INVARIANTS OF ALGEBRAIC TORI OF DEGREE AND WEIGHT 2

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Abstract. We determine the group of cohomological invariants of algebraic tori of degree and weight 2.

1. Introduction

The notion of *invariant* of a group scheme G over a field F was defined by Serre as follows (see [8]). Consider the functor

$$\mathrm{Tors}_G : \mathit{Fields}_F \longrightarrow \mathit{Sets},$$

where Fields_F is the category of field extensions of F and field homomorphisms over F, taking a field K to the set $\mathrm{Tors}_G(K)$ of isomorphism classes of G-torsors (principal homogeneous G-spaces) over $\operatorname{Spec} K$. Let

$$H : \mathit{Fields}_F \longrightarrow \mathit{Abelian\ Groups}$$

be another functor. An H-*invariant* of G is then a morphism of functors

$$\mathrm{Tors}_G \longrightarrow H,$$

where we view H as a functor with values in Sets. We denote the group of H-invariants of G by $\mathrm{Inv}(G, H)$.

An invariant in $\mathrm{Inv}(G, H)$ is called *normalized* if it takes the class of trivial G-torsors to 0. The normalized invariants form a subgroup $\mathrm{Inv}_{\mathrm{nm}}(G, H)$ of $\mathrm{Inv}(G, H)$ and there is a natural isomorphism

$$\mathrm{Inv}(G, H) \simeq H(F) \oplus \mathrm{Inv}_{\mathrm{nm}}(G, H),$$

where $H(F)$ can be identified with the subgroup of *constant* invariants.

Example 1.1. Let M be a complex of Galois modules over F and let H be the functor taking a field K over F to the Galois cohomology group $H^d(K, M)$ for a fixed integer d. We write $\mathrm{Inv}^d(G, M)$ for the group of *cohomological* invariants $\mathrm{Inv}(G, H)$. In particular, $\mathrm{Inv}^d(G, \mathbb{Z}/n\mathbb{Z}(j))$ for an integer $n > 0$ and $\mathrm{Inv}^d(G, \mathbb{Q}/\mathbb{Z}(j))$ are the group of *degree* d and *weight* j invariants. (For the definition of $\mathbb{Z}/n\mathbb{Z}(j)$ and $\mathbb{Q}/\mathbb{Z}(j)$ see [10]. If n is prime to $\mathrm{char}(F)$, then $\mathbb{Z}/n\mathbb{Z}(j) = \mu_n^{\otimes j}$) We call the *type* of such invariants the pair of integers (d, j).

The type $(1, 1)$ invariants of algebraic tori (and more generally, groups of multiplicative type) were considered in [18]. Cohomological invariants of types $(2, 1)$ (Brauer invariants) and $(3, 2)$ invariants were studied in [1].

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In the present paper we study type $(2,2)$-invariants of algebraic tori. Since

$$H^2(F,\mathbb{Z}/n\mathbb{Z}(2))) \simeq K_2(F)/nK_2(F) \quad \text{and} \quad H^2(F,\mathbb{Q}/\mathbb{Z}(2)) \simeq K_2(F)\otimes\mathbb{Q}/\mathbb{Z}$$

for a field F (see [2, §2] and [11, Theorem 11.5]), we study invariants with values in the functors K_2/nK_2 and $K_2 \otimes \mathbb{Q}/\mathbb{Z}$.

If $1 \to T \to P \to S \to 1$ is a coflasque resolution of a torus T, we prove that there is an exact sequence

$$P^\circ(F)[n] \to T^\circ(F)[n] \xrightarrow{\lambda} \operatorname{Inv}_{\mathrm{nm}}(T, K_2/nK_2) \to 0,$$

where P° and T° are the tori dual to P and T respectively. Note that $T^\circ(F)[n]$ is the group of F-points of the finite group $T^\circ[n] = \operatorname{Ker}(T^\circ \xrightarrow{n} T^\circ)$ of multiplicative type.

We give two formulas for the map λ. One formula uses the self-symmetric pairing

$$H^1(F, S[n]) \otimes H^1(F, S^\circ[n]) \xrightarrow{\cup} K_2(F)/nK_2(F).$$

The group $T^\circ(F)_{\mathrm{tors}}$ is the union of the subgroups $T^\circ(F)[n]$ of elements of exponent n. Passing to the colimit over all n we prove that there is an exact sequence

$$P^\circ(F)_{\mathrm{tors}} \to T^\circ(F)_{\mathrm{tors}} \to \operatorname{Inv}_{\mathrm{nm}}(T, K_2 \otimes \mathbb{Q}/\mathbb{Z}) \to 0.$$

We use the following notations in the paper.

For a field F write F_{sep} for a separable closure of F and $\Gamma = \Gamma_F$ for the (absolute) Galois group of F_{sep}/F.

If A is an abelian group and n an integer, we write $A[n]$ for the kernel of $A \xrightarrow{n} A$ and A_{tors} for the subgroup of elements of finite order in A.

If X is a scheme and M is an étale sheaf of abelian groups on X, we write $H^d(X,M)$ for the degree d étale cohomology group of X with values in M. The scheme $X \times_F \operatorname{Spec} F_{\mathrm{sep}}$ is denoted by X_{sep}.

$K_r(F)$ denotes Milnor's K-groups of F (see [14]).

The *multiplicative* group $\mathbb{G}_m$ over F is $\operatorname{Spec} F[t,t^{-1}]$.

We write μ_n for the Γ-module of all n-th roots of unity in $F_{\mathrm{sep}}^\times$ and $\mu_n(F)$ for $\mu_n \cap F^\times$.

2. Preliminary results

2a. **Invariants over fields of positive characteristic.** The following statement reduces to the study of invariants to the case when n is prime to $\operatorname{char}(F)$.

Lemma 2.1. *Let F be a field of characteristic $p > 0$ and $n = mp^s$ for integers $n > 0$ and $s \geqslant 0$. Then the map*

$$p^s : \operatorname{Inv}_{\mathrm{nm}}(G, K_r/mK_r) \to \operatorname{Inv}_{\mathrm{nm}}(G, K_r/nK_r)$$

is an isomorphism for every smooth group G and $r \geqslant 0$.

Proof. The group K_r of a field of characteristic p has no p-torsion by [9, Theorem A], hence the sequence

$$0 \to K_r(K)/mK_r(K) \xrightarrow{p^s} K_r(K)/nK_r(K) \to K_r(K)/p^sK_r(K)$$

is exact for every field extension K/F. It follows that the sequence

$$0 \to \operatorname{Inv}_{\mathrm{nm}}(G, K_r/mK_r) \xrightarrow{p^s} \operatorname{Inv}_{\mathrm{nm}}(G, K_r/nK_r) \to \operatorname{Inv}_{\mathrm{nm}}(G, K_r/p^sK_r)$$

is also exact. It suffices to show that the last group in the sequence is trivial. This follows from the fact that every G-torsor over K is split over K_{sep} (since G is smooth) and the natural homomorphism $K_r(K)/p^sK_r(K) \to K_r(K_{\mathrm{sep}})/p^sK_r(K_{\mathrm{sep}})$ is injective by [9, Corollary 6.5]. □

2b. **Groups of multiplicative type.** Let A be an algebraic group of multiplicative type over a field F (see [13, Chapter 12]). For a field extension K/F we write $A^*(K)$ for the group of *characters* $\mathrm{Hom}_F(A \otimes_F \mathrm{Spec}\, K, \mathbb{G}_m)$ of A over K and A^*_{sep} for the Γ-module $A^*(F_{\mathrm{sep}})$. The group A can be reconstructed out of the group ring of A^*_{sep} via the formula

$$A = \mathrm{Spec}(F_{\mathrm{sep}}[A^*_{\mathrm{sep}}])^{\Gamma}.$$

Let n be a positive integer prime to $\mathrm{char}(F)$ such that $nA^*_{\mathrm{sep}} = 0$. We have

$$A(F_{\mathrm{sep}}) = \mathrm{Hom}(A^*_{\mathrm{sep}}, F^{\times}_{\mathrm{sep}}) = \mathrm{Hom}(A^*_{\mathrm{sep}}, \mu_n) = A^{*\vee}_{\mathrm{sep}} \otimes \mu_n,$$

where $A^{*\vee}_{\mathrm{sep}} := \mathrm{Hom}(A^*_{\mathrm{sep}}, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(A^*_{\mathrm{sep}}, \mathbb{Z}/n\mathbb{Z})$ is the dual module.

The *dual group* A° is a group of multiplicative type with character module $A^{*\vee}_{\mathrm{sep}}$. Then $A^{\circ}(F_{\mathrm{sep}}) = A^*_{\mathrm{sep}} \otimes \mu_n$.

If T is an algebraic torus (i.e., the character group T^*_{sep} is a lattice), we write T° for the *dual torus* with character Galois module $T^{*\vee} = \mathrm{Hom}(T^*, \mathbb{Z})$.

The kernel $T[n]$ of $T \xrightarrow{n} T$ taking t to t^n is a finite group of multiplicative type with character Galois module $T^*_{\mathrm{sep}}/nT^*_{\mathrm{sep}}$. We have $T[n]^{\circ} = T^{\circ}[n]$ and $T^{\circ}[n](F) = (T^*_{\mathrm{sep}} \otimes \mu_n)^{\Gamma}$, and $T[n](K) = T(K)[n]$ for any field extension K/F.

2c. **Cohomology of BT.** Let T be an algebraic torus over a field F. Choose a representation $\tau : T \to \mathrm{GL}(V)$, where V is a finite dimensional vector space over F such that there exists a T-invariant open subscheme U of the affine space of V such that $U(F) \neq \emptyset$, $\mathrm{codim}_V(V - U) \geqslant 2$ and there is a T-torsor $U \to X$. Such representations exist (see [17, Remark 1.4]). We can view X as an approximation of the classifying space BT (which we don't define). We will sometimes write $X = U/T$.

Borel construction yields a homomorphism $T^*(F) \to \mathrm{Pic}(X)$ taking a character χ to the class of the line bundle $(U \times \mathbb{A}^1)/T \to X$, where T acts on the affine line $\mathbb{A}^1$ via χ.

Lemma 2.2. *The map $T^*(F) \to \mathrm{Pic}(X)$ is an isomorphism.*

Proof. By [15, Proposition 6.10] applied to the T-torsor $U \to X$ there is an exact sequence

$$F[X]^{\times}/F^{\times} \to T^*(F) \to \mathrm{Pic}(X) \to \mathrm{Pic}(U),$$

where the middle map is given by Borel construction. The group $F[X]^{\times}/F^{\times}$ is isomorphic to a subgroup of $F[U]^{\times}/F^{\times}$. The latter group is trivial as $F[V]^{\times} = F^{\times}$ and the divisor groups of U and V are canonically isomorphic by assumption. Moreover, $\mathrm{Pic}(U) = 0$ since the restriction homomorphism $0 = \mathrm{Pic}(V) \to \mathrm{Pic}(U)$ is surjective. □

For a point $x \in X$ of codimension 1 we write $[x]$ for the character in $T^*(F)$ corresponding to the class of x in $\mathrm{Pic}(X)$ under the isomorphism in Lemma 2.2.

Lemma 2.3. *If F is separably closed and n is a positive integer prime* $\mathrm{char}(F)$*, then there is a natural isomorphism*

$$H^2(X, \mu_n) \xrightarrow{\sim} T^*(F)/nT^*(F).$$

Proof. The Kummer short exact sequence $1 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 1$ yields an exact sequence

$$H^1(X, \mathbb{G}_m) \xrightarrow{n} H^1(X, \mathbb{G}_m) \to H^2(X, \mu_n) \to H^2(X, \mathbb{G}_m)[n] = \operatorname{Br}(X)[n],$$

where $\operatorname{Br}(X)$ is the Brauer group of X. In view of Lemma 2.2, $H^1(X, \mathbb{G}_m) \simeq \operatorname{Pic}(X) \simeq T^*(F)$. It suffices to show that the group $\operatorname{Br}(X)[n]$ is trivial. By [15, Proposition 6.10] applied to the T-torsor $U \to X$ there is an exact sequence

$$\operatorname{Pic}(T) \to \operatorname{Br}(X) \to \operatorname{Br}(U).$$

The Picard group of T is trivial as T is split. It remains to show that $\operatorname{Br}(U)[n] = 0$. By the homotopy invariance property for the étale cohomology (see [12, Ch. VI, Corollary 4.20]), $\operatorname{Br}(V)[n] \simeq \operatorname{Br}(F)[n] = 0$ as F is separably closed. It follows from [5, Corollary 3.4.2] that $\operatorname{Br}(U)[n] \simeq \operatorname{Br}(V)[n] = 0$. □

Let n be a positive integer prime to $\operatorname{char}(F)$. By Lemma 2.3, we have a natural composition

$$\alpha : H^2(X, \mu_n^{\otimes 2}) \to H^2(X_{\text{sep}}, \mu_n^{\otimes 2})^\Gamma = (H^2(X_{\text{sep}}, \mu_n) \otimes \mu_n)^\Gamma \xrightarrow{\sim} (T^*_{\text{sep}} \otimes \mu_n)^\Gamma = T^\circ(F)[n].$$

Proposition 2.4. *Let n be a positive integer prime* $\operatorname{char}(F)$. *Then the sequence*

$$0 \to H^2(F, \mu_n^{\otimes 2}) \xrightarrow{\iota} H^2(X, \mu_n^{\otimes 2}) \xrightarrow{\alpha} T^\circ(F)[n] \to 1,$$

where ι is the pull-back with respect to the structure morphism $X \to \operatorname{Spec} F$, is exact.

Proof. Consider the Hochschild–Serre spectral sequence (see [12, Chapter III, Theorem 2.20])

$$E_2^{p,q} = H^p(F, H^q(X_{\text{sep}}, \mu_n^{\otimes 2})) \Rightarrow H^{p+q}(X, \mu_n^{\otimes 2}).$$

The Kummer sequence yields

$$H^1(X_{\text{sep}}, \mu_n^{\otimes 2}) = H^1(X_{\text{sep}}, \mu_n) \otimes \mu_n \simeq \operatorname{Pic}(X)[n] \otimes \mu_n = 0$$

as $\operatorname{Pic}(X)$ is torsion free by Lemma 2.2, hence $E_2^{p,1} = 0$ for all p. In particular, $E_\infty^{1,1} = E_2^{1,1} = 0$ and $E_\infty^{2,0} = E_2^{2,0} = H^2(F, \mu_n^{\otimes 2})$. Since X has a point over F, the edge homomorphism $H^*(F, \mu_n^{\otimes 2}) \to H^*(X, \mu_n^{\otimes 2})$ is injective. In view of Lemma 2.3 it follows that

$$E_\infty^{0,2} = E_2^{0,2} = H^2(X_{\text{sep}}, \mu_n^{\otimes 2})^\Gamma \simeq (T^*_{\text{sep}} \otimes \mu_n)^\Gamma = T^\circ(F)[n].$$

The result follows. □

It can be deduced from Proposition 2.4 that the group $H^2(X, \mu_n^{\otimes 2})$ is canonically independent of the choice of the representation τ and the open subscheme U.

3. The pairings

Let n be a positive integer prime to $\operatorname{char}(F)$ and let A be a group of multiplicative type such that $nA^*_{\text{sep}} = 0$. The natural map $A^{*\vee}_{\text{sep}} \otimes A^*_{\text{sep}} \to \mathbb{Z}/n\mathbb{Z}$ yields a pairing

$$A(F_{\text{sep}}) \otimes A^\circ(F_{\text{sep}}) \to \mu_n^{\otimes 2} \tag{3.1}$$

of Galois modules.

Let T be a torus over F and let Y be a scheme over F. The pairing (3.1) for $A = T[n]$:

$$T[n](F_{\text{sep}}) \otimes T^\circ[n](F_{\text{sep}}) \to \mu_n^{\otimes 2}$$

yields the cup-product

$$H^p(Y,T[n]) \otimes H^q(Y,T^\circ[n]) \xrightarrow{\cup} H^{p+q}(Y,\mu_n^{\otimes 2}).$$

We define a pairing

$$H^1(Y,T) \otimes T^\circ(F)[n] \xrightarrow{\diamond} H^2(Y,\mu_n^{\otimes 2})$$

by the formula $a \diamond b := \delta(a) \cup b_Y$, where b_Y is the image of b under the natural homomorphism $T^\circ(F)[n] \to H^0(Y,T^\circ[n])$ and

$$\delta : H^1(Y,T) \to H^2(Y,T[n])$$

is the connecting homomorphism for the exact sequence

$$1 \to T[n] \to T \xrightarrow{n} T \to 1.$$

Example 3.2. The T-torsor $U \to X = U/G$ yields a canonical element $a_{\mathrm{can}} \in H^1(X,T)$. We have the homomorphism

$$\beta : T^\circ(F)[n] \to H^2(X,\mu_n^{\otimes 2}), \quad b \mapsto a_{\mathrm{can}} \diamond b.$$

Taking $Y = \operatorname{Spec} K$ for a field extension K/F, we get a homomorphism

$$\lambda_n : T^\circ(F)[n] \to \mathrm{Inv}^2_{\mathrm{nm}}(T,\mu_n^{\otimes 2}) = \mathrm{Inv}_{\mathrm{nm}}(T, K_2/nK_2)$$

sending an element $b \in T^\circ(F)[n]$ to the normalized invariant defined by

$$H^1(K,T) \to H^2(K,\mu_n^{\otimes 2}), \quad a \mapsto a \diamond b$$

for a field extension K/F.

Let

$$1 \to T \to P \to S \to 1$$

be an exact sequence of tori with P a quasi-split torus.

The diagram with exact rows and columns

$$\begin{array}{ccccc}
T[n] & \hookrightarrow & P[n] & \twoheadrightarrow & S[n] \\
\downarrow & & \downarrow & & \downarrow \\
T & \hookrightarrow & P & \twoheadrightarrow & S \\
\downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} \\
T & \hookrightarrow & P & \twoheadrightarrow & S
\end{array}$$

yields a diagram

(3.3)
$$\begin{array}{ccccc}
H^0(K,S) & \longrightarrow & H^1(K,T) & \longrightarrow & H^1(K,P) \\
\downarrow & & \downarrow{\scriptstyle \delta} & & \\
H^1(K,S[n]) & \xrightarrow{\delta'} & H^2(K,T[n]) & &
\end{array}$$

with an exact row and anti-commutative square (see [4, Chapter V, Proposition 4.1]). Since $H^1(K,P) = 0$, for every $a \in H^1(K,T)$, there is $a' \in H^1(K,S[n])$ such that $\delta'(a') = \delta(a)$. Let $b \in T^\circ(F)[n] = H^0(F,T^\circ[n])$ and let $b' := \delta''(b) \in H^1(F,S^\circ[n])$, where δ'' is the connecting homomorphism for the exact sequence $1 \to S^\circ[n] \to P^\circ[n] \to T^\circ[n] \to 1$.

The map λ_n can be expressed in terms of the pairing

$$H^1(K, S[n]) \otimes H^1(K, S^\circ[n]) \xrightarrow{\star} H^2(K, \mu_n^{\otimes 2})$$

as follows.

Proposition 3.4. *The map λ_n takes an element $b \in T^\circ(F)[n]$ to the invariant $a \mapsto a' \star b'_K$, where $a \in H^1(K, T)$.*

Proof. By [4, Chapter XII, Proposition 6.1],

$$\lambda_n(b)(a) = a \diamond b = \delta(a) \cup b_K = \delta'(a') \cup b_K = a' \star \delta''(b_K) = a' \star b'_K. \qquad \square$$

Recall that we have a T-torsor $U \to X = U/T$.

Lemma 3.5. *The composition $T^\circ(F)[n] \xrightarrow{\beta} H^2(X, \mu_n^{\otimes 2}) \xrightarrow{\alpha} T^\circ(F)[n]$ is the identity map.*

Proof. As the map $T^\circ[n](F) \to T^\circ[n](F_{\mathrm{sep}})$ is injective, we may assume that F is separably closed. Consider the commutative diagram

$$\begin{array}{ccccc}
T^* \otimes H^1(X, T) & \xrightarrow{\cup} & H^1(X, \mathbb{G}_m) & \xrightarrow{\sim} & T^*(F) \\
\downarrow & & \downarrow & & \downarrow \\
T^* \otimes H^2(X, T[n]) & \xrightarrow{\cup} & H^2(X, \mu_n) & \xrightarrow{\sim} & T^*(F)/nT^*(F),
\end{array}$$

where the isomorphisms are given by Lemmas 2.2 and 2.3. For a character $b \in T^*(F)$, the image of $a_{\mathrm{can}} \cup b$ in $H^1(X, \mathbb{G}_m) = \mathrm{Pic}(X)$ is given by Borel construction, i.e., coincides with the image of b under the isomorphism in Lemma 2.2. Hence, the image of $a_{\mathrm{can}} \otimes b$ under the composition in the top row in the diagram is equal to b. The statement follows from commutativity of the diagram after tensoring with μ_n. $\square$

By Lemma 3.5, the map β is a splitting for the exact sequence in Proposition 2.4. Hence we have a natural isomorphism

$$(\iota, \beta) : H^2(F, \mu_n^{\otimes 2}) \oplus T^\circ(F)[n] \xrightarrow{\sim} H^2(X, \mu_n^{\otimes 2}).$$

The Bloch-Ogus spectral sequence (see [3, §3] and [6, §1])

$$E_1^{p,q} = \coprod_{x \in X^{(p)}} H^{q-p}(F(x), \mu_n^{\otimes(2-p)}) \Rightarrow H^{p+q}(X, \mu_n^{\otimes 2})$$

for the sheaf $\mu_n^{\otimes 2}$ on X (here $X^{(p)}$ is the set of points of X of codimension p) gives the bottom exact sequence of the commutative diagram

$$\begin{array}{ccccccc}
 & & H^2(F, \mu_n^{\otimes 2}) \oplus T^\circ(F)[n] & \xrightarrow{(1,\lambda_n)} & H^2(F, \mu_n^{\otimes 2}) \oplus \mathrm{Inv}^2_{\mathrm{nm}}(T, \mu_n^{\otimes 2}) & & \\
 & & {\scriptstyle (\iota,\beta)}\downarrow\wr & & \downarrow{\scriptstyle \theta} & & \\
\coprod_{x \in X^{(1)}} \mu_n(F(x)) & \xrightarrow{\rho} & H^2(X, \mu_n^{\otimes 2}) & \longrightarrow & H^2(F(X), \mu_n^{\otimes 2})) & \xrightarrow{\partial} & \coprod_{x \in X^{(1)}} H^1(F(x), \mu_n),
\end{array}$$

where the homomorphism

$$\theta : H^2(F, \mu_n^{\otimes 2}) \oplus \mathrm{Inv}^2_{\mathrm{nm}}(T, \mu_n^{\otimes 2}) = \mathrm{Inv}^2(T, \mu_n^{\otimes 2}) \to H^2(F(X), \mu_n^{\otimes 2})$$

evaluates an invariant at the generic fiber of the T-torsor $U \to X$. By Totaro's theorem [8, Part 1, Appendix C], θ yields an isomorphism between $\mathrm{Inv}^2(T, \mu_n^{\otimes 2})$ and $\mathrm{Ker}\,\partial$. The natural homomorphism $H^2(F, \mu_n^{\otimes 2}) \to H^2(F(X), \mu_n^{\otimes 2})$ is injective since $X(F) \neq \emptyset$. It follows that $\mathrm{Im}(\rho) \subset \mathrm{Im}(\beta) \simeq T^\circ(F)[n]$ and the sequence

$$\coprod_{x \in X^{(1)}} \mu_n(F(x)) \xrightarrow{\rho} T^\circ(F)[n] \xrightarrow{\lambda_n} \mathrm{Inv}^2_{\mathrm{nm}}(T, \mu_n^{\otimes 2}) \to 0 \tag{3.6}$$

is exact.

Remark 3.7. It follows from the surjectivity and the definition of λ_n that every invariant I in $\mathrm{Inv}^2_{\mathrm{nm}}(T, \mu_n^{\otimes 2})$ is *homomorphic*, i.e., the map $I(K) : H^1(K, T) \to H^2(K, \mu_n^{\otimes 2})$ is a homomorphism for every field extension K/F.

Let

$$1 \to T \to P \to S \to 1 \tag{3.8}$$

be a coflasque resolution of T, i.e., P is a quasi-split torus and S is a coflasque torus (see [7, §1]).

Proposition 3.9. *The image of ρ in (3.6) coincides with the image of $P^\circ(F)[n] \to T^\circ(F)[n]$.*

Proof. In the commutative diagram

$$\begin{array}{ccccc} & & P^\circ(F)[n] & \longrightarrow & \mathrm{Inv}^2_{\mathrm{nm}}(P, \mu_n^{\otimes 2}) \\ & & \downarrow & & \downarrow \\ \coprod_{x \in X^{(1)}} \mu_n(F(x)) & \xrightarrow{\rho} & T^\circ(F)[n] & \xrightarrow{\lambda_n} & \mathrm{Inv}^2_{\mathrm{nm}}(T, \mu_n^{\otimes 2}) \end{array}$$

the group $\mathrm{Inv}^2_{\mathrm{nm}}(P, \mu_n^{\otimes 2})$ is trivial since every P-torsor over a field is trivial. It follows that the image of $P^\circ(F)[n] \to T^\circ(F)[n]$ is contained in $\mathrm{Im}(\rho)$.

In order to prove the opposite inclusion fix a point $x \in X^{(1)}$ and write ρ_x for the component $\mu_n(F(x)) \to T^\circ(F)[n]$ of ρ. Let K be the subfield of $F(x)$ generated by all roots of unity in $\mu_n(F(x))$ over F. The finite field extension K/F is separable and we can view K as a subfield of F_{sep}.

Let $y \in X_K := X \times_F \mathrm{Spec}\, K$ be (the only) point in the image of the canonical morphism $\mathrm{Spec}\, F(x) \to X_K$. This is a point of codimension 1 in X_K over $x \in X$ such that $K(y) = F(x)$. Note that $\mu_n(K(y)) = \mu_n(K)$.

It follows from [12, Chapter VI, §6] that the image of the composition

$$\mathbb{Z}/n\mathbb{Z} \to H^2(X_K, \mu_n) \to H^2(X_{\mathrm{sep}}, \mu_n) \xrightarrow{\sim} T^*_{\mathrm{sep}}/nT^*_{\mathrm{sep}},$$

where the first map is the y-component of the homomorphism

$$E_1^{1,1} \to H^2(X_K, \mu_n)$$

arising from the Bloch-Ogus spectral sequence

$$E_1^{p,q} = \coprod_{y \in X_K^{(p)}} H^{q-p}(K(y), \mu_n^{\otimes(1-p)}) \Rightarrow H^{p+q}(X_K, \mu_n)$$

for the sheaf μ_n on X_K, is contained in the subgroup generated by $[y]$ in $T^*_{\mathrm{sep}}/mT^*_{\mathrm{sep}}$. Therefore, as $\mu_n(K(y)) = \mu_n(K)$, the image of $\rho_y : \mu_n(K(y)) \to T^\circ(K)[n]$ is contained in $[y] \otimes \mu_n(K)$.

The natural morphism $f : X_K \to X$ yields a commutative diagram

$$\begin{array}{ccccc} \rho_y : \mu_n(K(y)) & \longrightarrow & H^2(X_K, \mu_n^{\otimes 2}) & \longrightarrow & T^\circ(K)[n] \\ \| & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle N_{K/F}} \\ \rho_x : \mu_n(F(x)) & \longrightarrow & H^2(X, \mu_n^{\otimes 2}) & \longrightarrow & T^\circ(F)[n]. \end{array}$$

It follows that

$$\mathrm{Im}(\rho_x) \subset N_{K/F}([y] \otimes \mu_n(K)) \subset T^\circ(F)[n].$$

As S is a coflasque torus, the last term in the exact sequence

$$P^*(K) \to T^*(K) \to H^1(K, S^*_{\mathrm{sep}})$$

is trivial, hence the first map is surjective. This implies that

$$[y] \otimes \mu_n(K) \subset \mathrm{Im}(P^\circ(K)[n] \to T^\circ(K)[n]).$$

Applying $N_{K/F}$ we see that

$$\mathrm{Im}(\rho_x) \subset N_{K/F}(\mathrm{Im}(P^\circ(K)[n] \to T^\circ(K)[n])) \subset \mathrm{Im}(P^\circ(F)[n] \to T^\circ(F)[n])$$

for all $x \in X^{(1)}$, hence $\mathrm{Im}(\rho) \subset \mathrm{Im}(P^\circ(F)[n] \to T^\circ(F)[n])$. □

4. Main results

Let T be a torus over F and let n be a positive integer. We define the homomorphism

$$\lambda_n : T^\circ(F)[n] \to \mathrm{Inv}_{\mathrm{nm}}(T, K_2/nK_2)$$

as follows. Recall that this map is defined in Section 3 in the case n is prime to p if $p = \mathrm{char}(F) > 0$. Write $n = mp^s$ for an integer m prime to p and define λ_n as the composition

$$T^\circ(F)[n] = T^\circ(F)[m] \xrightarrow{\lambda_m} \mathrm{Inv}_{\mathrm{nm}}(T, K_2/mK_2) \xrightarrow{p^s} \mathrm{Inv}_{\mathrm{nm}}(T, K_2/nK_2).$$

Now we are ready to compute the group of invariants $\mathrm{Inv}_{\mathrm{nm}}(T, K_2/nK_2)$.

Theorem 4.1. *Let T be a torus over a field F and let $1 \to T \xrightarrow{\varepsilon} P \to S \to 1$ be a coflasque resolution of T. Then for a positive integer n the sequence*

$$P^\circ(F)[n] \xrightarrow{\varepsilon^\circ} T^\circ(F)[n] \xrightarrow{\lambda_n} \mathrm{Inv}_{\mathrm{nm}}(T, K_2/nK_2) \to 0,$$

is exact.

Proof. By Lemma 2.1 we may assume that n is prime to $\mathrm{char}(F)$. Now the statement follows from (3.6) and Proposition 3.9. □

Example 4.2. Let L/F be a finite separable field extension and let T be the norm one torus for L/F, i.e., T is the kernel of the norm homomorphism $N_{L/F} : R_{L/F}(\mathbb{G}_m) \to \mathbb{G}_m$, so that we have the exact sequence (3.8) with $P = R_{L/F}(\mathbb{G}_m)$ and $S = \mathbb{G}_m$. For a positive integer n, we have $S[n] = S^\circ[n] = \mu_n$. The dual torus T° is $R_{L/F}(\mathbb{G}_m)/\mathbb{G}_m$, hence

$$T^\circ(F)[n] = \{x \in L^\times : x^n \in F^\times\}/F^\times.$$

To compute $\delta''(xF^\times)$ in the case when n is prime to $\mathrm{char}(F)$, where δ'' is the connecting map

$$T^\circ(F)[n] \to H^1(F, S^\circ[n]) = H^1(F, \mu_n),$$

let $a := x^n \in F^\times$ and find $y \in F_{\mathrm{sep}}^\times$ such that $y^n = a$. Then the element $z := xy^{-1} \in P^\circ[n](F_{\mathrm{sep}})$ goes to $xF^\times$ under $P^\circ[n](F_{\mathrm{sep}}) \to T^\circ[n](F_{\mathrm{sep}})$, hence $\delta''(xF^\times)$ is represented by the 1-cocycle $\gamma \mapsto \gamma(z)z^{-1} = y\gamma(y^{-1})$. The identification $F^\times/F^{\times n} = H^1(F, \mu_n)$ takes $aF^{\times n}$ to the class of the cocycle $\gamma \mapsto \gamma(y)y^{-1}$. It follows that δ'' viewed as map $T^\circ(F)[n] \to F^\times/F^{\times n}$ takes $xF^\times$ to $b' := x^{-n}F^{\times n}$.

If K/F is a field extension and

$$\bar{a} = aN_{KL/K}(KL^\times) \in K^\times/N_{KL/K}(KL^\times) = H^1(K, T)$$

for an element $a \in K^\times$, then we get from the anti-commutative diagram (3.3) that $\delta(\bar{a}) = \delta(a')$, where $a' = a^{-1}K^{\times n} \in K^\times/K^{\times n} = H^1(K, \mu_n) = H^1(K, S[n])$. It follows from the Proposition 3.4 that

$$\lambda_n(xF^\times)(\bar{a}) = a' \star b' = \{a^{-1}, x^{-n}\} = \{a, x^n\} \in K_2(K)/nK_2(K)$$

under the identification of $H^2(K, \mu_n^{\otimes 2})$ with $K_2(K)/nK_2(K)$.

We have $P^\circ(F)[n] = P(F)[n] = \mu_n(L)$ and the map $x \mapsto x^n$ identifies the cokernel of $P^\circ(F)[n] \to T^\circ(F)[n]$ with the group $F^\times \cap L^{\times n}$. The isomorphism

$$F^\times \cap L^{\times n} \xrightarrow{\sim} \mathrm{Inv}^2_{\mathrm{nm}}(T, \mu_n^{\otimes 2})$$

takes an element $b \in F^\times \cap L^{\times n}$ to the invariant $\bar{a} \mapsto \{a, b\} \in K_2(K)/nK_2(K)$. Note that this formula holds for all $n > 0$.

In the following theorem we determine the group of normalized $(K_2 \otimes \mathbb{Q}/\mathbb{Z})$-invariants of a torus.

Theorem 4.3. *Let T be a torus over field F and let $1 \to T \xrightarrow{\varepsilon} P \to S \to 1$ be a coflasque resolution of T. Then the sequence*

$$P^\circ(F)_{\mathrm{tors}} \xrightarrow{\varepsilon^\circ} T^\circ(F)_{\mathrm{tors}} \xrightarrow{\lambda} \mathrm{Inv}_{\mathrm{nm}}(T, K_2 \otimes \mathbb{Q}/\mathbb{Z}) \to 0,$$

where λ is the colimit of λ_n, is exact.

Proof. Let $X = U/T$ be as in Section 2c. According to [1, Theorem 3.4] and [16, Theorem 2.1] the group of invariants $\mathrm{Inv}(T, K_2/nK_2)$ is naturally isomorphic to the kernel of the homomorphism

$$p_1^* - p_2^* : K_2(F(X)) \otimes \mathbb{Z}/n\mathbb{Z} \to K_2(F((U \times U)/T)) \otimes \mathbb{Z}/n\mathbb{Z},$$

where p_1 and p_2 are the two projections of $(U \times U)/T$ onto X. A similar statement holds for the group $\mathrm{Inv}(T, K_2 \otimes \mathbb{Q}/\mathbb{Z})$. Since the étale cohomology commutes with colimits the natural map

$$\mathrm{colim}_n\, \mathrm{Inv}_{\mathrm{nm}}(T, K_2/nK_2) \to \mathrm{Inv}_{\mathrm{nm}}(T, K_2 \otimes \mathbb{Q}/\mathbb{Z})$$

is an isomorphism. Taking the colimit of the exact sequences in Theorem 4.1, we get the statement. □

Example 4.4. In the notation of Example 4.2, passing to the colimit over all integers n prime to $\mathrm{char}(F)$ we get an isomorphism

$$(F^\times \otimes \mathbb{Q}) \cap (L^\times/\mu(L)) \xrightarrow{\sim} \mathrm{Inv}_{\mathrm{nm}}(T, K_2 \otimes \mathbb{Q}/\mathbb{Z}),$$

where the intersection is taken in $L^\times \otimes \mathbb{Q}$. Note the kernel of the natural homomorphism

$$\mathrm{Inv}_{\mathrm{nm}}(T, K_2/nK_2) \to \mathrm{Inv}_{\mathrm{nm}}(T, K_2 \otimes \mathbb{Q}/\mathbb{Z})$$

is isomorphic to $\mu_n(F) \cap L^{\times n}$ and it is not trivial in general.

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