THE CHOW RING OF THE CLASSIFYING SPACE OF THE UNITARY GROUP

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Abstract. We fill a gap in the literature by computing the Chow ring of the classifying space of the unitary group of a hermitian form on a finite dimensional vector space over a separable quadratic field extension.

Contents

1. Introduction

Let K/F be a separable quadratic extension of fields and let V be a vector space over K of a finite dimension $n \geq 1$. A *hermitian form* h on V is a K-linear in the second argument map $V \times V \to K$ such that $\sigma(h(u,v)) = h(v,u)$ for any $u, v \in V$, where σ is the nontrivial automorphism of K/F.

The form h yields a K-linear map ${}^{\sigma}V \to V^{\vee}$, $v \mapsto h(v, -)$, where $V^{\vee}$ is the K-vector space dual to V and the vector space ${}^{\sigma}V$ is obtained from V via the base change by σ; h is called *non-degenerate* if the map ${}^{\sigma}V \to V^{\vee}$ is an isomorphism. Below we assume that this is the case.

The *unitary group* $\mathrm{U}(h)$, defined as in [10, Example 29.19], is an affine algebraic group over F. Its group of F-points $\mathrm{U}(h)(F)$ consists of the automorphisms of the vector space V (over K) preserving h. The isomorphism classes of $\mathrm{U}(h)$-torsors over F correspond bijectively to the isomorphism classes of all non-degenerate hermitian forms on V, [10, Example 29.19].

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The group $\mathrm{U}(h)$ is a closed subgroup of the Weil transfer $\mathcal{R}\,\mathrm{GL}(V) = \mathcal{R}_{K/F}\,\mathrm{GL}(V)$ of the general linear group of V. (See Section 2 for general references on Weil transfers.) We also note that the algebraic group $\mathrm{U}(h)_K$ over K, obtained from $\mathrm{U}(h)$ by the base change K/F, is canonically isomorphic to $\mathrm{GL}(V)$.

For any affine algebraic group G over a field F, the Chow ring $\mathrm{CH}(BG)$ of the classifying space BG of G has been defined in [17]. Among the split reductive groups G, the ring $\mathrm{CH}(BG)$ is computed for any *special* G, i.e., for G possessing no non-trivial G-torsors over field extensions of F (see [3]). In particular, $\mathrm{CH}(B\mathrm{GL}(V))$ for the general linear group is computed: as shown (already in [17]), it turns out to be the polynomial ring over the integers $\mathbb{Z}$ in the Chern classes $c_1, \dots, c_n$ of the tautological representation of $\mathrm{GL}(V)$.

For the orthogonal group $\mathrm{O}(q)$ of a non-degenerate quadratic form q, the Chow ring $\mathrm{CH}(B\mathrm{O}(q))$ is computed in [17] as well (see also [13] and [12]),[1] making the orthogonal group $\mathrm{O}(q)$ a rare example of a non-special group for which the Chow ring of its classifying space is understood.

However the ring $\mathrm{CH}(BG)$ for the unitary group $G = \mathrm{U}(h)$ seems not to be even discussed in the literature.[2]

In the present paper, this gap is being filled. Our main result is the following theorem. To state it, we consider the *norm homomorphism*

$$N\colon \mathbb{Z}[c_1, \dots, c_n] = \mathrm{CH}(B\mathrm{GL}(V)) \to \mathrm{CH}(BG),$$

given by the extension K/F. Note that N is a homomorphism of additive groups of the rings, but its image is an ideal so that its cokernel is a ring. The same situation occurs when we restrict to the subrings given by the sum of the even homogeneous components:

$$N\colon \mathbb{Z}[c_1, \dots, c_n]^{\mathrm{even}} \to \mathrm{CH}(BG)^{\mathrm{even}}.$$

Since $\mathcal{R}\,\mathrm{GL}(V)$ is a closed subgroup of $\mathrm{GL}(\mathcal{R}V)$, where $\mathcal{R}V$ is V viewed as the vector space over F, we have a ($2n$-dimensional) representation $G \hookrightarrow \mathrm{GL}(\mathcal{R}V)$ of G that we call *tautological* and write $d_i \in \mathrm{CH}^i(BG)$ for its ith Chern class.

For any odd $m = 2i+1 \geq 1$, we define an element

$$e_m := c_0c_m + c_1c_{m-1} + \cdots + c_ic_{i+1} \in \mathbb{F}_2[c_1, \dots, c_n]^{\mathrm{odd}},$$

where $c_0 = 1$ and $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$. The superscript $(-)^{\mathrm{odd}}$ stands for the sum of odd degree homogeneous components of the corresponding graded ring (the degree of c_i is defined to be i). Since $c_i = 0$ for $i > n$, we have $e_m = 0$ for $m > 2n-1$. Let $E \subset \mathbb{F}_2[c_1, \dots, c_n]^{\mathrm{odd}}$ be the subgroup, generated by all e_jc^2, where $j \geq 1$ is odd and $c \in \mathbb{F}_2[c_1, \dots, c_n]$ is any monomial.

Theorem 1.1. (1) *The group* $\mathrm{CH}^{\mathrm{even}}(BG)$ *is free of torsion. The map*

$$N\colon \mathbb{Z}[c_1, \dots, c_n]^{\mathrm{even}} \to \mathrm{CH}^{\mathrm{even}}(BG)$$

is injective and its cokernel is the polynomial ring over $\mathbb{F}_2$ *in the even Chern classes* $d_2, d_4, \dots, d_{2n}$.

[1] For characteristic 2 see [9, Appendix B]. Note that the answer in characteristic 2 differs from the one in other characteristics.

[2] Unfortunately, the Chow ring with coefficients $\mathbb{Z}/2\mathbb{Z}$ of the *Nisnevich* classifying space of $\mathrm{U}(h)$, computed in [16] (for split h and in characteristic different from 2), is only a distant relative of our $\mathrm{CH}(B\mathrm{U}(h))$.

(2) *The group* $\mathrm{CH}^{\mathrm{odd}}(BG)$ *is* 2*-torsion. The map*

$$N' \colon \mathbb{F}_2[c_1, \ldots, c_n]^{\mathrm{odd}} \to \mathrm{CH}^{\mathrm{odd}}(BG),$$

induced by $N \colon \mathbb{Z}[c_1, \ldots, c_n]^{\mathrm{odd}} \to \mathrm{CH}^{\mathrm{odd}}(BG)$*, is surjective and* E *is its kernel.*

The tautological representation of the group G factors through the symplectic group of certain non-degenerate alternating bilinear form on $\mathcal{R}V$. By this reason, $d_m = 0$ for all odd $m \geq 1$ in any characteristic (Proposition 6.6).

In order to verify that there are no further relations between the elements of $\mathrm{CH}(BG)$, we consider their images under the ring homomorphism $\mathrm{CH}(BG) \to \mathrm{CH}(BT)$ for the torus $T \subset G$, given by a diagonalization of h. The ring $\mathrm{CH}(BT)$ is easy to compute (Lemma 6.2) and the homomorphism turns out to be injective (Theorem 7.6).

As an intermediate step, the Chow ring $\mathrm{CH}(\mathcal{R}\,\mathrm{GL}(V))$ of the classifying space of the Weil transfer of the general linear group (this Weil transfer is a special quasi-split non-split reductive algebraic group) is computed on the way (see Propositions 4.1 and 4.3).

2. Approximations of Weil transfers

Let K/F be a finite separable extension of fields. We write $\mathcal{R} = \mathcal{R}_{K/F}$ for the Weil transfer with respect to it (see [1, §7.6], [15, §4], [5, §1] for Weil transfer of varieties, [11, §2i] for Weil transfer of algebraic groups). Note that for any (quasi-)projective variety X over K, its Weil transfer $\mathcal{R}X$ exists (see [15, Corollary 4.8.1]) and is a (quasi-)projective F-variety.

For an arbitrary affine algebraic group G over the field K and an integer $l \geq 1$, let us consider a quasi-projective l*-approximation* U/G of the classifying space BG, see [9]. Thus U is an open G-invariant subset in a (finite dimensional) vector space V with a linear G-action such that $\mathrm{codim}_V(V \setminus U)$ is at least l and U is a G-torsor over a quasi-projective K-variety U/G. As shown in [17] (see also [9]), a quasi-projective l-approximation exists for any G and any l.

Recall that a morphism of (right) G-varieties $X \to Y$ over K with the trivial G-action on Y is called a G-torsor if it is faithfully flat and the induced morphism $X \times G \to X \times_Y X$, $(x, g) \mapsto (xg, x)$ is an isomorphism. So, applying the Weil transfer to the morphism of the K-varieties $U \to U/G$, we get a morphism of F-varieties $\mathcal{R}U \to \mathcal{R}(U/G)$ which is a torsor under the affine algebraic group $\mathcal{R}G$ over F.

Moreover, $\mathcal{R}U$ is an open $\mathcal{R}G$-invariant subset of the vector space $\mathcal{R}V$ over F (the space $\mathcal{R}V$ is endowed with the linear action of $\mathcal{R}G$ given by the action of G on V). Note that the Weil transfer $\mathcal{R}V$ of the vector space V over K is V itself considered as a vector space over F.

Finally, the codimension of $\mathcal{R}V \setminus \mathcal{R}U$ in $\mathcal{R}V$ coincides with $\mathrm{codim}_V(V \setminus U)$. Indeed, over K the variety $\mathcal{R}V$ and its open piece $\mathcal{R}U$ become the product ${}^{\sigma}V \times V$ and its open subset ${}^{\sigma}U \times U$, where given a K-variety X, we write ${}^{\sigma}X$ for the K-variety obtained by the base change via the non-trivial automorphism σ of K/F.

We have proved

Lemma 2.1. *If* U/G *is a quasi-projective* l*-approximation of* BG*, then* $\mathcal{R}(U/G) = \mathcal{R}U/\mathcal{R}G$ *is a quasi-projective* l*-approximation of* $B\mathcal{R}G$*.* □

3. Weil transfers of grassmannians

We fix a separable *quadratic* field extension K/F and write $\mathcal{R} = \mathcal{R}_{K/F}$ for the Weil transfer with respect to it. Let X be the grassmannian of the subspaces of a fixed dimension in a fixed (finite dimensional) vector space V over K.

The K-variety $(\mathcal{R}X)_K$ is identified with the product ${}^\sigma X \times X$. Thus ${}^\sigma X$ is the "same" as X grassmannian given (in place of V) by the vector space ${}^\sigma V$ obtained from V by the base change via σ. The isomorphism of the Chow rings $\mathrm{CH}({}^\sigma X) \simeq \mathrm{CH}(X)$ given by the isomorphism of the K-varieties ${}^\sigma X \simeq X$ given by any isomorphism of the vector spaces ${}^\sigma V \simeq V$ does not depend on the latter (see, e.g., [7, Corollary 4.2]). Therefore the rings $\mathrm{CH}({}^\sigma X)$ and $\mathrm{CH}(X)$ are canonically isomorphic.

Since the variety X is (absolutely) cellular (in the sense of [4, §66]), the external product homomorphism $\mathrm{CH}({}^\sigma X) \otimes \mathrm{CH}(X) \to \mathrm{CH}({}^\sigma X \times X)$ is an isomorphism. It follows that the Chow ring $\mathrm{CH}((\mathcal{R}X)_K)$ is identified with the tensor square $\mathrm{CH}(X) \otimes \mathrm{CH}(X)$.

The non-trivial automorphism σ of K/F induces an involution on the ring $\mathrm{CH}((\mathcal{R}X)_K)$, which we will denote by the same letter σ. Under the identification $\mathrm{CH}((\mathcal{R}X)_K) = \mathrm{CH}(X) \otimes \mathrm{CH}(X)$, the involution $\sigma : \mathrm{CH}(X) \otimes \mathrm{CH}(X) \to \mathrm{CH}(X) \otimes \mathrm{CH}(X)$ is given by the switch of the factors.

We work with the category of Chow motives defined as in [4, §64]. By [2] (see also [6]), the Chow motive of the projective F-variety $\mathcal{R}X$ decomposes into a finite direct sum, where each summand is a shift of the motive of $\operatorname{Spec} F$ or of the motive of $\operatorname{Spec} K$ (where $\operatorname{Spec} K$ is considered as an F-variety). This proves

Lemma 3.1. *The ring* $\mathrm{CH}((\mathcal{R}X))$ *is identified with the subring* $(\mathrm{CH}(X) \otimes \mathrm{CH}(X))^\sigma$ *of* σ*-invariant elements in* $\mathrm{CH}(X) \otimes \mathrm{CH}(X)$*, where* $\sigma\colon \mathrm{CH}(X) \otimes \mathrm{CH}(X) \to \mathrm{CH}(X) \otimes \mathrm{CH}(X)$ *is the switch involution.* □

Remark 3.2. Lemma 3.1 and its proof generalize to an arbitrary étale F-algebra K (in place of the quadratic separable field extension K/F) as follows (cf. [7, §3] treating the case where X is a projective space). Let us fix a separable closure $\bar{F}$ of the field F, write Γ for the Galois group of $\bar{F}/F$, and write S for the finite Γ-set of all F-algebra homomorphisms $K \to \bar{F}$. The ring $\mathrm{CH}((\mathcal{R}X)_{\bar{F}})$ is identified with the tensor product $\bigotimes_S \mathrm{CH}(X)$ of copies of $\mathrm{CH}(X)$ numbered by S, the change of field homomorphism

$$\mathrm{CH}(\mathcal{R}X) \to \mathrm{CH}((\mathcal{R}X)_{\bar{F}}) = \bigotimes_S \mathrm{CH}(X)$$

is injective, and its image is the subring of Γ-invariant elements in the product, where Γ acts by the permutations of the factors.

4. Weil transfer of the general linear group

We fix a separable quadratic field extension K/F, write $\mathcal{R} = \mathcal{R}_{K/F}$ for Weil transfer with respect to it, and consider the group $H = \mathcal{R}\,\mathrm{GL}(V)$, where V is a vector space over K of a finite dimension n. In this section, we investigate the Chow ring $\mathrm{CH}(BH)$ of the classifying space BH of H. Recall that for any algebraic group G and any i, the ith homogeneous component $\mathrm{CH}^i(BG)$ of the graded ring $\mathrm{CH}(BG)$ is defined in [17] as the ith Chow group $\mathrm{CH}^i(U/G)$ of an l-approximation U/G of BG, introduced in §2, with $l > i$.

The ring $\mathrm{CH}(B\mathrm{GL}(V))$ is computed in [17] (see also [9, Example 4.1]) using approximations of $B\mathrm{GL}(V)$ by the grassmannians of n-dimensional subspaces in vector spaces of higher dimensions. The answer is: $\mathrm{CH}(B\mathrm{GL}(V))$ is the polynomial ring $\mathbb{Z}[c_1,\dots,c_n]$, where $c_1,\dots,c_n$ are the Chern classes of the tautological (also called standard) representation of $\mathrm{GL}(V)$.

It follows by Lemma 2.1 that $\mathrm{CH}(BH)$ can be computed using Weil transfers of the grassmannians. By Lemma 3.1, we get

Proposition 4.1. *The change of field homomorphism*

$$\mathrm{CH}(BH)\to\mathrm{CH}(BH_K)=\mathrm{CH}(B\mathrm{GL}(V))\otimes\mathrm{CH}(B\mathrm{GL}(V))$$

is injective and identifies $\mathrm{CH}(BH)$ *with the subring* $(\mathrm{CH}(B\mathrm{GL}(V))\otimes\mathrm{CH}(B\mathrm{GL}(V)))^\sigma$ *of* σ*-invariant elements, where* σ *is the switch involution on the tensor product. In particular, the group* $\mathrm{CH}(BH)$ *is free of torsion.* □

Remark 4.2. Proposition 4.1 and its proof generalize to an arbitrary étale F-algebra K (in place of the quadratic separable field extension K/F) as follows (cf. [7, §3] treating the case of $n=1$). In notation of Remark 3.2, for $H=\mathcal{R}\,\mathrm{GL}_n$, the ring $\mathrm{CH}(BH_{\bar F})$ is identified with the tensor product $\bigotimes_S\mathrm{CH}(B\mathrm{GL}_n)$ of copies of the ring $\mathrm{CH}(B\mathrm{GL}_n)$ numbered by S, the change of field homomorphism $\mathrm{CH}(BH)\to\mathrm{CH}(BH_{\bar F})$ is injective, and its image is the subring of Γ-invariant elements in the tensor product, where Γ acts by the permutations of the factors.

Returning to the case of a separable quadratic field extension K/F and the group $H=\mathcal{R}\,\mathrm{GL}(V)$, the subring of σ-invariant elements from Proposition 4.1 is better understood as follows. Let N be the norm map $\mathrm{CH}(BH_K)\to\mathrm{CH}(BH)$. Its image $N\,\mathrm{CH}(BH_K)$ is an ideal of the ring $\mathrm{CH}(BH)$ containing the ideal $2\,\mathrm{CH}(BH)$. Under the identification of Proposition 4.1, the norm ideal $N\,\mathrm{CH}(BH_K)$ is the image of the self-map $\mathrm{id}+\sigma$ of the tensor product

$$\mathrm{CH}(B\mathrm{GL}(V))\otimes\mathrm{CH}(B\mathrm{GL}(V))=\mathbb{Z}[c'_1,\dots,c'_n,c_1,\dots,c_n].$$

By [7, Lemma 2.1], the quotient

$$\mathrm{CH}(BH)/N\,\mathrm{CH}(BH_K)$$

is the polynomial ring over $\mathbb{F}_2=\mathbb{Z}/2\mathbb{Z}$ in the monomials $c'_1c_1,\dots,c'_nc_n$. Finally, for any $i=1,\dots,n$, the $2i$th Chern class d_{2i} of the tautological representation of H equals

$$c'_0c_{2i}+c'_1c_{2i-1}+\dots+c'_ic_i+\dots+c'_{2i-1}c_1+c'_{2i}c_0=\\ N(c'_0c_{2i}+c'_1c_{2i-1}+\dots+c'_{i-1}c_{i+1})+c'_ic_i$$

and therefore is congruent modulo $N\,\mathrm{CH}(BH_K)$ to the product c'_ic_i. We proved

Proposition 4.3. *Let* K/F *be a separable quadratic extension of fields,* V *a finite-dimensional* K*-vector space,* H *the group* $\mathcal{R}\,\mathrm{GL}(V)$*, and* N *the norm map* $\mathrm{CH}(BH_K)\to\mathrm{CH}(BH)$*. The cokernel of* N *is a polynomial ring over* $\mathbb{F}_2$ *in the even Chern classes of the tautological representation* $H\hookrightarrow\mathrm{GL}(\mathcal{R}V)$*.* □

5. The unitary group

Let K/F be a separable quadratic field extension, V a vector space over K of a finite dimension $n \geq 1$, and h a non-degenerate hermitian form on V. Let G be the unitary group $\mathrm{U}(h)$. It is a closed subgroup of $\mathcal{R}\,\mathrm{GL}(V)$ and the quotient $\mathcal{R}\,\mathrm{GL}(V)/G$ is the F-variety of all non-degenerate hermitian forms on V, which is an open subset of the affine space (over F) of *all* (not necessarily non-degenerate) hermitian forms on V. This is similar to the situation with the orthogonal group described in [17, §15] (see also [8, §3]). Using [17, Proposition 14.2] (see also [8, Proposition 5.1])), we get the following result:

Proposition 5.1. *The ring homomorphism* $\mathrm{CH}(B\mathcal{R}\,\mathrm{GL}(V)) \to \mathrm{CH}(BG)$*, induced by the embedding* $G \hookrightarrow \mathcal{R}\,\mathrm{GL}(V)$*, is surjective.* □

We are now in a position to prove Theorem 1.1.

Beginning of Proof of Theorem 1.1. The cartesian square

$$\begin{array}{ccc} G_K = \mathrm{GL}(V) & \longrightarrow & (\mathcal{R}\,\mathrm{GL}(V))_K = \mathrm{GL}(V^\vee) \times \mathrm{GL}(V) \\ \big\downarrow & & \big\downarrow \\ G & \longrightarrow & \mathcal{R}\,\mathrm{GL}(V) \end{array}$$

(the horizontal arrows are the natural closed embeddings, the top one maps f to the pair $((f^\vee)^{-1}, f)$) yields a similar cartesian square of l-approximations of the classifying spaces (for any $l \geq 1$) and therefore a commutative square

$$\begin{array}{ccc} \mathrm{CH}(B\mathrm{GL}(V)) & \longleftarrow & \mathrm{CH}(B(\mathrm{GL}(V^\vee) \times \mathrm{GL}(V))) \\ \big\downarrow{\scriptstyle N} & & \big\downarrow{\scriptstyle N} \\ \mathrm{CH}(BG) & \longleftarrow & \mathrm{CH}(B\mathcal{R}\,\mathrm{GL}(V)). \end{array}$$

Under the identifications $\mathrm{CH}(B\mathrm{GL}(V)) = \mathbb{Z}[c_1, \dots, c_n]$ and

$$\mathrm{CH}(B(\mathrm{GL}(V^\vee) \times \mathrm{GL}(V))) = \mathrm{CH}(B\mathrm{GL}(V^\vee)) \otimes \mathrm{CH}(B\mathrm{GL}(V)) = \mathbb{Z}[c'_1, \dots, c'_n, c_1, \dots, c_n],$$

the top arrow of the square is the ring homomorphism mapping c'_i to $(-1)^i c_i$ and mapping c_i to c_i.

By Proposition 5.1, the bottom arrow in the square is surjective so that the induced ring homomorphism

$$\mathrm{CH}(B\mathcal{R}\,\mathrm{GL}(V))/N\,\mathrm{CH}(B\mathrm{GL}(V^\vee) \times B\mathrm{GL}(V)) \to \mathrm{CH}(BG)/N\,\mathrm{CH}(B\mathrm{GL}(V))$$

is surjective as well. It follows by Proposition 4.3 that the quotient

$$\mathrm{CH}(BG)/N\,\mathrm{CH}(B\mathrm{GL}(V))$$

is generated by the even Chern classes of the tautological representation. In particular, the norm map is surjective in odd degrees.

Replacing the norm maps by the change of field homomorphisms in the last square, we get another commutative square

$$\begin{array}{ccc} \mathbb{Z}[c_1,\dots,c_n] & \longleftarrow & \mathbb{Z}[c'_1,\dots,c'_n,c_1,\dots,c_n] \\ \uparrow{\scriptstyle \mathrm{res}} & & \uparrow{\scriptstyle \mathrm{res}} \\ \mathrm{CH}(BG) & \longleftarrow & \mathrm{CH}(B\mathcal{R}\,\mathrm{GL}(V)). \end{array}$$

Using it, we see that the composition $\mathrm{res}\circ N\colon \mathbb{Z}[c_1,\dots,c_n]\to\mathbb{Z}[c_1,\dots,c_n]$ coincides with $\mathrm{id}+\sigma$, where σ is the involution of the ring $\mathbb{Z}[c_1,\dots,c_n]$ mapping c_i to $(-1)^ic_i$ for any i. Therefore this composition is 0 in odd degrees and is injective (being the multiplication by 2) in even degrees. Since the composition $N\circ\mathrm{res}$ in the opposite order is also multiplication by 2, it follows that $2\,\mathrm{CH}^{\mathrm{odd}}(BG)=0$.

It also follows that in even degrees the homomorphism res is injective on the norm ideal.

The norm ideal vanishes under the composition

$$\mathrm{CH}(BG)\xrightarrow{\mathrm{res}}\mathrm{CH}(B(\mathrm{GL}(V)))\longrightarrow\mathrm{CH}(B\mathrm{GL}(V))/2\,\mathrm{CH}(B\mathrm{GL}(V)).$$

The even Chern classes $d_2,\dots,d_{2n}$ are mapped under this composition to the classes of the squares $c_1^2,\dots,c_n^2$ which are algebraically independent (over $\mathbb{F}_2$). Therefore the classes $d_2,\dots,d_{2n}$, viewed in the cokernel of the norm map, are algebraically independent as well. Since we already know that res is injective on $N\mathbb{Z}[c_1,\dots,c_n]^{\mathrm{even}}$, it follows that it is injective on the entire ring $\mathrm{CH}(BG)^{\mathrm{even}}$. In particular, the additive group of this ring is torsion free.

We proved Theorem 1.1 almost completely. The only missing part is the description of $\mathrm{Ker}\,N'$ and will be done in the end of §7. □

Remark 5.2. For given K/F and V, the ring $\mathrm{CH}(BG)$ does not depend on h. A general (and an a priori) explanation of this phenomenon is given in [12, Remark 4.2].

Of course, with Theorem 1.1 (as well as with Theorem 7.6) we also see that the ring $\mathrm{CH}(BG)$ does not depend on the fields K and F and, in particular, on their characteristic.

6. Torsion

Theorem 1.1 in particular claims that the subgroup $\mathrm{CH}(BG)_{\mathrm{tors}}$ of the elements of finite order has exponent 2 and coincides with the image of $\mathbb{Z}[c_1,\dots,c_n]^{\mathrm{odd}}$ under the norm map N. Here comes our first example of nonzero torsion:

Proposition 6.1. *For any $i\in\{2,\dots,n\}$, the element $N(c_i)\in\mathrm{CH}(BG)$ is nonzero.*

Proof. For even i, the order of the element $N(c_i)$ is infinite, because

$$\mathrm{res}(N(c_i))=2c_i.$$

Our main case of interest is the case of odd i: here $\mathrm{res}(N(c_i))=0$ implying that $2N(c_i)=0$. However the proof of $N(c_i)\neq 0$ we give below does not depend on the parity of i.

Choosing a diagonalization of h (which exists in any characteristic, see [14, Theorem 7.6.3]), we get an embedding $T\hookrightarrow G$ of the direct product T of n copies of the group U_1 – the unitary group of a 1-dimensional hermitian form. (The group U_1 can be also viewed as the *norm* 1 *torus* $\mathcal{R}^{(1)}(\mathbb{G}_{\mathrm{m}})$ of the separable quadratic extension K/F, defined as the kernel of the norm map $\mathcal{R}\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$.) Note that T is a maximal torus of the reductive group G. We will see later on (Theorem 7.6) that the induced homomorphism $\mathrm{CH}(BG) \to \mathrm{CH}(BT)$ injective.

Lemma 6.2. *The embedding $T \hookrightarrow \mathcal{R}\mathbb{G}_{\mathrm{m}}^{\times n}$ yields a surjective ring homomorphism*

$$\mathrm{CH}(B\mathcal{R}\mathbb{G}_{\mathrm{m}}^{\times n}) \to \mathrm{CH}(BT).$$

Its kernel is generated by $x'_1 + x_1, \dots, x'_n + x_n$. Here we identify $\mathrm{CH}(B\mathcal{R}\mathbb{G}_{\mathrm{m}}^{\times n})$ with the subring of the polynomial ring

$$\mathbb{Z}[x'_1, \dots, x'_n, x_1, \dots, x_n] = \mathrm{CH}(B\mathbb{G}_{\mathrm{m}}^{\times 2n}) = \mathrm{CH}(B(\mathcal{R}\mathbb{G}_{\mathrm{m}}^{\times n})_K)$$

consisting of the polynomials invariant under the involution σ exchanging x'_i with x_i for every i (cf. §4, see [7, §3]*).*

Proof. Since $(\mathcal{R}\mathbb{G}_{\mathrm{m}}^{\times n})/T = \mathbb{G}_{\mathrm{m}}^{\times n}$, we are in the situation of [8, Proposition 4.1]. □

Example 6.3. For $n = 1$, we have $G = T$. It follows that the ring $\mathrm{CH}(BG)$ is the polynomial ring over $\mathbb{Z}$ in the second Chern class of the tautological (2-dimensional) representation of G. The change of field homomorphism

$$\mathrm{CH}(BG) \to \mathrm{CH}(B\mathbb{G}_{\mathrm{m}})$$

is injective. (The first Chern class of the representation vanishes.)

We are now in a position to prove that $N(c_n) \neq 0$ (provided that $n \geq 2$). The image of $N(c_n)$ in $\mathrm{CH}(BT)$ coincides with the image of the polynomial $x'_1 \dots x'_n + x_1 \dots x_n$ under the homomorphism of Lemma 6.2. So, it suffices to show that this polynomial cannot be written in the form

$$x'_1 \dots x'_n + x_1 \dots x_n = (x'_1 + x_1)f_1 + \dots + (x'_n + x_n)f_n \tag{6.4}$$

with σ-invariant $f_1, \dots, f_n \in \mathbb{Z}[x'_1, \dots, x'_n, x_1, \dots, x_n]$. Recall that the subring of σ-invariant polynomials is generated by the products $x'_1x_1, \dots, x'_nx_n$ modulo the norm ideal, where the norm ideal here is the image of the norm homomorphism

$$\mathbb{Z}[x'_1, \dots, x'_n, x_1, \dots, x_n] = \mathrm{CH}(B(\mathcal{R}\mathbb{G}_{\mathrm{m}}^{\times n})_K) \to \mathrm{CH}(B\mathcal{R}\mathbb{G}_{\mathrm{m}}^{\times n}) = \mathbb{Z}[x'_1, \dots, x'_n, x_1, \dots, x_n]^{\sigma}$$

mapping $f \in \mathbb{Z}[x'_1, \dots, x'_n, x_1, \dots, x_n]$ to $f + \sigma(f)$.

Setting in (6.4) $x'_i = x_i$ for all i and dividing by 2 we get the relation

$$x_1 \dots x_n = x_1g_1 + \dots + x_ng_n \in \mathbb{Z}[x_1, \dots, x_n].$$

Because of the above description of σ-invariant elements, the polynomials $g_1, \dots, g_n$ reduced modulo 2 become polynomials in the squares of the variables. Switching to $\mathbb{F}_2[x_1, \dots, x_n]$ and taking the derivative in x_1, we therefore get the relation

$$x_2 \dots x_n = g_1 \in \mathbb{F}_2[x_1, \dots, x_n]$$

which cannot hold for $n \geq 2$.

We proved that $N(c_n) \neq 0$ (for $n \geq 2$). To prove that $N(c_i) \neq 0$ for any $2 \leq i \leq n$, we simply choose a non-degenerate i-dimensional subform h' of h and apply the homomorphism $\mathrm{CH}(BG) \to \mathrm{CH}(BG')$ given by the embedding $G' := \mathrm{U}(h') \hookrightarrow \mathrm{U}(h)$. The image of $N(c_i) \in \mathrm{CH}(BG)$ is the nonzero $N(c_i) \in \mathrm{CH}(BG')$. □

Recall that for any $m \geq 0$, we write $d_m \in \mathrm{CH}^m(BG)$ for the mth Chern class of the tautological representation $G \hookrightarrow \mathrm{GL}(\mathcal{R}V)$.

Lemma 6.5. *For any odd $m = 2i+1 \geq 1$ we have*

$$d_m = d_{2i+1} = N(c_{2i+1} + c_1 c_{2i} + \cdots + c_i c_{i+1}).$$

Proof. Recall the commutative square

$$\begin{array}{ccc} \mathbb{Z}[c_1, \ldots, c_n] & \longleftarrow & \mathbb{Z}[c_1', \ldots, c_n', c_1, \ldots, c_n] \\ \big\downarrow N & & \big\downarrow N \\ \mathrm{CH}(BG) & \longleftarrow & \mathrm{CH}(B\mathcal{R}\,\mathrm{GL}(V)). \end{array}$$

The element d_m is the image of

$$N(c_{2i+1}' + c_{2i}' c_1 + \cdots + c_i' c_{i+1}).$$

Since the upper arrow maps c_j' to $(-1)^j c_j$ for any $j \geq 1$, we get that

$$d_m = N(-c_{2i+1} + c_{2i} c_1 + \cdots + (-1)^i c_i c_{i+1}).$$

Finally, since the norm of every summand is killed by 2, the signs do not matter. □

Proposition 6.6. $d_m = 0$ *for any odd $m \geq 1$.*

Proof. Let us consider the trace homomorphism (of the additive groups) $\mathrm{Tr}\colon K \to F$. Let $a \in K$ be a nonzero element of trace 0. The bilinear form

$$b\colon (u, v) \mapsto \mathrm{Tr}(ah(u, v))$$

on $\mathcal{R}V$ is then alternating and non-degenerate. The group G embeds into the symplectic group $\mathrm{Sp}(b)$ and the tautological representation of G factors through this embedding. By [17, §15] (see also [12]), all odd Chern classes of the tautological representation of $\mathrm{Sp}(b)$ vanish, giving the desired result. □

Remark 6.7. Note that the element $a \in K$ from the proof of Proposition 6.6, is unique up to multiplication by a nonzero element of F so that the described embedding $G \hookrightarrow \mathrm{Sp}(b)$ is canonical. In characteristic 2, one can take $a = 1$ in which case b becomes the (symmetric) bilinear form associated with the quadratic form q on $\mathcal{R}V$ given by h via the formula $q(v) = h(v, v)$. (The values of q, living a priori in K, are σ-invariant and therefore live in F.) In characteristic not 2, a is a generator of K/F with $a^2 \in F$.

Example 6.8. For $n = 2$, the group $\mathrm{CH}(BG)$ is torsion free. Indeed, by projection formula, since c_1^2 and c_2^2 are in the image of res, the ideal of the elements of finite order is generated by $N(c_1) = d_1 = 0$ and $N(c_1 c_2) = d_3 = 0$.

Since for $n = 1$ the group $\mathrm{CH}(BG)$ is also torsion free (cf. Example 6.3) whereas for any $n \geq 3$ it is not (by Proposition 6.1), we have determined the exact value of n starting from which the non-trivial torsion appears.

7. Norm kernel

In order to better understand the norm kernel for the unitary group $G = \mathrm{U}(h)$, we study the norm kernel for the torus T of Lemma 6.2.

We have two commutative squares with the same vertexes, the same upper arrows and the same lower arrows:

$$\begin{tikzcd} \mathrm{CH}(BT_K) \arrow[d, "N"] & \mathrm{CH}(B(\mathcal{R}\mathbb{G}_{\mathrm{m}}^{\times n})_K) \arrow[l] \arrow[d, "N"] \\ \mathrm{CH}(BT) & \mathrm{CH}(B(\mathcal{R}\mathbb{G}_{\mathrm{m}}^{\times n})) \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{CH}(BT_K) & \mathrm{CH}(B(\mathcal{R}\mathbb{G}_{\mathrm{m}}^{\times n})_K) \arrow[l] \\ \mathrm{CH}(BT) \arrow[u, "\mathrm{res}"] & \mathrm{CH}(B(\mathcal{R}\mathbb{G}_{\mathrm{m}}^{\times n})). \arrow[l] \arrow[u, "\mathrm{res}"] \end{tikzcd}$$

Both res are ring homomorphisms, the right one is injective. Both (the upper and the lower) horizontal arrows are surjective ring homomorphisms.

Let A be the polynomial ring $\mathbb{Z}[x_1', \dots, x_n', x_1, \dots, x_n]$ (which we view as the upper right vertex of the squares) and σ the involution on A satisfying $\sigma(x_i') = x_i$ for every i (viewed as the involution of the upper right vertex given by the non-trivial automorphism of K/F). The subgroup $A^\sigma \subset A$ of the σ-invariant elements in A is generated by $N(A)$, where

$$N := \mathrm{id}_A + \sigma \colon A \to A^\sigma$$

is the norm (group) homomorphism, and the monomials $(x_1'x_1)^{a_1} \dots (x_n'x_n)^{a_n}$ with all $a_1 \geq 0, \dots, a_n \geq 0$. (By Lemma 6.2, A^σ is the subring of A corresponding to the lower right vertex.) Let C be the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ (viewed as the upper left vertex of the squares). Consider the ideal $J \subset A^\sigma$ generated by $x_i' + x_i$ for all i. By Lemma 6.2, this ideal is the kernel of the lower arrow of the squares. We want to understand the torsion subgroup in the factor ring $B = A^\sigma/J$ (which – again by Lemma 6.2 – is the lower left vertex $\mathrm{CH}(BT)$). Note that the kernel of the upper arrow of the squares is also the ideal (now of the ring A instead of A^σ) generated by $x_i' + x_i$ for all i. The norm homomorphism $N \colon C \to B$, corresponding to $N \colon \mathrm{CH}(BT_K) \to \mathrm{CH}(BT)$, can be defined directly in terms of $N \colon A \to A^\sigma$ using the commutative square.

We write B_{tors} for the torsion subgroup of B. Besides, as usual, we write B^{odd} and C^{odd} for the sum of the odd degree components of the graded rings B and C.

Lemma 7.1. *The kernel of $N \colon C \to B$ is contained in C^{odd}. The group B_{tors} is 2-torsion and coincides with B^{odd} and with $N(C^{\mathrm{odd}})$.*

Proof. The ring homomorphism $\mathrm{res} \colon B \to C$, induced by the embedding $A^\sigma \hookrightarrow A$, takes the class of x_i' to $-x_i$ and takes the class of x_i to x_i for every i. Since C is free of torsion and $N \circ \mathrm{res}$ is multiplication by 2, the torsion subgroup of B is 2-torsion.

The composition $\mathrm{res} \circ N$ coincides with $\mathrm{id}_C + \sigma$, where the ring homomorphism $\sigma \colon C \to C$ is induced by the ring homomorphism $\sigma \colon A \to A$ and maps x_i to $(-1)^i x_i$ for any i. So, $\mathrm{res} \circ N$ vanishes on the odd degree homogeneous components and is injective (multiplication by 2) on the even degree components. In particular, the kernel of N is contained in C^{odd}. Since A^σ is additively generated by $N(A)$ and monomials in $x_1y_1, \dots, x_ny_n$, the additive group of B is generated by $N(g)$ and $g \cdot \sigma(g)$ for all $g \in C$. This implies that $B^{\mathrm{odd}} = N(C^{\mathrm{odd}})$. This also implies that res is trivial in odd degrees (hence $B_{\mathrm{tors}} \supset B^{\mathrm{odd}}$) and that res is injective in even degrees (hence B has no torsion in even degrees and so $B_{\mathrm{tors}} = B^{\mathrm{odd}}$). □

A monomial $x_1^{a_1}\dots x_n^{a_n}$ of odd degree is called an *almost square*, if all a_i but one are even. We are going to describe the kernel of the norm homomorphism $N\colon C\to B$. By Lemma 7.1, this kernel is concentrated in odd degrees.

Proposition 7.2. *Let $f\in C^{\mathrm{odd}}$. Then $N(f)=0$ if and only if the reduction $\bar f\in\mathbb{F}_2[x_1,\dots,x_n]$ of f modulo 2 is a sum of almost square monomials.*

Proof. Suppose $N(f)=0$ and consider f as a polynomial in A. Then $N(f)\in J$, i.e.,

$$N(f)=\sum_{i=1}^n (x_i'+x_i)g_i,\quad \text{where } g_i\in A^\sigma. \tag{7.3}$$

Note that if $g\in A^\sigma$ and we plug in $x_i'=x_i$ in g (for all i), we get a monomial in C that is contained in $2C+$ a sum of square monomials. (A square monomial is a monomial that is the square of a monomial.) So, plugging in $x_i'=x_i$ for all i in (7.3), we get

$$2f\in\sum_{i=1}^n 2x_i(2C+\text{a sum of square monomials}).$$

Dividing by 2 and reducing modulo 2 we see that $\bar f$ is a sum of almost square monomials.

Conversely, for $f\in C^{\mathrm{odd}}$, suppose $\bar f$ is a sum of almost square monomials. We will show that $N(f)=0$. If $f\in 2C$, i.e., $f=2g$ for some $g\in C^{\mathrm{odd}}$, then, since $N(g)$ is a 2-torsion element in B by Lemma 7.1, we have $N(f)=2N(g)=0$.

It remains to consider the case when f is an almost square monomial. We have $f=x_i\cdot x^2$ for some i and monomial x. Since x_j^2 is the image of the class of $-x_j'x_j$ under $\mathrm{res}\colon B\to C$, the monomial x^2 is in the image of res. The projection formula relating N and res shows that $N(f)$ is a multiple of $N(x_i)=0$. $\square$

We identify the polynomial ring $\mathbb{Z}[c_1,\dots,c_n]$ with the subring of symmetric polynomials in the polynomial ring $\mathbb{Z}[x_1,\dots,x_n]$ by viewing c_i as the ith elementary symmetric function in $x_1,\dots,x_n$. We think of $\mathbb{Z}[c_1,\dots,c_n]$ as of $\mathrm{CH}(B\mathrm{GL}(V))$ and we think of $\mathbb{Z}[x_1,\dots,x_n]$ as of $\mathrm{CH}(B\mathbb{G}_{\mathrm{m}}^{\times n})$. We have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{Z}[c_1,\dots,c_n]^{\mathrm{odd}} \arrow[r] \arrow[d, "N_c"'] & \mathbb{Z}[x_1,\dots,x_n]^{\mathrm{odd}} \arrow[d, "N_x"] \\
\mathrm{CH}^{\mathrm{odd}}(BG) \arrow[r] & \mathrm{CH}^{\mathrm{odd}}(BT),
\end{tikzcd}$$

where we added the indexes in the notation N_c and N_x for the norm maps in order to distinguish between them. We would like to determine the kernel of N_c. By Proposition 7.2, we know the kernel of N_x: it is (additively) generated by $2\mathbb{Z}[x_1,\dots,x_n]^{\mathrm{odd}}$ and the almost square monomials. Since $\mathrm{Ker}(N_c)$ also contains $2\mathbb{Z}[c_1,\dots,c_n]^{\mathrm{odd}}$, we may switch to the coefficients $\mathbb{F}_2$ and work with the commutative diagram

$$\begin{tikzcd}
\mathbb{F}_2[c_1,\dots,c_n]^{\mathrm{odd}} \arrow[r] \arrow[d, "N_c"'] & \mathbb{F}_2[x_1,\dots,x_n]^{\mathrm{odd}} \arrow[d, "N_x"] \\
\mathrm{CH}^{\mathrm{odd}}(BG) \arrow[r] & \mathrm{CH}^{\mathrm{odd}}(BT).
\end{tikzcd}$$

So, the kernel of the new N_x is generated by the almost square monomials. Let $D \subset \mathbb{F}_2[c_1,\dots,c_n]^{\mathrm{odd}}$ be the inverse image of $\mathrm{Ker}(N_x)$. Certainly, $\mathrm{Ker}(N_c) \subset D$. First we are going to construct generators for D and then we will see if each of these generators is in $\mathrm{Ker}(N_c)$.

For every odd $m = 2i+1 \geq 1$, let us consider the element

$$e_m := c_m + c_1c_{2i} + \ldots + c_ic_{i+1} \in \mathbb{F}_2[c_1,\dots,c_n]^{\mathrm{odd}}$$

introduced in §1. (In particular, $e_1 = c_1$.) Note that $N_c(e_m) = d_m \in \mathrm{CH}^m(BG)$ by Lemma 6.5.

By Proposition 6.6 and projection formula (used as in Remark 6.8), the group D contains the elements $e_m \cdot c^2$, where c is a monomial of the polynomial ring $\mathbb{F}_2[c_1,\dots,c_n]$ and $m \geq 1$ is odd.

Proposition 7.4. *The group D is generated by the elements $e_m \cdot c^2$ with odd $m \geq 1$.*

Proof. Let D' be the image of D in $\mathbb{F}_2[x_1,\dots,x_n]^{\mathrm{odd}}$. Thus, D' consists of all symmetric polynomials whose monomials are almost squares. The natural map $D \to D'$ is an isomorphism. We want to find bases for D and D'. For any odd $m \geq 1$, we write D_m for the degree m component of D and we write D'_m for the degree m component of D'.

A basis for D'_m consists of the symmetrizations of the monomials $x_1^{a_1}x_2^{a_2}x_3^{a_3}\dots x_n^{a_n}$ of degree m, where a_1 is odd and $a_2 \leq a_3 \leq \dots$ is a sequence of even numbers. Note that these symmetrizations are indeed linearly independent since they share no common monomial. For a given odd a_1 the number of such basis vectors is $l(\frac{m-a_1}{2})$, where $l(k)$ is the number of all partitions $k = b_2+\ldots+b_n$ with $0 \leq b_2 \leq \ldots \leq b_n$. (These are partitions of a length $\leq n-1$.) Thus,

$$\dim(D'_m) = l\big(\tfrac{m-1}{2}\big) + l\big(\tfrac{m-3}{2}\big) + \cdots + l(0).$$

Let us compute the number of polynomials $e_j \cdot c^2$ in D_m, where the index j here runs over all odd integers from 1 to (including) $2n-1$ (We exclude other j because $e_j = 0$ for $j \geq 2n+1$.) Note that these polynomials are linearly independent since they share no common monomial. The pairs (j,a) with $a \geq 0$ are in bijection $(j,a) \mapsto j+2na$ with all odd numbers ≥ 1. Therefore the number of polynomials $e_j \cdot c^2$ in D_m is

$$s\big(\tfrac{m-1}{2}\big) + s\big(\tfrac{m-3}{2}\big) + \cdots + s(0),$$

where $s(k)$ is the number of monomials in $c_1,\dots,c_{n-1}$ of degree k. (We use the integer a as exponent of c_n in the monomial c.) It is well known (see Lemma 7.5) that $s(k) = l(k)$. Therefore the polynomials $e_j \cdot c^2$ form a basis for D_m. □

Lemma 7.5. *For any $k \geq 0$ and any $n \geq 1$, the numbers $s(k)$ and $l(k)$, defined in the proof of Proposition 7.4, coincide.*

Proof. A monomial $c_1^{a_1}\dots c_{n-1}^{a_{n-1}}$ of degree $k = \sum_{i=1}^{n-1} ia_i$ determines a partition of k with a_i summands equal to i for $i = 1,\dots,n-1$. The conjugate partition of k has length at most $n-1$. The resulting map of the set of degree k monomials in $c_1,\dots,c_{n-1}$ into the set of partitions of length $\leq n-1$ of k is bijective. □

End of Proof of Theorem 1.1. Recall that the only missing part of Theorem 1.1 after the partial proof given in §5 is the equality $\mathrm{Ker}\, N_c = D$. Since $\mathrm{Ker}\, N_c \subset D$, we only need to check that the generators $e_m \cdot c^2$ of D, given in Proposition 7.4, are in $\operatorname{Ker} N_c$. By Proposition 6.6, the element e_m is in $\operatorname{Ker} N_c$. Since the element c^2 is in the image of res: $\mathrm{CH}(BG) \to \mathbb{Z}[c_1, \dots, c_n]$, the inclusion $e_m \cdot c^2 \in \operatorname{Ker} N_c$ follows from projection formula. $\square$

It is easy to see that the cokernel of the norm map $\mathbb{Z}[x_1, \dots, x_n] \to \mathrm{CH}(BT)$ can be viewed as the polynomial ring over $\mathbb{F}_2$ in the squares of the variables. Similarly the cokernel of the norm map $\mathbb{Z}[c_1, \dots, c_n] \to \mathrm{CH}(BG)$ can be viewed as the polynomial ring over $\mathbb{F}_2$ in the squares of $c_1, \dots, c_n$. The ring homomorphism $\mathrm{CH}(BG) \to \mathrm{CH}(BT)$ induces the ring homomorphism of the cokernels, mapping c_i^2 to the ith elementary symmetric polynomial in $x_1^2, \dots, x_n^2$. In particular, the homomorphism of the cokernels is injective.

From the commutative square

$$\begin{tikzcd}
\mathbb{Z}[c_1, \dots, c_n]^{\mathrm{even}} \arrow[r] \arrow[d] & \mathbb{Z}[x_1, \dots, x_n]^{\mathrm{even}} \arrow[d] \\
\mathrm{CH}^{\mathrm{even}}(BG) \arrow[r] & \mathrm{CH}^{\mathrm{even}}(BT)
\end{tikzcd}$$

with injective upper and (both) side maps, we see that the lower map is injective on the norm ideal.

Finally, we have just seen that the map $\mathrm{CH}^{\mathrm{odd}}(BG) \to \mathrm{CH}^{\mathrm{odd}}(BT)$ is injective.

Summarizing, we see that $\mathrm{CH}(BG)$ can be identified with a subring of

$$\mathrm{CH}(BT) = \mathbb{Z}[x_1', \dots, x_n', x_1, \dots, x_n]^\sigma / J :$$

Theorem 7.6. *The ring homomorphism $\mathrm{CH}(BG) \to \mathrm{CH}(BT)$ is injective. Moreover, $\mathrm{CH}(BG)$ is naturally isomorphic to the image of the homomorphism*

$$\mathbb{Z}[c_1', \dots, c_n', c_1, \dots, c_n]^\sigma \to \mathbb{Z}[x_1', \dots, x_n', x_1, \dots, x_n]^\sigma / J$$

taking c_i' (respectively, c_i) to the class of the ith elementary symmetric function in $x_1', \dots, x_n'$ (respectively, $x_1, \dots, x_n$). $\square$

Remark 7.7. Using Theorem 1.1 (describing the additive structure of $\mathrm{CH}(BG)$) and projection formula, it is also easy to understand the multiplicative structure of $\mathrm{CH}(BG)$. Indeed,

$$N(c) \cdot d_{2m} = N(c \cdot \operatorname{res}(d_{2m}))$$

for any $c \in \mathbb{Z}[c_1, \dots, c_n]$ and any $m \geq 0$, where

$$\operatorname{res}(d_{2m}) = c_0 c_{2m} - c_1 c_{2m-1} + \dots + (-1)^m c_m^2 + \dots - c_{2m-1} c_1 + c_{2m} c_0.$$

Similarly, one shows for any monomials $a, b \in \mathbb{Z}[c_1, \dots, c_n]$ that

$$N(a) \cdot N(b) = \begin{cases} 2N(ab), & \text{if } a \text{ and } b \text{ are of even degrees;} \\ 0, & \text{otherwise.} \end{cases}$$

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