ESSENTIAL DIMENSION OF FINITE p-GROUPS

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ABSTRACT. We prove that the essential dimension and p-dimension of a p-group G over a field F containing a primitive p-th root of unity is equal to the least dimension of a faithful representation of G over F.

The notion of the essential dimension $\mathrm{ed}(G)$ of a finite group G over a field F was introduced in [5]. The integer $\mathrm{ed}(G)$ is equal to the smallest number of algebraically independent parameters required to define a Galois G-algebra over any field extension of F. If V is a faithful linear representation of G over F then $\mathrm{ed}(G) \leq \dim(V)$ (cf. [2, Prop. 4.15]). The essential dimension of G can be smaller than $\dim(V)$ for every faithful representation V of G over F. For example, we have $\mathrm{ed}(\mathbb{Z}/3\mathbb{Z}) = 1$ over $\mathbb{Q}$ or any field F of characteristic 3 (cf. [2, Cor. 7.5]) and $\mathrm{ed}(S_3) = 1$ over $\mathbb{C}$ (cf. [5, Th. 6.5]).

In this paper we prove that if G is a p-group and F is a field of characteristic different from p containing p-th roots of unity, then $\mathrm{ed}(G)$ coincides with the least dimension of a faithful representation of G over F (cf. Theorem 4.1).

We also compute the essential p-dimension of a p-group G introduced in [15]. We show that $\mathrm{ed}_p(G) = \mathrm{ed}(G)$ over a field F containing p-th roots of unity.

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1. Preliminaries

In the paper the word "scheme" means a separated scheme of finite type over a field and "variety" an integral scheme.

1.1. **Severi-Brauer varieties.** (cf. [1]) Let A be a central simple algebra of degree n over a field F. The *Severi-Brauer variety* $P = \mathrm{SB}(A)$ of A is the variety of right ideals in A of dimension n. For a field extension L/F, the algebra A is split over L if and only if $P(L) \neq \emptyset$ if and only if $P_L \simeq \mathbb{P}^{n-1}_L$.

The change of field map $\deg : \mathrm{Pic}(P) \to \mathrm{Pic}(P_L) = \mathbb{Z}$ for a splitting field extension L/F identifies $\mathrm{Pic}(P)$ with $e\mathbb{Z}$, where e is the exponent (period) of

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A. In particular, P has divisors of degree e. The algebra A is split over L if and only if P_L has a prime divisor of degree 1 (a hyperplane).

1.2. **Groupoids and gerbes.** (cf. [4]) Let $\mathcal{X}$ be a groupoid over F in the sense of [19]. We assume that for any field extension L/F, the isomorphism classes of objects in the category $\mathcal{X}(L)$ form a set which we denote by $\widehat{\mathcal{X}}(L)$. We can view $\widehat{\mathcal{X}}$ as a functor from the category *Fields*$/F$ of field extensions of F to *Sets*.

Example 1.2.1. If G is an algebraic group over F, then the groupoid BG is defined as the category of G-torsors over a scheme over F. Hence the functor $\widehat{BG}$ takes a field extension L/F to the set of all isomorphism classes of G-torsors over L.

Special examples of groupoids are *gerbes banded by a commutative group scheme* C over F. There is a bijection between the set of isomorphism classes of gerbes banded by C and the Galois cohomology group $H^2(F, C)$ (cf. [7, Ch. 4] and [13, Ch. 4, §2]). The split gerbe BC corresponds to the trivial element of $H^2(F, C)$.

Example 1.2.2. (Gerbes banded by μ_n) Let A be a central simple F-algebra and n an integer with $[A] \in \mathrm{Br}_n(F) = H^2(F, \mu_n)$. Let P be the Severi-Brauer variety of A and S a divisor on P of degree n. Denote by $\mathcal{X}_A$ the gerbe banded by μ_n corresponding to $[A]$. For a field extension L/F, the set $\widehat{\mathcal{X}}_A(L)$ has the following explicit description (cf. [4]): $\widehat{\mathcal{X}}_A(L)$ is nonempty if and only if P is split over L. In this case $\widehat{\mathcal{X}}_A(L)$ is the set of equivalence classes of the set

$$\left\{f \in L(P)^\times : \ \ \mathrm{div}(f) = nH - S_L, \ \text{ where } H \text{ is a hyperplane in } P_L\right\},$$

and two functions f and f' are equivalent if $f' = fh^n$ for some $h \in L(P)^\times$.

1.3. **Essential dimension.** Let T : *Fields*$/F \to$ *Sets* be a functor. For a field extension L/F and an element $t \in T(L)$, the *essential dimension of* t, denoted $\mathrm{ed}(t)$, is the least $\mathrm{tr.deg}_F(L')$ over all subfields $L' \subset L$ over F such that t belongs to the image of the map $T(L') \to T(L)$. The *essential dimension* $\mathrm{ed}(T)$ *of the functor* T is the supremum of $\mathrm{ed}(t)$ over all $t \in T(L)$ and field extensions L/F.

Let p be a prime integer and $t \in T(L)$. The *essential p-dimension of* t, denoted $\mathrm{ed}_p(t)$, is the least $\mathrm{tr.deg}_F(L'')$ over all subfields $L'' \subset L'$ over F, where L' is a finite field extension of L of degree prime to p such that the image of t in $T(L')$ belongs to the image of the map $T(L'') \to T(L')$. The *essential p-dimension* $\mathrm{ed}_p(T)$ *of the functor* T is the supremum of $\mathrm{ed}_p(t)$ over all $t \in T(L)$ and field extensions L/F. Clearly, $\mathrm{ed}(T) \geq \mathrm{ed}_p(T)$.

Let G be an algebraic group over F. The *essential dimension* $\mathrm{ed}(G)$ *of* G (respectively the *essential p-dimension* $\mathrm{ed}(G)$) is the essential dimension (respectively the essential p-dimension) of the functor taking a field extension L/F to the set of isomorphism classes of G-torsors over $\mathrm{Spec}\, L$.

If G is a finite group, we view G as a constant group over a field F. Every G-torsor over $\operatorname{Spec} L$ has the form $\operatorname{Spec} K$ where K is a Galois G-algebra over L. Therefore, $\operatorname{ed}(G)$ is the essential dimension of the functor taking a field L to the set of isomorphism classes of Galois G-algebras over L.

Example 1.3.1. Let $\mathcal{X}$ be a groupoid over F. The *essential dimension of* $\mathcal{X}$, denoted by $\operatorname{ed}(\mathcal{X})$, is the essential dimension $\operatorname{ed}(\widehat{\mathcal{X}})$ of the functor $\widehat{\mathcal{X}}$ defined in §1.2. The *essential p-dimension of* $\operatorname{ed}_p(\mathcal{X})$ is defined similarly. In particular, $\operatorname{ed}(BG) = \operatorname{ed}(G)$ and $\operatorname{ed}_p(BG) = \operatorname{ed}_p(G)$ for an algebraic group G over F.

1.4. **Canonical dimension.** (cf. [3], [11]) Let F be a field and $\mathcal{C}$ a class of field extensions of F. A field $E \in \mathcal{C}$ is called *generic* if for any $L \in \mathcal{C}$ there is an F-place $E \rightsquigarrow L$.

The *canonical dimension* $\operatorname{cdim}(\mathcal{C})$ of the class $\mathcal{C}$ is the minimum of the $\operatorname{tr.deg}_F E$ over all generic fields $E \in \mathcal{C}$.

Let p be a prime integer. A field E in a class $\mathcal{C}$ is called *p-generic* if for any $L \in \mathcal{C}$ there is a finite field extension L' of L of degree prime to p and an F-place $E \rightsquigarrow L'$. The *canonical p-dimension* $\operatorname{cdim}_p(\mathcal{C})$ of the class $\mathcal{C}$ is the least $\operatorname{tr.deg}_F E$ over all p-generic fields $E \in \mathcal{C}$. Obviously, $\operatorname{cdim}(\mathcal{C}) \geq \operatorname{cdim}_p(\mathcal{C})$.

Let $T : \textsf{Fields}/F \to \textsf{Sets}$ be a functor. Denote by $\mathcal{C}_T$ the class of *splitting fields of* T, i.e., the class of field extensions L/F such that $T(L) \neq \emptyset$. The *canonical dimension (p-dimension) of* T, denoted $\operatorname{cdim}(T)$ (respectively $\operatorname{cdim}_p(T)$), is the canonical dimension (p-dimension) of the class $\mathcal{C}_T$.

If X is a scheme over F, we write $\operatorname{cdim}(X)$ and $\operatorname{cdim}_p(X)$ for the canonical dimension and p-dimension of X viewed as a functor $L \mapsto X(L) = \operatorname{Mor}_F(\operatorname{Spec} L, X)$.

Example 1.4.1. Let $\mathcal{X}$ be a groupoid over F. We define the *canonical dimension* $\operatorname{cdim}(\mathcal{X})$ *and p-dimension* $\operatorname{cdim}_p(\mathcal{X})$ *of* $\mathcal{X}$ as the canonical dimension and p-dimension of the functor $\widehat{\mathcal{X}}$.

Example 1.4.2. If X is a regular and complete variety over F viewed as a functor then $\operatorname{cdim}(X)$ is equal to the smallest dimension of a closed subvariety $Z \subset X$ such that there is a rational morphism $X \dashrightarrow Z$ (cf. [11, Cor. 4.6]). If p is a prime integer then $\operatorname{cdim}_p(X)$ is equal to the smallest dimension of a closed subvariety $Z \subset X$ such that there are dominant rational morphisms $X' \dashrightarrow X$ of degree prime to p and $X' \dashrightarrow Z$ for some variety X' (cf. [11, Prop. 4.10]).

Remark 1.4.3. (A relation between essential and canonical dimension) Let $T : \textsf{Fields}/F \to \textsf{Sets}$ be a functor. We define the "contraction" functor $T^c : \textsf{Fields}/F \to \textsf{Sets}$ as follows. For a field extension L/F, we have $T^c(L) = \emptyset$ if $T(L)$ is empty and $T^c(L)$ is a one element set otherwise. If X is a regular and complete variety over F viewed as a functor then one can show that $\operatorname{ed}(X^c) = \operatorname{cdim}(X)$ and $\operatorname{ed}_p(X^c) = \operatorname{cdim}_p(X)$.

1.5. **Valuations.** Let K/F be a regular field extension, i.e., for any field extension L/F, the ring $K \otimes_F L$ is a domain. We write KL for the quotient field of $K \otimes_F L$.

Let v be a valuation on L over F with residue field R. Let O be the associated valuation ring and M its maximal ideal. As $K \otimes_F R$ is a domain, the ideal $\widetilde{M} := K \otimes_F M$ in the ring $\widetilde{O} := K \otimes_F O$ is prime. The localization ring $\widetilde{O}_{\widetilde{M}}$ is a valuation ring in KL with residue field KR. The corresponding valuation $\tilde{v}$ of KL is called the *canonical extension of v on KL*. Note that the groups of values of v and $\tilde{v}$ coincide.

We shall need the following lemma.

Lemma 1.1 (cf. [11, Lemma 3.2])**.** *Let v be a discrete valuation (of rank 1) of a field L with residue field R and L'/L a finite field extension of degree prime to p. Then v extends to a discrete valuation of L' with residue field R' such that the ramification index and the degree $[R' : R]$ are prime to p.*

Proof. If L'/L is separable and $v_1, \dots, v_k$ are all the extensions of v on L' then $[L' : L] = \sum e_i[R_i : R]$ where e_i is the ramification index and R_i is the residue field of v_i (cf. [20, Ch. VI, Th. 20 and p. 63]). It follows that the integer $e_i[R_i : R]$ is prime to p for some i.

If L'/L is purely inseparable of degree q then the valuation v' of L' defined by $v'(x) = v(x^q)$ satisfies the desired properties. The general case follows. □

2. Canonical dimension of a subgroup of $\mathrm{Br}(F)$

Let F be an arbitrary field, p a prime integer and D a finite subgroup of $\mathrm{Br}_p(F)$ of dimension r over $\mathbb{Z}/p\mathbb{Z}$. In this section we determine the canonical dimension $\operatorname{cdim} D$ and the canonical p-dimension $\operatorname{cdim}_p D$ of the class of common splitting fields of all elements of D. We say that a basis $\{a_1, a_2, \dots, a_r\}$ of D is *minimal* if for any $i = 1, \dots, r$ and any element $d \in D$ outside of the subgroup generated by $a_1, \dots, a_{i-1}$, we have $\operatorname{ind} d \geq \operatorname{ind} a_i$.

One can construct a minimal basis of D by induction as follows. Let a_1 be a nonzero element of D of minimal index. If the elements $a_1, \dots, a_{i-1}$ are already chosen for some $i \leq r$, we take for the a_i an element of D of the minimal index among the elements outside of the subgroup generated by $a_1, \dots, a_{i-1}$.

In this section we prove the following

Theorem 2.1. *Let F be an arbitrary field, p a prime integer, $D \subset \mathrm{Br}_p(F)$ a subgroup of dimension r and $\{a_1, a_2, \dots, a_r\}$ a minimal basis of D. Then*

$$\operatorname{cdim}_p(D) = \operatorname{cdim}(D) = \Big(\sum_{i=1}^{r} \operatorname{ind} a_i\Big) - r\ .$$

We prove Theorem 2.1 in several steps.

Let $\{a_1, a_2, \dots, a_r\}$ be a minimal basis of D. For every $i = 1, 2, \dots, r$, let P_i be the Severi-Brauer variety of a central division F-algebra A_i representing the element $a_i \in \mathrm{Br}_p F$. We write P for the product $P_1 \times P_2 \times \cdots \times P_r$. We have

$$\dim P = \sum_{i=1}^{r} \dim P_i = \Big(\sum_{i=1}^{r} \operatorname{ind} a_i\Big) - r.$$

Moreover, the classes of splitting fields of P and D coincide, hence $\operatorname{cdim}(D) = \operatorname{cdim}(P)$ and $\operatorname{cdim}_p(D) = \operatorname{cdim}_p(P)$. Thus, the statement of Theorem 2.1 is equivalent to the equality $\operatorname{cdim}_p(P) = \operatorname{cdim}(P) = \dim(P)$.

Let $r \geq 1$ and $0 \leq n_1 \leq n_2 \leq \cdots \leq n_r$ be integers and $K = K(n_1, \dots, n_r)$ the subgroup of the polynomial ring $\mathbb{Z}[x]$ in r variables $x = (x_1, \dots, x_r)$ generated by the monomials $p^{e(j_1,\dots,j_r)} x_1^{j_1} \dots x_r^{j_r}$ for all $j_1, \dots, j_r \geq 0$, where the exponent $e(j_1, \dots, j_r)$ is 0 if all the $j_1, \dots, j_r$ are divisible by p, otherwise $e(j_1, \dots, j_r) = n_k$ with the maximum k such that j_k is not divisible by p. In fact, K is a subring of $\mathbb{Z}[x]$.

Remark 2.2. Let $A_1, \dots, A_r$ be central division algebras over some field such that for any non-negative integers $j_1, \dots, j_r$, the index of the tensor product $A_1^{\otimes j_1} \otimes \cdots \otimes A_r^{\otimes j_r}$ is equal to $p^{e(j_1,\dots,j_r)}$. The group K can be interpreted as the colimit of the Grothendieck groups of the product over $i = 1, \dots, r$ of the Severi-Brauer varieties of the matrix algebras $M_{l_i}(A_i)$ over all positive integers $l_1, \dots, l_r$.

We set $h = (h_1, \dots, h_r)$ with $h_i = 1 - x_i \in \mathbb{Z}[x]$.

Proposition 2.3. *Let $bh_1^{i_1} \dots h_r^{i_r}$ be a monomial of the lowest total degree of a polynomial f in the variables h lying in K. Assume that the integer b is not divisible by p. Then $p^{n_1} \mid i_1, \dots, p^{n_r} \mid i_r$.*

Proof. We recast the proof for $r = 1$ given in [8, Lemma 2.1.2] to the case of arbitrary r.

We proceed by induction on $m = r + n_1 + \cdots + n_r$. The case $m = 1$ is trivial. If $m > 1$ and $n_1 = 0$, then $K = K(n_2, \dots, n_r)[x_1]$ and we are done by induction applied to $K(n_2, \dots, n_r)$. In what follows we assume that $n_1 \geq 1$.

Since $K(n_1, n_2, \dots, n_r) \subset K(n_1 - 1, n_2, \dots, n_r)$, by the induction hypothesis $p^{n_1-1} \mid i_1,\ p^{n_2} \mid i_2, \dots, p^{n_r} \mid i_r$. It remains to show that i_1 is divisible by p^{n_1}.

Consider the additive operation $\varphi \colon \mathbb{Z}[x] \to \mathbb{Q}[x]$ which takes a polynomial $g \in \mathbb{Z}[x]$ to the polynomial $p^{-1} x_1 \cdot g'$, where g' is the partial derivative of g with respect to x_1. We have

$$\varphi(K) \subset K(n_1 - 1, n_2 - 1, \dots, n_r - 1) \subset K(n_1 - 1)[x_2, \dots, x_r]$$

and

$$\varphi(h_1^{j_1} h_2^{j_2} \cdots h_r^{j_r}) = -p^{-1} j_1 h_1^{j_1 - 1} h_2^{j_2} \cdots h_r^{j_r} + p^{-1} j_1 h_1^{j_1} h_2^{j_2} \cdots h_r^{j_r}.$$

Since $bh_1^{i_1} \cdots h_r^{i_r}$ is a monomial of the lowest total degree of the polynomial f, it follows that $-bp^{-1} i_1 h_1^{i_1 - 1} h_2^{i_2} \cdots h_r^{i_r}$ is a monomial of $\varphi(f)$ considered as a polynomial in h. As

$$\varphi(f) \in K(n_1 - 1)[x_2, \dots, x_r]\ ,$$

we see that $-bp^{-1}i_1h_1^{i_1-1}$ is a monomial of a polynomial from $K(n_1-1)$. It follows that $p^{-1}i_1$ is an integer and by Lemma 2.4 below, this integer is divisible by p^{n_1-1}. Therefore $p^{n_1} \mid i_1$. □

Lemma 2.4. *Let g be a polynomial in h_1 lying in $K(m)$ for some $m \geq 0$. Let bh_1^{i-1} be a monomial of g such that i is divisible by p^m. Then b is divisible by p^m.*

Proof. We write h for h_1 and x for x_1. Note that $h^i \in K(m)$ since i is divisible by p^m. Moreover, the quotient ring $K(m)/(h^i)$ is additively generated by $p^{e(j)}x^j$ with $j < i$. Indeed, the polynomial $x^i - (-h)^i = x^i - (x-1)^i$ is a linear combination with integer coefficients of $p^{e(j)}x^j$ with $j < i$. Consequently, for any $k \geq 0$, multiplying by $p^{e(k)}x^k$, we see that the polynomial $p^{e(i+k)}x^{i+k} = p^{e(k)}x^{i+k}$ modulo the ideal (h^i) is a linear combination with integer coefficients of the $p^{e(j)}x^j$ with $j < i + k$.

Thus, $K(m)/(h^i)$ is additively generated by $p^{e(j)}(1-h)^j$ with $j < i$. Only the generator $p^{e(i-1)}(1-h)^{i-1} = p^m(1-h)^{i-1}$ has a nonzero h^{i-1}-coefficient and that coefficient is divisible by p^m. □

Let Y be a scheme over the field F. We write $\mathrm{CH}(Y)$ for the Chow group of Y and set $\mathrm{Ch}(Y) = \mathrm{CH}(Y)/p\,\mathrm{CH}(Y)$. We define $\mathrm{Ch}(\overline{Y})$ as the colimit of $\mathrm{Ch}(Y_L)$ where L runs over all field extensions of F. Thus for any field extension L/F, we have a canonical homomorphism $\mathrm{Ch}(Y_L) \to \mathrm{Ch}(\overline{Y})$. This homomorphism is an isomorphism if $Y = P$, the variety defined above, and L is a splitting field of P.

We define $\overline{\mathrm{Ch}}(Y)$ to be the image of the homomorphism $\mathrm{Ch}(Y) \to \mathrm{Ch}(\overline{Y})$.

Proposition 2.5. *We have $\overline{\mathrm{Ch}}^j(P) = 0$ for any $j > 0$.*

Proof. Let $K_0(P)$ be the Grothendieck group of P. We write $K_0(\overline{P})$ for the colimit of $K_0(P_L)$ taken over all field extensions L/F. The group $K_0(\overline{P})$ is canonically isomorphic to $K_0(P_L)$ for any splitting field L of P. Each of the groups $K_0(P)$ and $K_0(\overline{P})$ is endowed with the topological filtration. The subsequent factor groups $G^jK_0(P)$ and $G^jK_0(\overline{P})$ of these filtrations fit into the commutative square

$$\begin{array}{ccc} \mathrm{CH}^j(\overline{P}) & \longrightarrow & G^jK_0(\overline{P}) \\ \uparrow & & \uparrow \\ \mathrm{CH}^j(P) & \longrightarrow & G^jK_0(P) \end{array}$$

where the top map is an isomorphism. Therefore it suffices to show that the image of the homomorphism $G^jK_0(P) \to G^jK_0(\overline{P})$ is divisible by p for any $j > 0$.

The ring $K_0(\overline{P})$ is identified with the quotient of the polynomial ring $\mathbb{Z}[h]$ by the ideal generated by $h_1^{\operatorname{ind} a_1}, \dots, h_r^{\operatorname{ind} a_r}$. Under this identification, the element h_i is the pull-back to P of the class of a hyperplane in P_i over a splitting field and the j-th term $K_0(\overline{P})^{(j)}$ of the filtration is generated by the classes of monomials of degree at least j. The group $G^jK_0(\overline{P})$ is identified with the group of all homogeneous polynomials of degree j.

The group $K_0(P)$ is isomorphic to the direct sum of $K_0(B)$, where $B = A_1^{\otimes j_1} \otimes \cdots \otimes A_r^{\otimes j_r}$, over all j_i with $0 \leq j_i < \operatorname{ind} a_i$ (cf. [14, §9]). The image of the natural map $K_0(B) \to K_0(B_L) = \mathbb{Z}$, where L is a splitting field of B, is equal to $\operatorname{ind}(a_1^{j_1} \cdots a_r^{j_r})\mathbb{Z}$. The image of the homomorphism $K_0(P) \to K_0(\overline{P})$ (which is in fact an injection) is generated by

$$\operatorname{ind}(a_1^{j_1} \cdots a_r^{j_r})(1-h_1)^{j_1} \cdots (1-h_r)^{j_r}$$

over all $j_1, \dots, j_r \geq 0$.

We embed $K_0(\overline{P})$ into the polynomial ring $\mathbb{Z}[x] = \mathbb{Z}[x_1, \dots, x_r]$ as a subgroup by identifying a monomial $h_1^{j_1} \cdots h_r^{j_r}$ where $0 \leq j_i < \operatorname{ind} a_i$ with the polynomial $(1-x_1)^{j_1} \cdots (1-x_r)^{j_r}$. As the elements $a_1, \dots, a_r$ form a minimal basis of D, the index $\operatorname{ind}(a_1^{j_1} \cdots a_r^{j_r})$ is a power of p with the exponent at least $e(\log_p \operatorname{ind} a_1, \dots, \log_p \operatorname{ind} a_r)$. Therefore,

$$K_0(P) \subset K(\log_p \operatorname{ind} a_1, \dots, \log_p \operatorname{ind} a_r) \subset \mathbb{Z}[x].$$

An element of $K_0(P)^{(j)}$ with $j > 0$ is a polynomial f in h of degree at least j. The image of f in $G^jK_0(\overline{P})$ is the j-th homogeneous part f_j of f. As the degree of f with respect to h_i is less than $\operatorname{ind} a_i$, it follows from Proposition 2.3 that all the coefficients of f_j are divisible by p. □

Let $d = \dim P$ and $\alpha \in \mathrm{CH}^d(P \times P)$. The *first multiplicity* $\operatorname{mult}_1(\alpha)$ *of* α is the image of α under the push-forward map $\mathrm{CH}^d(P \times P) \to \mathrm{CH}^0(P) = \mathbb{Z}$ given by the first projection $P \times P \to P$ (cf. [10]). Similarly, we define the *second multiplicity* $\operatorname{mult}_2(\alpha)$.

Corollary 2.6. *For any element* $\alpha \in \mathrm{CH}^d(P \times P)$*, we have*

$$\operatorname{mult}_1(\alpha) \equiv \operatorname{mult}_2(\alpha) \quad \textit{modulo } p.$$

Proof. We follow the proof of [9, Th. 2.1]. The homomorphism

$$f \colon \mathrm{CH}^d(P \times P) \to (\mathbb{Z}/p\mathbb{Z})^2,$$

taking an $\alpha \in \mathrm{CH}^d(P \times P)$ to $\big(\operatorname{mult}_1(\alpha), \operatorname{mult}_2(\alpha)\big)$ modulo p, factors through the group $\overline{\mathrm{Ch}}^d(P \times P)$. Since for any i, any projection $P_i \times P_i \to P_i$ is a projective bundle, the Chow group $\overline{\mathrm{Ch}}^d(P \times P)$ is a direct some of several copies of $\overline{\mathrm{Ch}}^i(P)$ for some i's and the value $i = 0$ appears once. By Proposition 2.5, the dimension over $\mathbb{Z}/p\mathbb{Z}$ of the vector space $\overline{\mathrm{Ch}}^d(P \times P)$ is equal to 1 and consequently the dimension of the image of f is at most 1. Since the image of the diagonal class under f is $(1,1)$, the image of f is generated by $(1,1)$. □

Corollary 2.7. *Any rational map* $P \dashrightarrow P$ *is dominant.*

Proof. Let $\alpha \in \mathrm{CH}^d(P \times P)$ be the class of the closure of the graph of a rational map $P \dashrightarrow P$. We have $\operatorname{mult}_1(\alpha) = 1$. Therefore, by Corollary 2.6, $\operatorname{mult}_2(\alpha) \neq 0$, and it follows that the rational map is dominant. □

Corollary 2.8. $\operatorname{cdim}_p P = \operatorname{cdim} P = \dim P$.

Proof. As $\operatorname{cdim}_p P \leq \operatorname{cdim} P \leq \dim P$, it suffices to show that $\operatorname{cdim}_p P = \dim P$. Let $Z \subset P$ be a closed subvariety and $f : P' \dashrightarrow P$ and $g : P' \dashrightarrow Z$ dominant rational morphisms such that $\deg f$ is prime to p. Let α be the class in $\mathrm{CH}^d(P \times P)$ of the closure in $P \times P$ of the image of $f \times g : P' \dashrightarrow P \times Z$. As $\operatorname{mult}_1(\alpha) = \deg f$ is prime to p, by Corollary 2.6, we have $\operatorname{mult}_2(\alpha) \neq 0$, i.e., $Z = P$. By Example 1.4.2, $\operatorname{cdim}_p P = \dim P$. □

The corollary completes the proof of Theorem 2.1.

Remark 2.9. Theorem 2.1 can be generalized to the case of any finite subgroup $D \subset \operatorname{Br}(F)$ consisting of elements of p-primary orders. Let $\{a_1, a_2, \dots, a_r\}$ be elements of D such that their images $\{a'_1, a'_2, \dots, a'_r\}$ in D/D^p form a minimal basis, i.e., for any $i = 1, \dots r$ and any element $d \in D$ with the class in D/D^p outside of the subgroup generated by $a'_1, \dots, a'_{i-1}$, the inequality $\operatorname{ind} d \geq \operatorname{ind} a_i$ holds. In particular, $\{a_1, a_2, \dots, a_r\}$ generate D. Then, as in Theorem 2.1, we have

$$\operatorname{cdim}_p(D) = \operatorname{cdim}(D) = \Big(\sum_{i=1}^{r} \operatorname{ind} a_i\Big) - r\ .$$

Indeed, the group D and the variety $P = P_1 \times \cdots \times P_r$, where P_i for every $i = 1, \dots, r$ is the Severi-Brauer variety of a central division algebra representing the element a_i, have the same splitting fields. Therefore, $\operatorname{cdim}(D) = \operatorname{cdim}(P)$ and $\operatorname{cdim}_p(D) = \operatorname{cdim}_p(P)$. Corollaries 2.6, 2.7 and 2.8 hold for P since $K_0(P) \subset K(\log_p \operatorname{ind} a_1, \dots, \log_p \operatorname{ind} a_r)$.

Remark 2.10. One can compute the canonical p-dimension of an arbitrary finite subgroup of $D \subset \operatorname{Br}(F)$ as follows. Let D' be the Sylow p-subgroup of D. Write $D = D' \oplus D''$ for a subgroup $D'' \subset D$ and let L/F be a finite field extension of degree prime to p such that D'' is split over L. Then $D_L = D'_L$ and $\operatorname{cdim}_p(D) = \operatorname{cdim}_p(D_L) = \operatorname{cdim}_p(D'_L) = \operatorname{cdim}_p(D') = \operatorname{cdim}(D')$.

3. Essential and canonical dimension of gerbes banded by $(\boldsymbol{\mu}_p)^s$

In this section we relate the essential and canonical (p-)dimensions of gerbes banded by $(\boldsymbol{\mu}_p)^s$ where $s \geq 0$. The following statement is a generalization of [4, Th. 7.1].

Theorem 3.1. *Let p be a prime integer and $\mathcal{X}$ a gerbe banded by $(\boldsymbol{\mu}_p)^s$ over an arbitrary field F. Then*

$$\operatorname{ed}(\mathcal{X}) = \operatorname{ed}_p(\mathcal{X}) = \operatorname{cdim}_p(\mathcal{X}) + s = \operatorname{cdim}(\mathcal{X}) + s.$$

Proof. The gerbe $\mathcal{X}$ is given by an element in $H^2\big(F, (\boldsymbol{\mu}_p)^s\big) = \operatorname{Br}_p(F)^s$, i.e., by an s-tuple of central simple algebras $A_1, A_2, \dots, A_s$ with $[A_i] \in \operatorname{Br}_p(F)$. Let P be the product of the Severi-Brauer varieties $P_i := \operatorname{SB}(A_i)$ and D the subgroup of $\operatorname{Br}_p(F)$ generated by the $[A_i]$, $i = 1, \dots, s$. As the classes of splitting fields for $\mathcal{X}$, D and P coincide, we have

$$\text{(1)}\quad \operatorname{cdim}(\mathcal{X}) = \operatorname{cdim}(P) = \operatorname{cdim}(D) = \operatorname{cdim}_p(D) = \operatorname{cdim}_p(P) = \operatorname{cdim}_p(\mathcal{X})$$

by Theorem 2.1. We shall prove the inequalities $\mathrm{ed}_p(\mathcal{X}) \geq \mathrm{cdim}(P) + s \geq \mathrm{ed}(\mathcal{X})$.

Let S_i be a divisor on P_i of degree p. Let L/F be a field extension and $f_i \in L(P_i)^\times$ with $\mathrm{div}(f_i) = pH_i - (S_i)_L$, where H_i is a hyperplane in $(P_i)_L$ for $i = 1, \dots, s$. We write $\langle f_i \rangle_{i=1}^s$ for the corresponding element in $\widehat{\mathcal{X}}(L)$ (cf. §1.2).

By Example 1.4.2, there is a closed subvariety $Z \subset P$ and a rational dominant morphism $P \dashrightarrow Z$ with $\dim(Z) = \mathrm{cdim}(P) = \mathrm{cdim}_p(P)$. We view $F(Z)$ as a subfield of $F(P)$. As $P(L) \neq \emptyset$ and P is regular, there is an F-place $\gamma : F(P) \rightsquigarrow L$ (cf. [11, §4.1]). Since Z is complete, the valuation ring of the restriction $\gamma|_{F(Z)} : F(Z) \rightsquigarrow L$ dominates a point in Z. It follows that $Z(L) \neq \emptyset$. Choose a point $y \in Z$ such that $F' := F(y) \subset L$.

Since $P(F') \neq \emptyset$, the P_i are split over F', hence $\mathrm{Pic}(P_i)_{F'} = \mathbb{Z}$ and there are functions $g_i \in F'(P_i)^\times$ with $\mathrm{div}(g_i) = pH'_i - (S_i)_{F'}$, where H'_i is a hyperplane in P_i for $i = 1, \dots, s$. As $\mathrm{Pic}(P_i)_L = \mathbb{Z}$, there are functions $h_i \in L(P_i)^\times$ with $\mathrm{div}(h_i) = (H'_i)_L - H_i$. We have

$$\mathrm{div}(g_i)_L = \mathrm{div}(f_i) + \mathrm{div}(h_i^p),$$

hence

$$a_i g_i = f_i h_i^p$$

for some $a_i \in L^\times$. It follows that $\langle f_i \rangle_{i=1}^s = \langle a_i g_i \rangle_{i=1}^s$ in $\mathcal{X}(L)$, therefore $\langle f_i \rangle_{i=1}^s$ is defined over the field $F'(a_1, a_2, \dots, a_s)$. Hence

$$\mathrm{ed}\langle f_i \rangle_{i=1}^s \leq \mathrm{tr.deg}_F(F') + s \leq \dim(Z) + s = \mathrm{cdim}(P) + s,$$

and therefore $\mathrm{ed}(\mathcal{X}) \leq \mathrm{cdim}(P) + s$.

We shall prove the inequality $\mathrm{ed}_p(\mathcal{X}) \geq \mathrm{cdim}(P) + s$. As $P\big(F(Z)\big) \neq \emptyset$, there are functions $f_i \in F(Z)(P_i)^\times$ with $\mathrm{div}(f_i) = pH_i - (S_i)_{F(Z)}$, where H_i is a hyperplane in $(P_i)_{F(Z)}$. Let $L := F(Z)(t_1, t_2, \dots, t_s)$, where the t_i are variables, and consider the point $\langle t_i f_i \rangle_{i=1}^s \in \widehat{\mathcal{X}}(L)$.

We claim that $\mathrm{ed}_p\langle t_i f_i \rangle_{i=1}^s \geq \mathrm{cdim}(P) + s$. Let L' be a finite extension of L of degree prime to p and $L'' \subset L'$ a subfield such that the image of $\langle t_i f_i \rangle_{i=1}^s$ in $\widehat{\mathcal{X}}(L')$ is defined over L'', i.e., there are functions $g_i \in L''(P_i)^\times$ and $h_i \in L'(P_i)^\times$ with $t_i f_i = g_i h_i^p$. We shall show that $\mathrm{tr.deg}_F(L'') \geq \mathrm{cdim}(P) + s$.

Let $L_i := F(Z)(t_i, \dots, t_s)$ and v_i be the discrete valuation of L_i corresponding to the variable t_i for $i = 1, \dots, s$. We construct a sequence of field extensions L'_i/L_i of degree prime to p and discrete valuations v'_i of L'_i for $i = 1, \dots, s$ by induction on i as follows. Set $L'_1 = L'$. Suppose the fields $L'_1, \dots, L'_i$ and the valuations $v'_1, \dots, v'_{i-1}$ are constructed. By Lemma 1.1, there is a valuation v'_i of L'_i with residue field L'_{i+1} extending the discrete valuation v_i of L'_i with the ramification index e_i and the degree $[L'_{i+1} : L_{i+1}]$ prime to p.

The composition v' of the discrete valuations v'_i is a valuation of L' with residue field of degree over $F(Z)$ prime to p. A choice of prime elements in all the L'_i identifies the group of values of v' with $\mathbb{Z}^s$. Moreover, for every

$i = 1, \dots, s$, we have

$$v'(t_i) = e_i\varepsilon_i + \sum_{j>i} a_{ij}\varepsilon_j$$

where the ε_i's denote the standard basis elements of $\mathbb{Z}^s$ and $a_{ij} \in \mathbb{Z}$.

Write v'' for the restriction of v' on L''. Let $K = F(P)$. We extend canonically the valuations v' and v'' to valuations $\tilde{v}'$ and $\tilde{v}''$ of KL' and KL'' respectively (cf. §1.5). Note that $f_i \in K(Z)^\times$, $g_i \in (KL'')^\times$ and $h_i \in (KL')^\times$. We have

$$e_i\varepsilon_i + \sum_{j>i} a_{ij}\varepsilon_j = v'(t_i) = \tilde{v}'(t_i f_i) \equiv \tilde{v}''(g_i) \pmod{p}.$$

Since e_i are prime to p, the elements $\tilde{v}''(g_i)$ generate a subgroup of $\mathbb{Z}^s$ of finite index. It follows that the value group of $\tilde{v}''$ is of rank s, hence $\operatorname{rank}(v'') = \operatorname{rank}(\tilde{v}'') = s$.

Let R'' and R' be residue fields of v'' and v' respectively. We have the inclusions $R'' \subset R' \supset F(Z)$ and $[R' : F(Z)]$ is prime to p. By [20, Ch. VI, Th. 3, Cor. 1],

$$\operatorname{tr.deg}_F(L'') \geq \operatorname{tr.deg}_F(R'') + \operatorname{rank}(v'') = \operatorname{tr.deg}_F(R'') + s. \tag{2}$$

As $P(L'') \neq \emptyset$, there is an F-place $F(P) \rightsquigarrow L''$. Composing it with the place $L'' \rightsquigarrow R''$ given by v'', we get an F-place $F(P) \rightsquigarrow R''$. As P is complete, we have $P(R'') \neq \emptyset$, i.e., R'' is a splitting field of P.

We prove that R'' is a p-generic splitting field of P. Let M be a splitting field of P. A regular system of parameters at the image of a morphism $\alpha : \operatorname{Spec} M \to P$ yields an F-place $F(P) \rightsquigarrow M$ that is a composition of places associated with discrete valuations (cf. [11, §1.4]). By [11, Lemma 3.2] applied to the restriction of α to $F(Z)$, there is a finite field extension M' of M and an F-place $R' \rightsquigarrow M'$. Restricting to R'' we get an F-place $R'' \rightsquigarrow M'$, i.e., R'' is a p-generic splitting field of P.

By the definition of the canonical p-dimension,

$$\operatorname{cdim}(P) = \operatorname{tr.deg}_F F(Z) = \operatorname{tr.deg}_F R' \geq \operatorname{tr.deg}_F(R'') \geq \operatorname{cdim}_p(P).$$

It follows that $\operatorname{tr.deg}_F(R'') = \operatorname{cdim}(P)$ by (1) and therefore, $\operatorname{tr.deg}_F(L'') \geq \operatorname{cdim}(P) + s$ by (2). The claim is proved.

It follows from the claim that $\operatorname{ed}_p(\mathcal{X}) \geq \operatorname{cdim}(P) + s$. □

4. Main theorem

The main result of the paper is the following

Theorem 4.1. *Let G be a p-group and F a field of characteristic different from p containing a primitive p-th root of unity. Then $\operatorname{ed}_p(G)$ over F is equal to $\operatorname{ed}(G)$ over F and coincides with the least dimension of a faithful representation of G over F.*

The rest of the section is devoted to the proof of the theorem. As was mentioned in the introduction, we have $\operatorname{ed}_p(G) \leq \operatorname{ed}(G) \leq \dim(V)$ for any faithful representation V of G over F. We shall construct a faithful representation V of G over F with $\operatorname{ed}_p(G) \geq \dim(V)$.

Denote by C the subgroup of all central elements of G of exponent p and set $H = G/C$, so we have an exact sequence

$$1 \to C \to G \to H \to 1. \tag{3}$$

Let $E \to \operatorname{Spec} F$ be an H-torsor and $\operatorname{Spec} F \to BH$ be the corresponding morphism. Set $\mathcal{X}^E := BG \times_{BH} \operatorname{Spec} F$. Then $\mathcal{X}^E$ is a gerbe over F banded by C and its class in $H^2(F, C)$ coincides with the image of the class of E under the connecting map $H^1(F, H) \to H^2(F, C)$ (cf. [13, Ch. 4, §2]). An object of $\mathcal{X}^E$ over a field extension L/F is a pair (E', α), where E' is a G-torsor over L and $\alpha : E'/C \xrightarrow{\sim} E_L$ is an isomorphism of H-torsors over L.

Alternatively, $\mathcal{X}^E = [E/G]$ with objects (over L) G-equivariant morphisms $E' \to E_L$, where E' is a G-torsor over L (cf. [19]).

A lower bound for $\operatorname{ed}(G)$ was established in [4, Prop. 2.20]. We give a similar bound for $\operatorname{ed}_p(G)$.

Theorem 4.2. *For any H-torsor E over F, we have $\operatorname{ed}_p(G) \geq \operatorname{ed}_p(\mathcal{X}^E)$.*

Proof. Let L/F be a field extension and $x = (E', \alpha)$ an object of $\mathcal{X}^E(L)$. Choose a field a field extension L'/L of degree prime to p and a subfield $L'' \subset L'$ over F such that $\operatorname{tr.deg}(L'') = \operatorname{ed}_p(E')$ and there is a G-torsor E'' over L'' with $E''_{L'} \simeq E'_{L'}$.

We shall write Z for the (zero-dimensional) scheme of isomorphisms $\operatorname{Iso}_{L''}(E''/C, E_{L''})$ of H-torsors over L''. The image of the morphism $\operatorname{Spec} L' \to Z$ over L'' representing the isomorphism $\alpha_{L'}$ is a one point set $\{z\}$ of Z. The field extension $L''(z)/L''$ is algebraic since $\dim Z = 0$.

The isomorphism $\alpha_{L'}$ descends to an isomorphism of the H-torsors E''/C and E over $L''(z)$. Hence the isomorphism class of $x_{L'}$ belongs to the image of the map $\widehat{\mathcal{X}}^E\big(L''(z)\big) \to \widehat{\mathcal{X}}^E(L')$. Therefore,

$$\operatorname{ed}_p(G) \geq \operatorname{ed}_p(E') = \operatorname{tr.deg}(L'') = \operatorname{tr.deg}\big(L''(z)\big) \geq \operatorname{ed}_p(x).$$

It follows that $\operatorname{ed}_p(G) \geq \operatorname{ed}_p(\mathcal{X}^E)$. □

Let $C^* := \operatorname{Hom}(C, \mathbf{G}_{\mathrm{m}})$ denote the character group of C. An H-torsor E over F yields a homomorphism

$$\beta^E : C^* \to \operatorname{Br}(F)$$

taking a character $\chi : C \to \mathbf{G}_{\mathrm{m}}$ to the image of the class of E under the composition

$$H^1(F, H) \xrightarrow{\partial} H^2(F, C) \xrightarrow{\chi_*} H^2(F, \mathbf{G}_{\mathrm{m}}) = \operatorname{Br}(F),$$

where ∂ is the connecting map for the exact sequence (3). Note that as $\mu_p \subset F^\times$, the intersection of $\operatorname{Ker}(\chi_*)$ over all characters $\chi \in C^*$ is trivial. It follows that the classes of splitting fields of the gerbe $\mathcal{X}^E$ and the subgroup $\operatorname{Im}(\beta^E)$ coincide. It follows that

$$\operatorname{cdim}_p(\mathcal{X}^E) = \operatorname{cdim}_p\big(\operatorname{Im}(\beta^E)\big). \tag{4}$$

Let $\chi_1, \chi_2, \dots, \chi_s$ be a basis of C^* over $\mathbb{Z}/p\mathbb{Z}$ such that $\{\beta^E(\chi_1), \dots, \beta^E(\chi_r)\}$ is a minimal basis of $\mathrm{Im}(\beta^E)$ for some r and $\beta^E(\chi_i) = 1$ for $i > r$. By Theorem 2.1, we have

$$\mathrm{cdim}_p\big(\mathrm{Im}(\beta^E)\big) = \Big(\sum_{i=1}^{r} \mathrm{ind}\,\beta^E(\chi_i)\Big) - r = \Big(\sum_{i=1}^{s} \mathrm{ind}\,\beta^E(\chi_i)\Big) - s. \tag{5}$$

In view of (4) and Theorems 3.1 and 4.2, we shall find an H-torsor E (over a field extension of F) so that the integer in (5) is as large as possible. Let U be a faithful representation of H and X an open subset of the affine space $\mathbb{A}(U)$ of U where H acts freely. Set $Y := X/H$. Let E be the generic fiber of the H-torsor $\pi : X \to Y$. It is a "generic" H-torsor over the function field $L := F(Y)$.

Let $\chi : C \to \mathbf{G}_{\mathrm{m}}$ be a character and $\mathrm{Rep}^{(\chi)}(G)$ the category of all finite dimensional representations ρ of G such that $\rho(c)$ is multiplication by $\chi(c)$ for any $c \in C$. Fix a representations $\rho : G \to \mathbf{GL}(W)$ in $\mathrm{Rep}^{(\chi)}(G)$. The conjugation action of G on $B := \mathrm{End}(W)$ factors through an H-action. By descent (cf. [13, Ch. 1, §2]), there is (a unique up to canonical isomorphism) Azumaya algebra $\mathcal{A}$ over Y and an H-equivariant algebra isomorphism $\pi^*(\mathcal{A}) \simeq B_X := B \times X$. Let A be the generic fiber of $\mathcal{A}$; it is a central simple algebra over $L = F(Y)$.

Consider the homomorphism $\beta^E : C^* \to \mathrm{Br}(L)$.

Lemma 4.3. *The class of A in $\mathrm{Br}(L)$ coincides with $\beta^E(\chi)$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & C & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \alpha} & & \\
1 & \longrightarrow & \mathbf{G}_{\mathrm{m}} & \longrightarrow & \mathbf{GL}(W) & \longrightarrow & \mathbf{PGL}(W) & \longrightarrow & 1
\end{array}$$

The image of the H-torsor $\pi : X \to Y$ under α is the $\mathbf{PGL}(W)$-torsor

$$E' := \mathbf{PGL}(W)_X/H \to Y$$

where $\mathbf{PGL}(W)_X := \mathbf{PGL}(W) \times X$ and H acts on $\mathbf{PGL}(W)_X$ by $h(a,x) = (ah^{-1}, hx)$. The conjugation action of $\mathbf{PGL}(W)$ on B gives rise to an isomorphism between $\mathbf{PGL}(W)_X$ and the H-torsor $\mathrm{Iso}_X\big(B_X, \mathrm{End}(W)_X\big)$ of isomorphisms between the (split) Azumaya $\mathcal{O}_X$-algebras B_X and $\mathrm{End}(W)_X$. Note that this isomorphism is H-equivariant if H acts by conjugation on B_X and trivially on $\mathrm{End}(W)_X$. By descent,

$$E' \simeq \mathrm{Iso}_Y\big(\mathcal{A}, \mathrm{End}(W)_Y\big).$$

Therefore, the image of the class of the torsor $E' \to Y$ under the connecting map for the bottom row of the diagram coincides with the class of the Azumaya algebra $\mathcal{A}$. Restricting to the generic fiber yields $[A] = \beta^E(\chi)$. □

Theorem 4.4. *For any character $\chi \in C^*$, we have $\mathrm{ind}\,\beta^E(\chi) = \min\dim(V)$ over all representations V in $\mathrm{Rep}^{(\chi)}(G)$.*

Proof. We follow the approach given in [12]. Let H act on a scheme Z over F. We also view Z as a G-scheme. Denote by $\mathcal{M}(G,Z)$ the (abelian) category of left G-modules on Z that are coherent $\mathcal{O}_Z$-modules (cf. [18, §1.2]). In particular, $\mathcal{M}(G,\operatorname{Spec} F) = \operatorname{Rep}(G)$, the category off all finite dimensional representations of G.

Note that C acts trivially on Z. For a character $\chi : C \to \mathbf{G}_{\mathrm{m}}$, let $\mathcal{M}^{(\chi)}(G,Z)$ be the full subcategory of $\mathcal{M}(G,Z)$ consisting of G-modules on which C acts via χ. For example, $\mathcal{M}^{(\chi)}(G,\operatorname{Spec} F) = \operatorname{Rep}^{(\chi)}(G)$.

We write $K_0(G,Z)$ and $K_0^{(\chi)}(G,Z)$ for the Grothendieck groups of $\mathcal{M}(G,Z)$ and $\mathcal{M}^{(\chi)}(G,Z)$ respectively.

Every M in $\mathcal{M}(G,Z)$ is a direct sum of unique submodules $M^{(\chi)}$ of M in $\mathcal{M}^{(\chi)}(G,Z)$ over all characters χ of C. It follows that

$$K_0(G,Z) = \coprod K_0^{(\chi)}(G,Z).$$

Let q be the order of G. By [17, Th. 24], every irreducible representation of G is defined over the field $F(\mu_q)$. Since F contains p-th roots of unity, the degree $[F(\mu_q) : F]$ is a power of p. Hence the dimension of any irreducible representation of G over F is a power of p. It follows by Lemma 4.3 that it suffices to show $\operatorname{ind}(A) = \gcd \dim(V)$ over all representations V in $\operatorname{Rep}^{(\chi)}(G)$.

The image of the map $\dim : K_0(A) \to \mathbb{Z}$ given by the dimension over L is equal to $\operatorname{ind}(A) \cdot \dim(W) \cdot \mathbb{Z}$. To finish the proof of the theorem it suffices to construct a surjective homomorphism

$$K_0\big(\operatorname{Rep}^{(\chi)}(G)\big) \to K_0(A) \tag{6}$$

such that the composition $K_0\big(\operatorname{Rep}^{(\chi)}(G)\big) \to K_0(A) \xrightarrow{\dim} \mathbb{Z}$ is given by the dimension times $\dim(W)$.

First of all we have

$$K_0\big(\operatorname{Rep}^{(\chi)}(G)\big) \simeq K_0^{(\chi)}(G,\operatorname{Spec} F). \tag{7}$$

Recall that X an open subset of $\mathbb{A}(U)$ where H acts freely. By homotopy invariance in the equivariant K-theory [18, Cor. 4.2],

$$K_0(G,\operatorname{Spec} F) \simeq K_0\big(G,\mathbb{A}(U)\big).$$

It follows that

$$K_0^{(\chi)}(G,\operatorname{Spec} F) \simeq K_0^{(\chi)}\big(G,\mathbb{A}(U)\big). \tag{8}$$

By localization [18, Th. 2.7], the restriction homomorphism

$$K_0^{(\chi)}\big(G,\mathbb{A}(U)\big) \to K_0^{(\chi)}(G,X). \tag{9}$$

is surjective.

Denote by $\mathcal{M}^{(1)}(G,X,B_X)$ the category of left G-modules M on X that are coherent $\mathcal{O}_X$-modules and right B_X-modules such that C acts trivially on M and the G-action on M and the conjugation G-action on B_X agree. The corresponding Grothendieck group is denoted by $K_0^{(1)}(G,X,B_X)$. For any object L in $\mathcal{M}^{(x)}(G,X)$, the group C acts trivially on $L\otimes_F W^*$ and B acts on the right on $L\otimes_F W^*$. We have Morita equivalence

$$\mathcal{M}^{(x)}(G,X) \xrightarrow{\sim} \mathcal{M}^{(1)}(G,X,B_X)$$

given by $L\mapsto L\otimes_F W^*$ (with the inverse functor $M\mapsto M\otimes_B W$). Hence

$$K_0^{(x)}(G,X)\simeq K_0^{(1)}(G,X,B_X). \tag{10}$$

Now, as C acts trivially on X and B_X, the category $\mathcal{M}^{(1)}(G,X,B_X)$ is equivalent to the category $\mathcal{M}(H,X,B_X)$ of left H-modules M on X that are coherent $\mathcal{O}_X$-modules and right B_X-modules such that the G-action on M and the conjugation G-action on B_X agree. Hence

$$K_0^{(1)}(G,X,B_X)\simeq K_0(H,X,B_X). \tag{11}$$

Recall that $Y=X/H$. By descent, the category $\mathcal{M}(H,X,B_X)$ is equivalent to the category $\mathcal{M}(Y,\mathcal{A})$ of coherent $\mathcal{O}_Y$-modules that are right $\mathcal{A}$-modules. Hence

$$K_0(H,X,B_X)\simeq K_0(Y,\mathcal{A}). \tag{12}$$

The restriction to the generic point of Y gives a surjective homomorphism

$$K_0(Y,\mathcal{A})\to K_0(A). \tag{13}$$

The homomorphism (6) is the composition of (7), (8), (9), (10), (11), (12) and (13). It takes the class of a representation V to the class in $K_0(A)$ of the generic fiber of the vector bundle $\big((V\otimes W^*)\times X\big)/H$ over Y of rank $\dim(V)\cdot\dim(W)$. □

Remark 4.5. The theorem holds with min replaced by the gcd (with the same proof) in a more general context when the sequence (3) is an arbitrary exact sequence of algebraic groups with C a central diagonalizable subgroup of G.

Example 4.6 (cf. [6], [4, §14], [16, Th. 7.3.8])**.** Let p be a prime integer, F be a field of characteristic different from p and C_m the cyclic group $\mathbb{Z}/p^m\mathbb{Z}$. Let $K=F(t_1,\dots,t_{p^m})$ and C_m act on the variables $t_1,\dots,t_{p^m}$ by cyclic permutations. Then K is a Galois C_m-algebra over K^{C_m}. Assume that F contains a primitive root of unity ξ_{p^k} for some k. The image of the class of K under the connecting map $H^1(F,C_m)\to H^2(F,C_k)\simeq \mathrm{Br}_{p^k}(F)$ for the exact sequence

$$1\to C_k\to C_n\to C_m\to 1,$$

where $n=k+m$, is the class of the cyclic algebra $A=\big(K/K^{C_m},\xi_{p^k}\big)$. The group C_n acts F-linearly on $F(\xi_{p^n})$ by multiplication by roots of unity making the F-space $F(\xi_{p^n})$ a faithful representation of C_n of the smallest dimension. By Theorem 4.4 and Remark 4.5, we have

$$\mathrm{ind}(A)=\big[F(\xi_{p^n}):F\big].$$

We can now complete the proof of Theorem 4.1. By Theorem 4.4, there are representations V_i in $\mathrm{Rep}^{(\chi_i)}(G)$ such that $\operatorname{ind}\beta^E(\chi_i) = \dim(V_i)$, $i = 1, \dots, s$. Let V be the direct sum of all the V_i. By Theorem 4.2 (applied to the group G over L and the generic torsor E), Theorem 3.1, (4) and (5), we have

$$\begin{aligned}\mathrm{ed}_p(G) \geq \mathrm{ed}_p(G_L) \geq \mathrm{ed}_p(\mathcal{X}^E) = \mathrm{cdim}_p(\mathcal{X}^E) + s = \mathrm{cdim}_p\big(\mathrm{Im}(\beta^E)\big) + s \\ = \sum_{i=1}^{s} \operatorname{ind}\beta^E(\chi_i) = \sum_{i=1}^{s} \dim(V_i) = \dim(V).\end{aligned}$$

Since $\chi_1, \chi_2, \dots, \chi_s$ generate C^*, the restriction of V on C is faithful. As every nontrivial normal subgroup of G intersects C nontrivially, the G-representation V is faithful. We have constructed a faithful representation V of G over F with $\mathrm{ed}_p(G) \geq \dim(V)$. The theorem is proved.

Remark 4.7. The proof of Theorem 4.1 shows how to compute the essential dimension of G over F. For every character $\chi \in C^*$ choose a representation $V_\chi \in \mathrm{Rep}^{(\chi)}(G)$ of the smallest dimension. It appears as an irreducible component of the smallest dimension of the induced representation $\mathrm{Ind}_C^G(\chi)$. We construct a basis $\chi_1, \dots, \chi_s$ of C^* by induction as follows. Let χ_1 be a nonzero character with the smallest $\dim(V_{\chi_1})$. If the characters $\chi_1, \dots, \chi_{i-1}$ are already constructed for some $i \leq s$, then we take for χ_i a character with minimal $\dim(V_{\chi_i})$ among all the characters outside of the subgroup generated by $\chi_1, \dots, \chi_{i-1}$. Then V is a faithful representation of the least dimension and $\mathrm{ed}(G) = \sum_{i=1}^{s} \dim(V_{\chi_i})$.

Remark 4.8. We can compute the essential p-dimension of an arbitrary finite group G over a field F of characteristic different from p. (We don't assume that F contains p-th roots of unity.) Let G' be a Sylow p-subgroup of G. One can prove that $\mathrm{ed}_p(G) = \mathrm{ed}_p(G')$ and $\mathrm{ed}_p(G')$ does not change under field extensions of degree prime to p. In particular $\mathrm{ed}_p(G') = \mathrm{ed}_p(G'_{F'})$ where $F' = F(\mu_p)$. It follows from Theorem 4.1 that $\mathrm{ed}_p(G)$ coincides with the least dimension of a faithful representation of G' over F'.

5. An application

Theorem 5.1. *Let G_1 and G_2 be two p-groups and F a field of characteristic different from p containing a primitive p-th root of unity. Then*

$$\mathrm{ed}(G_1 \times G_2) = \mathrm{ed}(G_1) + \mathrm{ed}(G_2).$$

Proof. The index j in the proof takes the values 1 and 2. If V_j is a faithful representation of G_j then $V_1 \oplus V_2$ is a faithful representation of $G_1 \times G_2$. Hence $\mathrm{ed}(G_1 \times G_2) \leq \mathrm{ed}(G_1) + \mathrm{ed}(G_2)$ (cf. [5, Lemma 4.1(b)]).

Denote by C_j the subgroup of all central elements of G_j of exponent p. Set $C = C_1 \times C_2$. We identify C^* with $C_1^* \oplus C_2^*$.

For every character $\chi \in C^*$ choose a representation $\rho_\chi : G_1 \times G_2 \to \mathbf{GL}(V_\chi)$ in $\mathrm{Rep}^{(\chi)}(G_1 \times G_2)$ of the smallest dimension. We construct a basis $\{\chi_1, \chi_2, \dots, \chi_s\}$ of C^* following Remark 4.7. We claim that all the χ_i can be chosen in one of the C_j^*. Indeed, suppose the characters $\chi_1, \dots, \chi_{i-1}$ are already constructed, and let χ_i be a character with minimal $\dim(V_{\chi_i})$ among the characters outside of the subgroup generated by $\chi_1, \dots, \chi_{i-1}$. Let $\chi_i = \chi_i^{(1)} + \chi_i^{(2)}$ with $\chi_i^{(j)} \in C_j^*$. Denote by ε_1 and ε_2 the endomorphisms of $G_1 \times G_2$ taking (g_1, g_2) to $(g_1, 1)$ and $(1, g_2)$ respectively. The restriction of the representation $\rho_{\chi_i} \circ \varepsilon_j$ on C is given by the character $\chi_i^{(j)}$. We replace χ_i by $\chi_i^{(j)}$ with j such that $\chi_i^{(j)}$ does not belong to the subgroup generated by $\chi_1, \dots, \chi_{i-1}$. The claim is proved.

Let W_j be the direct sum of all the V_{χ_i} with $\chi_i \in C_j^*$. Then the restriction of W_j on C_j is faithful, hence so is the restriction of W_j on G_j. It follows that $\operatorname{ed}(G_j) \le \dim(W_j)$. As $W_1 \oplus W_2 = V$, we have

$$\operatorname{ed}(G_1) + \operatorname{ed}(G_2) \le \dim(W_1) + \dim(W_2) = \dim(V) = \operatorname{ed}(G_1 \times G_2). \qquad \square$$

Corollary 5.2. *Let F be a field as in Theorem 5.1. Then*

$$\operatorname{ed}\big(\mathbb{Z}/p^{n_1}\mathbb{Z} \times \mathbb{Z}/p^{n_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{n_s}\mathbb{Z}\big) = \sum_{i=1}^{s} \big[F(\xi_{p^{n_i}}) : F\big].$$

Proof. By Theorem 5.1, it suffices to consider the case $s = 1$. This case has been done in [6]. It is also covered by Theorem 4.1 as the natural representation of the group $\mathbb{Z}/p^n\mathbb{Z}$ in the F-space $F(\xi_{p^n})$ is faithful irreducible of the smallest dimension (cf. Remark 4.6). $\square$

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