valuation on $F(T)$ over F. The residue field $F(v)$ is a field extension of F. By [5, §5], there is the residue homomorphism

$$\partial_v : \operatorname{Br}\big(L(T)/F(T)\big) \to G^*,$$

where G^* is the character group of G. An element α in $\operatorname{Br}\big(L(T)/F(T)\big)$ is called *unramified with respect to* v if $\partial_v(\alpha) = 0$ and *(totally) unramified* if α is unramified with respect to every discrete valuation of $F(T)$ over F.

Proposition 2.3. *For any $t \in T\big(F(T)\big)$, the element $q_{F(T)}(t)$ in $\operatorname{Br}\big(L(T)/F(T)\big)$ is unramified.*

Proof. Write K for $F(T)$, so $L(T) = K\otimes L = KL$. As the character group G^* is of exponent 2, it suffices to show that $q_K(t)$ is divisible by 2 in $\operatorname{Br}(KL/K)$. By Hilbert Theorem 90, there are elements $z \in K_a^\times$ and $w \in K_b^\times$ such that $t\cdot\sigma(t) = z\cdot\tau(z)^{-1}$ and $t\cdot\tau(t) = w^{-1}\cdot\sigma(w)$. Consider the cross product central simple K-algebra (cf. [6, §12]):

$$A = KL1 \oplus KLu_\sigma \oplus KLu_\tau \oplus KLu_\tau u_\sigma$$

with multiplication table:

$$u_\sigma^2 = z, \quad u_\tau^2 = w, \quad u_\sigma u_\tau = tu_\tau u_\sigma.$$

As KL is a maximal subalgebra of A, the Brauer class of A belongs to $\mathrm{Br}(KL/K)$.

The centralizer C of the quadratic subalgebra $K_a \subset KL \subset A$ in A is generated by KL and u_σ and hence is isomorphic to the quaternion algebra $(z,b)_{K_a}$. It follows from [6, §7] that

$$[A \otimes_K K_a] = [\big(z,b\big)_{K_a}] \quad \text{in}\, \mathrm{Br}(KL/K_a),$$

hence

$$q_K(t) = [\big(z\cdot\sigma(z),b\big)_K] = \mathrm{cor}_{K_a/K}[\big(z,b\big)_{K_a}] = \mathrm{cor}_{K_a/K}[A\otimes_K K_a] = 2[A]. \quad \square$$

We write α_T for the element $q_{F(T)}(t)$ in $\mathrm{Br}\big(L(T)/F(T)\big)$, where t is the generic point in $T\big(F(T)\big)$. As $2\alpha_T = 0$, we can view α_T as an element of the group $H^2\big(F(T),\mathbb{Z}/2\mathbb{Z}\big) = \mathrm{Br}_2\big(F(T)\big)$. By Proposition 2.3, α_T is an unramified element of $H^2\big(F(T),\mathbb{Z}/2\mathbb{Z}\big)$ in the sense of [1] (cf. [10, 2.2]).

Remark 2.4. If L/F is a field extension, by [3, Prop. 9.5], the factor group of the group of unramified elements in $\mathrm{Br}\big(F(T)\big)$ modulo $\mathrm{Br}(F)$ is canonically isomorphic to $H^2(G,\widehat{T}) \simeq H^3(G,\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$, where $\widehat{T}$ is the Galois module of characters of T. The class α_T corresponds to the only nontrivial element of the group $H^2(G,\widehat{T})$.

Choose a smooth compactifications X of T, so we can view α as an unramified element of $H^2\big(F(X),\mathbb{Z}/2\mathbb{Z}\big)$. Let $x\in X(F)$ be any point over F. We write $\alpha(x)\in H^2\big(F,\mathbb{Z}/2\mathbb{Z}\big)$ for the *value of* α *at* x (cf. [10, 2.1]). If $x\in T(F)$, then $\alpha(x) = q_F(x)$. In particular, we have $\alpha(1) = 0$ and $\alpha(i) = (a)\cup(b)$ by Example 2.2 if F contains a square root i of -1.

Let $L' = F_{a'}\otimes F_{b'}$ be another biquadratic F-algebra and $T' := \mathrm{R}^1_{L'/F}(\mathbf{G}_\mathrm{m})$ and let $\alpha_{T'}\in H^2\big(F(T'),\mathbb{Z}/2\mathbb{Z}\big)$ be the element as above. Choose also a smooth compactification X' of T'. Restricting α and α' to $F(X\times X')$ and taking the cup-product, we get the unramified element

$$\beta = \alpha\times\alpha' \in H^4\big(F(X\times X'),\mathbb{Z}/2\mathbb{Z}\big).$$

Let $\mathrm{Z}_0(X\times X')$ be the group of zero-dimensional cycles on $X\times X'$. The map $\mathrm{Z}_0(X\times X') \to H^4(F,\mathbb{Z}/2\mathbb{Z})$ taking the class of a closed point $z\in X\times X'$ to $N_{F(z)/F}\big(\beta(z)\big)$ factors through a homomorphism

$$\rho : \mathrm{CH}_0(X\times X') \to H^4(F,\mathbb{Z}/2\mathbb{Z})$$

(cf. [10, 2.4]). Note that for every $t\in T(F)$ and $t'\in T'(F)$ we have

$$\rho([t]\times[t']) = \beta(t,t') = \alpha(t)\cup\alpha'(t') \in H^4(F,\mathbb{Z}/2\mathbb{Z}).$$

It follows that

$$\rho(([t]-[1])\times([t']-[1])) = \big(\alpha(t)-\alpha(1)\big)\cup\big(\alpha'(t')-\alpha'(1)\big) = \alpha(t)\cup\alpha'(t')$$

in $H^4(F,\mathbb{Z}/2\mathbb{Z})$.

Assume that F contains a square root i of -1, so $i\in T(F)$. We then have

$$\rho\big(([i]-[1])\times([i]-[1])\big) = (a)\cup(b)\cup(a')\cup(b') \in H^4(F,\mathbb{Z}/2\mathbb{Z}).$$

One can easily find a field F and elements a,b,a',b' with $(a)\cup(b)\cup(a')\cup(b')\neq 0$ in $H^4(F,\mathbb{Z}/2\mathbb{Z})$. For example, one can take $F = k(a,b,a',b')$, where a,b,a',b' are variables over a field k. This contradicts (2). Hence $\varphi_{T\times T'}$ is not a homomorphism.

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