COHOMOLOGICAL INVARIANTS OF CENTRAL SIMPLE ALGEBRAS

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1. Introduction

Cohomological invariants, introduced by J.-P. Serre in [10], allow us to study algebraic objects by means of Galois cohomology groups.

In this paper we study cohomological invariants of central simple algebras over field extensions of a base field F. The tautological degree 2 invariant takes a cental simple algebra A over a field K to its class $[A]$ in the Brauer group $\mathrm{Br}(K) = H^2(K, \mathbb{Q}/\mathbb{Z}(1))$. Using cup-products, one can construct invariants of higher degree: if $a \in F^\times$, then the cup-product $(a) \cup [A]$ yields a degree 3 invariant of A in $H^3(K, \mathbb{Q}/\mathbb{Z}(2))$. We call such degree 3 invariants decomposable. Are there indecomposable degree 3 invariants of central simple algebras?

We also study cohomological invariants of tuples of central simple algebras with linear relations in the Brauer group. For example, consider k-tuples of quaternion algebras $Q = (Q_1, Q_2, \dots, Q_k)$ over a field K such that

$$[Q_1] + [Q_2] + \cdots + [Q_k] = 0$$

in $\mathrm{Br}(K)$. It turns out that if $k \geq 3$, then there is a nontrivial degree 3 and exponent 2 indecomposable invariant of such tuples defined as follows. Let φ_j be the reduced norm quadratic form of Q_j. The sum φ of the forms φ_j in the Witt group $W(K)$ of K belongs to the cube of the fundamental ideal of $W(K)$. The Arason invariant of φ in $H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ yields a nontrivial degree 3 and exponent 2 nontrivial invariant Ar_k of k-tuples Q (see Example 7.2).

Let $n_1, n_2, \dots, n_k$ be a sequence of positive integers and $D \subset \coprod_{j=1}^{k} (\mathbb{Z}/n_j\mathbb{Z})$ a subgroup. For a field extension K/F, let $\mathrm{CSA}_D(K)$ be the set of isomorphism classes of k-tuples of central simple K-algebras $A = (A_1, A_2, \dots, A_k)$ with $\deg(A_j) = n_j$ such that $\sum_j d_j[A_j] = 0$ in the Brauer group $\mathrm{Br}(K)$ for all tuples $d = (d_j + n_j\mathbb{Z}) \in D$. Thus, D is the group of *relations* between the classes of the algebras A_j.

Let $d \in D$ be a relation such that $2d_j$ is divisible by n_j for every j, i.e., d is of exponent 2 in D. For every $A \in \mathrm{CSA}_D(K)$, the class $d_j[A_j]$ in $\mathrm{Br}(K)$ is represented by a quaternion algebra Q_j. Then the relation d yields a degree 3 nontrivial indecomposable invariant of $\mathrm{CSA}_D(K)$ taking a tuple A to the cohomology class $\mathrm{Ar}_k(Q)$. In Theorem 7.1, we prove that every degree 3 indecomposable invariant of CSA_D is of this form and compute the group of all

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invariants. In particular, we show that there are no nontrivial indecomposable invariants of k-tuples of simple algebras with relations for $k \leq 2$ and there are no nontrivial indecomposable invariants of CSA_D of odd exponent.

This result is similar to the one on the invariants of étale algebras: Serre proved (see [6, Part 1, Chapter VII]) that étale algebras have no cohomological invariants modulo odd primes, but there are nontrivial invariants of exponent 2 (the Stiefel-Whitney classes of the trace form of the algebra).

We use the following approach to the problem. For every group of relations D there is a reductive algebraic group G_{red} such that the set of isomorphism classes of G_{red}-torsors over an arbitrary field extension K over F is bijective to the set $\mathrm{CSA}_D(K)$. We study degree 3 cohomological invariants of CSA_D via the invariants of G_{red} using earlier results on degree 3 cohomological invariants of algebraic groups.

Note that every split semisimple group of type A (i.e., every connected component of the Dynkin diagram of G is A_n for some n) is embedded to a reductive group G_{red} corresponding to some group of relations D. Moreover, the group of invariants of G_{red} is identified with the subgroup of *reductive* invariants of G (see Section 3). Thus, we study degree three reductive cohomological invariants of all split semisimple groups of type A.

2. Preliminaries

2.1. **Symmetric square and Tor groups.** Let A be an abelian group. We write $\mathsf{S}^2(A)$ for the *symmetric square* of A, the factor group of $A \otimes A$ by the image of $1-\sigma$, where $\sigma : A \otimes A \to A \otimes A$ is the *exchange* map, $\sigma(a \otimes a') = a' \otimes a$. We write aa' for the image of $a \otimes a'$ in $\mathsf{S}^2(A)$.

The *polar* homomorphism

$$\mathrm{pol}_A : \mathsf{S}^2(A) \to A \otimes A$$

is defined by $\mathrm{pol}(aa') = a \otimes a' + a' \otimes a$.

Let A and B be two abelian groups. We write $A*B$ for the group $\mathrm{Tor}_1^{\mathbb{Z}}(A, B)$. The group $(\mathbb{Z}/n\mathbb{Z}) * (\mathbb{Z}/m\mathbb{Z})$ is cyclic of order $\gcd(n, m)$ with a canonical generator. Write w_n for the canonical generator of $(\mathbb{Z}/n\mathbb{Z}) * (\mathbb{Z}/n\mathbb{Z})$. If $a \in A$ and $b \in B$ are two elements of exponent n, we write $[a, n, b]$ for the image of w_n under the homomorphism $(\mathbb{Z}/n\mathbb{Z}) * (\mathbb{Z}/n\mathbb{Z}) \to A * B$ given by the maps $a : \mathbb{Z}/n\mathbb{Z} \to A$ and $b : \mathbb{Z}/n\mathbb{Z} \to B$. The elements $[a, n, b]$ generate $A * B$ and are subject to the following relations (see [5, §62]):

1. $[a, n, b]$ is bi-additive in a and b,
2. $[a, nm, b] = [a, n, mb]$ if $na = 0$ and $nmb = 0$.

Let $\tau : A*A \to A*A$ be the exchange map. Write $\Delta^2(A)$ for the factor group of $A*A$ by $\mathrm{Ker}(1-\tau)$ and $\Sigma^2(A)$ for $(A*A)/\mathrm{Im}(1-\tau)$. If A is a cyclic group, we have $\tau = 1$ and $\Delta^2(A) = 0$. Moreover, $\Delta^2(A \oplus B) \simeq \Delta^2(A) \oplus (A * B) \oplus \Delta^2(B)$. It follows that if A is a direct sum of cyclic groups of order $n_1, n_2, \dots, n_k$, respectively, then $|\Delta^2(A)| = \prod_{i<j} d_{ij}$, where $d_{ij} = \gcd(n_i, n_j)$.

If A is a cyclic group, we have $\Sigma^2(A) = A * A$.

Lemma 2.1. *Let a be an element of prime order p in an abelian group A. Then $[a,p,a] \notin \operatorname{Im}(1-\tau)$, i.e., the coset of $[a,p,a]$ in $\Sigma^2(A)$ is not trivial.*

Proof. Let A' be the cyclic subgroup of A of order p generated by a. Choose a homomorphism $f: A \to C$ to a cyclic group C such that $f(a) \neq 0$, i.e., the composition $A' \hookrightarrow A \xrightarrow{f} C$ is injective. Then the composition

$$A' * A' = \Sigma^2(A') \to \Sigma^2(A) \to \Sigma^2(C) = C * C$$

is injective since Tor is a left exact functor. As $[a,p,a]$ is a generator of the cyclic group $A' * A'$ of order p, the class of $[a,p,a]$ in $\Sigma^2(A)$ is not trivial. □

An exact sequence of abelian groups

$$0 \to A \to B \xrightarrow{\varphi} C \to 0$$

(we identify A with a subgroup of B) yields a commutative diagram with the exact column

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \Delta^2(C) & \xrightarrow{\alpha} & A \otimes C & \xrightarrow{\beta} & S^2(B)/S^2(A) & \longrightarrow & S^2(C) & \longrightarrow & 0 \\
 & & {\scriptstyle\rho}\big\downarrow & & \big\| & & {\scriptstyle\delta}\big\downarrow & & {\scriptstyle\mathrm{pol}_C}\big\downarrow & & \\
0 & \longrightarrow & C * C & \xrightarrow{\gamma} & A \otimes C & \longrightarrow & B \otimes C & \longrightarrow & C \otimes C & \longrightarrow & 0 \\
 & & \big\downarrow & & & & & & & & \\
 & & \Sigma^2(C) & & & & & & & &
\end{array}$$

where $\rho = 1-\tau$, $\gamma(\varphi(b), n, c) = nb \otimes c$, $\alpha = \gamma \circ \rho$, $\beta(a \otimes \varphi(b)) = ab + S^2(A)$ and δ is given by the composition $(1_B \otimes \varphi) \circ \mathrm{pol}_B$.

Lemma 2.2. *If C is finite and B is a free abelian group of finite rank, then the two rows of the diagram are exact.*

Proof. The lower sequence is exact since B is free. The map α is injective since so are γ and ρ. The map $C \otimes C \to \operatorname{Coker}(\beta)$ taking $\varphi(b) \otimes \varphi(b')$ to the coset of bb' yields an inverse of the map $\operatorname{Coker}(\beta) \to S^2(C)$, whence the exactness of the top row in the term $S^2(B)/S^2(A)$. The top row is a complex that is acyclic in all terms but possibly $A \otimes C$.

Choose a $\mathbb{Z}$-basis $x_1, x_2, \dots, x_s$ for B such that $n_1x_1, n_2x_2, \dots, n_sx_s$ is a basis for A for some positive integers n_i. Then x_ix_j with $i \leq j$ is a basis for $S^2(B)$ and $n_in_jx_ix_j$ is a basis for $S^2(A)$. It follows that $|S^2(B)/S^2(A)| = (\prod_i n_i)^{s+1}$. We also have $|\Delta^2(A)| = \prod_{i<j} d_{ij}$, where $d_{ij} = \gcd(n_i, n_j)$ and $|A \otimes C| = (\prod_i n_i)^s$, $|S^2(C)| = (\prod_i n_i) \cdot \prod_{i<j} d_{ij}$. A calculation implies that $|\Delta^2(A)| \cdot |S^2(B)/S^2(A)| = |A \otimes C| \cdot |S^2(C)|$. This proves the exactness of the top row in the term $A \otimes C$. □

Suppose the conditions of Lemma 2.2 hold and we are given an element $q \in S^2(B)$ and we would like to know the conditions under which $q \in S^2(A)$. Clearly, the image of q in $S^2(C)$ should be trivial and $\delta(q)$ should be zero.

If these two conditions hold, a diagram chase yields a unique element $\varepsilon(q)$ in $\Sigma^2(C)$ such that $q \in S^2(A)$ if and only if $\varepsilon(q) = 0$.

Example 2.3. Let $B = \mathbb{Z}^n/\mathbb{Z}$ with $\mathbb{Z}$ embedded diagonally into $\mathbb{Z}^n$. Write $x_1, \dots, x_n$ for the canonical generators for B, so that $x_1 + \dots + x_n = 0$. Consider the quadratic form $q = -\sum_{i<j} x_i x_j \in S^2(B)$. Note that $2q = \sum_i x_i^2$. Let $A \subset B$ be a subgroup containing $x_i - x_j$ for all i and j and set $C = B/A$. Write $\bar{x}$ for the common image of the x_i's in C, so C is a cyclic group of exponent n generated by $\bar{x}$. Since $\sum_i (x_i - x_1)^2 = 2q + nx_1^2$ and $x_i - x_1 \in A$, we have $2q = -nx_1^2$ in $S^2(B)/S^2(A)$. Therefore, $\beta(nx_1 \otimes \bar{x}) = nx_1^2 = -2q$ and $nx_1 \otimes \bar{x} = \gamma[\bar{x}, n, \bar{x}]$. Thus, $\varepsilon(2q) = -[\bar{x}, n, \bar{x}]$ in $\Sigma^2(C)$.

Let A be an additively written abelian group. We write elements of the group ring $\mathbb{Z}[A]$ in the exponential form: $u = \sum_i r_i e^{a_i}$ for $r_i \in \mathbb{Z}$ and $a_i \in A$. The *rank* $\operatorname{rank}(u)$ of u is $\sum_i r_i$.

2.2. **Chern classes.** Let A be a lattice. There are (abstract) *Chern class* maps (see [8, §3c])

$$c_i : \mathbb{Z}[A] \to S^i(A).$$

The first Chern class $c_1 : \mathbb{Z}[A] \to A$ is a homomorphism, $c_1(\sum_i r_i e^{a_i}) = \sum_i r_i a_i$. The second Chern class satisfies

$$c_2\Big(\sum_i e^{a_i}\Big) = \sum_{i<j} a_i a_j.$$

and $c_2(u+v) = c_2(u) + c_1(u)c_1(v) + c_2(v)$.

Suppose A is a W-module for a finite group W. Then the Chern classes are W-equivariant. If $a \in A$, we write We^a for the sum $e^{a_1} + e^{a_2} + \dots + e^{a_k}$ in $\mathbb{Z}[A]$, where $\{a_1, a_2, \dots, a_k\}$ is the W-orbit of a. Then $We^a \in \mathbb{Z}[A]^W$ and

$$c_2(We^a) = \sum_{i<j} a_i a_j.$$

We write $\operatorname{Dec}(A)$ for the subgroup of $S^2(A)^W$ generated by $c_2(\mathbb{Z}[A]^W)$. The group $\operatorname{Dec}(A)$ is generated by elements of the following types (see [7, §5]):

1) $\sum_{i<j} a_i a_j$, where $\{a_i\}$ is the W-orbit of an element in A,

2) aa', where $a, a' \in A^W$.

Thus, $\operatorname{Dec}(G)$ is the subgroup of the "obvious" elements in $S^2(A)^W$.

If $A^W = 0$, the first Chern class is trivial on $\mathbb{Z}[A]^W$, hence the restriction $\mathbb{Z}[A]^W \to S^2(A)^W$ of c_2 is a homomorphism. We will use the following formula proved in [6, §10.14].

Lemma 2.4. *If $A^W = 0$, we have $c_2(uv) = c_2(u)\operatorname{rank}(v) + c_2(v)\operatorname{rank}(u)$ for all $u, v \in \mathbb{Z}[A]^W$.*

3. Cohomological invariants

Let $\Phi : \mathit{Fields}_F \longrightarrow \mathit{PSets}$ be a functor, where Fields_F is the category of field extensions of F, and PSets is the category of pointed sets. Let n and j be two integers. For a field extension K/F, write $H^n\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$ for the Galois cohomology group of the absolute Galois group of K with values in $\mathbb{Q}/\mathbb{Z}(j)$. If $p \neq \operatorname{char}(F)$, the p-primary component of $\mathbb{Q}/\mathbb{Z}(j)$ is defined as the colimit over n of the twisted groups of roots of unity $\mu_{p^n}^{\otimes j}$. If $p = \operatorname{char}(F) > 0$, then the definition of the p-primary component of the cohomology group $H^n\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$ requires special care (e.g., see [3, §3b]).

A *normalized degree n cohomological invariant of Φ with coefficients in* $\mathbb{Q}/\mathbb{Z}(j)$ is collection of maps of pointed sets

$$\Phi(K) \to H^n\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$$

for all field extensions K/F, natural in K, i.e., an invariant is a morphism of functors $\Phi \to H^n\big(-, \mathbb{Q}/\mathbb{Z}(j)\big)$. We write $\operatorname{Inv}^n\big(\Phi, \mathbb{Q}/\mathbb{Z}(j)\big)$ for the group of all normalized cohomological invariants of Φ of degree n with coefficients in $\mathbb{Q}/\mathbb{Z}(j)$.

The cup-product in cohomology yields a pairing

$$F^\times \otimes \operatorname{Inv}^{n-1}\big(\Phi, \mathbb{Q}/\mathbb{Z}(j-1)\big) \to \operatorname{Inv}^n\big(\Phi, \mathbb{Q}/\mathbb{Z}(j)\big).$$

The cokernel $\operatorname{Inv}^n\big(\Phi, \mathbb{Q}/\mathbb{Z}(j)\big)_{\mathrm{ind}}$ of this pairing is the group of *indecomposable* invariants.

Let G be a linear algebraic group over a field F and Φ_G is the functor taking a field K to the set $H^1(K, G)$ of isomorphism classes of principal homogeneous G-spaces (G-torsors) over K. A cohomological invariant of Φ_G is also called an *invariant of* G.

We write $\operatorname{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big)$ for $\operatorname{Inv}^n\big(\Phi_G, \mathbb{Q}/\mathbb{Z}(j)\big)$ and $\operatorname{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big)_{\mathrm{ind}}$ for $\operatorname{Inv}^n\big(\Phi_G, \mathbb{Q}/\mathbb{Z}(j)\big)_{\mathrm{ind}}$. By [3, Theorem 2.4], the group $\operatorname{Inv}^2\big(G, \mathbb{Q}/\mathbb{Z}(1)\big)$ is isomorphic to $\operatorname{Pic}(G)$ if G is reductive. In this paper we consider cohomological invariants of degree 3. The group of degree 3 indecomposable invariants of split reductive groups was computed in [7, Theorem 5.1.]:

Theorem 3.1. *Let G_{red} be a split reductive group, $T \subset G_{\mathrm{red}}$ a split maximal torus. Then there is an isomorphism*

$$\operatorname{Inv}^3\big(G_{\mathrm{red}}, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{ind}} \simeq S^2(T^*)^W / \operatorname{Dec}(T^*),$$

where W is the Weyl group of G and $\operatorname{Dec}(T^*)$ is the subgroup of $S^2(T^*)^W$ defined in Section 2.2.

Let G be a split semisimple group over a field F, $S \subset G$ a split maximal torus. A *reductive envelope* of G is a split reductive group G_{red} over F with the commutator subgroup G. Choose a split maximal $T \subset G_{\mathrm{red}}$ such that $T \cap G = S$. We have a natural homomorphism

$$\varphi : S^2(T^*)^W \to S^2(S^*)^W.$$

A reductive envelope $G_{\rm red}$ of G is called *strict* if the center of $G_{\rm red}$ is a torus (see [9, Section 9]). If $G_{\rm red}$ is strict, the image of φ is the smallest possible and it is independent of the choice of the strict envelope $G_{\rm red}$. We write $S^2(S^*)^W_{\rm red}$ for $\operatorname{Im}(\varphi)$.

We have the following commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[T^*]^W & \longrightarrow & \mathbb{Z}[S^*]^W \\ \downarrow{\scriptstyle c_2} & & \downarrow{\scriptstyle c_2} \\ S^2(T^*)^W & \xrightarrow{\varphi} & S^2(S^*)^W. \end{array}$$

The top homomorphism in the diagram is surjective (see the proof of [9, Lemma 5.2]). Hence, we have

$$\operatorname{Dec}(S^*) \subset S^2(S^*)^W_{\rm red} \subset S^2(S^*)^W.$$

By [9, Proposition 6.1], the restriction homomorphism

$$\operatorname{Inv}^3\big(G_{\rm red}, \mathbb{Q}/\mathbb{Z}(2)\big)_{\rm ind} \to \operatorname{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\rm ind}$$

is injective (see [9]). Its image is the subgroup $\operatorname{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\rm red}$ of *reductive* invariants. Thus, the reductive invariants of G are those indecomposable invariants of G that can be extended to indecomposable invariants of a strict envelope $G_{\rm red}$ of G.

Theorem 3.1 identifies $\operatorname{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\rm red}$ with the subgroup $S^2(S^*)^W_{\rm red}/\operatorname{Dec}(S^*)$ of $S^2(S^*)^W/\operatorname{Dec}(S^*)$.

Let G be a split semisimple simply connected group over F. Then $G = G_1 \times G_2 \times \cdots \times G_k$, where G_j are (almost) simple simply connected groups. Let $S_j \subset G_j$ be a maximal torus, W_j the Weyl group of G_j. Then $S = S_1 \times S_2 \times \cdots \times S_k$ is a maximal torus of G and $W = W_1 \times W_2 \times \cdots \times W_k$ is the Weyl group of G.

The group $S^2(S^*)^W$ can be viewed as the group of W-invariant integer quadratic forms on the lattice of co-characters S_*. The group $S^2(S^*)^W$ is free with a canonical basis $q_1, q_2, \dots, q_k$, where q_j is (the only) W_j-invariant quadratic form on $(S_j)_*$ that has value 1 on a short co-root of G_j (see [6, Part 2, §10]).

We have the second Chern class homomorphism (note that $(S^*)^W = 0$)

$$c_2 : \mathbb{Z}[S^*]^W \to S^2(S^*)^W.$$

If $u \in \mathbb{Z}[S^*]^W$, we write

$$c_2(u) = N_1(u)q_1 + N_2(u)q_2 + \cdots + N_k(u)q_k \tag{1}$$

with unique $N_j(u) \in \mathbb{Z}$.

4. Central simple algebras with relations

Let $n_1, n_2, \dots, n_k$ be positive integers and D a subgroup of $\coprod_{j=1}^k (\mathbb{Z}/n_j\mathbb{Z})$. Consider a functor

$$\operatorname{CSA}_D : \mathit{Fields}_F \longrightarrow \mathit{PSets}$$

that takes a field extension K/F to the set $\mathrm{CSA}_D(K)$ of k-tuples of central simple K-algebras $(A_1, A_2, \dots, A_k)$ with $\deg(A_j) = n_j$ such that $\sum_j d_j[A_j] = 0$ in the Brauer group $\mathrm{Br}(K)$ for all tuples $(d_j + n_j\mathbb{Z}) \in D$. We call D the *group of relations* between classes of central simple algebras.

We show that the functor Φ is isomorphic to the functor $\Phi_{G_{\mathrm{red}}}$ for a reductive group G_{red}. The group $\coprod_{j=1}^{k}(\mathbb{Z}/n_j\mathbb{Z})$ is the character group of $\boldsymbol{\mu} := \prod_{j=1}^{k} \boldsymbol{\mu}_{n_j}$. Let $Z \subset \boldsymbol{\mu}$ be a subgroup such that $Z^* = \boldsymbol{\mu}^*/D$.

Write G for the factor group of the product $\prod_{j=1}^{k} \mathbf{SL}_{n_j}$ by Z and set $G_{\mathrm{red}} = (\prod_{j=1}^{k} \mathbf{GL}_{n_j})/Z$. Then G_{red} is a strict envelope of G. Note that D is naturally isomorphic to the character group of the center of G.

The natural surjection $G_{\mathrm{red}} \to \prod_{j=1}^{k} \mathbf{PGL}_{n_j}$ yields a map

$$\rho : H^1(K, G_{\mathrm{red}}) \to \prod_{j=1}^{k} H^1(K, \mathbf{PGL}_{n_j})$$

for every field extension K/F. Recall that the set $H^1(K, \mathbf{PGL}_n)$ is naturally bijective to the set of isomorphism classes of central simple algebras of degree n. Therefore, a G_{red}-torsor over K yields a tuple of central simple K-algebras $(A_1, A_2, \dots, A_k)$ with $\deg(A_j) = n_j$.

Proposition 4.1. [4, Theorem A1] *The map ρ establishes a bijection between $\Phi_{G_{\mathrm{red}}}(K) = H^1(K, G_{\mathrm{red}})$ and the set $\mathrm{CSA}_D(K)$ for every field extension K/F.*

The group of invariants $\mathrm{Inv}^n\big(G_{\mathrm{red}}, \mathbb{Q}/\mathbb{Z}(j)\big)$ is identified with the subgroup of reductive invariants $\mathrm{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big)_{\mathrm{red}}$ in $\mathrm{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big)$. Thus, we can view $\mathrm{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big)_{\mathrm{red}}$ as the group of cohomological invariants of the set of k-tuples of central simple algebras of given degrees n_j and satisfying linear relations given by the group of relations D.

5. Simple groups of type A

5.1. **Case $G = \mathbf{SL}_n$.** Write B for the character group of the maximal torus of diagonal matrices. Then $B = \mathbb{Z}^n/\mathbb{Z} = \sum_i \mathbb{Z}x_i$ (see Example 2.3) and B is the weight lattice of the root system A_{n-1}. The root sublattice $\Lambda_r \subset B$ is generated by roots $x_i - x_j$. The Weyl group W is the symmetric group S_n acting by permutations on the x_i's. The factor group B/Λ_r is equal to $(\mathbb{Z}/n\mathbb{Z})\hat{x}$, where $\hat{x}$ is the class of x_i (it is independent of i). For every character $y \in B$ we write $\hat{y} = a\hat{x}$ for its residue in $(\mathbb{Z}/n\mathbb{Z})\hat{x}$.

Choose a character $y \in B$. Some of the components of y may coincide. Let y have distinct components $a_1 > a_2 > \cdots > a_k$ which repeat $r_1, r_2, \dots, r_k$ times respectively, so that $n = \sum r_i$ and $\hat{y} = a\hat{x}$ with $a = \sum_i r_i a_i$. We denote the character y by $(r_1, \dots, r_k; a_1, \dots, a_k)$ or simply by $(\mathbf{r}, \mathbf{a})$ (see [6, Part 2, §11]).

The stabilizer of $y = (\mathbf{r}, \mathbf{a})$ in the Weyl group $W = S_n$ is isomorphic to the product $S_{r_1} \times S_{r_2} \times \cdots \times S_{r_k}$ of symmetric groups. Hence the rank of We^y, i.e., the number of characters in the W-orbit of y is equal to

$$\operatorname{rank}(We^y) = \frac{n!}{r_1!\, r_2! \cdots r_k!}. \tag{2}$$

Write v_p for the p-adic valuation for a prime p.

Lemma 5.1. *Let $\hat{y} = a\hat{x}$ for $y \in B$. Then $v_p(\operatorname{rank}(We^y)) \geq v_p(n) - v_p(a)$ for every prime p.*

Proof. Write $y = (\mathbf{r}, \mathbf{a})$ as above. Let $l = \min_i v_p(r_i)$. Since $a \equiv \sum_i r_i a_i$ modulo n and $n = \sum_i r_i \in p^l\mathbb{Z}$, we have $v_p(a) \geq l$. By [6, Lemma 11.3],

$$v_p\Big(\frac{n!}{r_1!\, r_2! \cdots r_k!}\Big) \geq v_p(n) - l.$$

The result follows from (2). □

Recall that $c_2(We^y) = N(We^y)q$, where $q = -\sum_{i<j} x_i x_j \in S^2(B)^W$ (see (1)).

Lemma 5.2. *Let $y \in B$ be such that $\hat{y} = a\hat{x}$ with $v_p(a) \leq v_p(n)$, then $v_p(N(We^y)) \geq v_p(a)$.*

Proof. Write $y = (\mathbf{r}, \mathbf{a})$. By [6, Lemma 11.4], the gcd of $\sum_i r_i a_i$ and n divides $v_p(N(We^y))$. Since $a \equiv \sum_i r_i a_i$ modulo n, the result follows from the assumption on a. □

The following statement shows that the inequalities in Lemmas 5.1 and 5.2 are sharp.

Lemma 5.3. *Let a be an integer with $v_p(a) < v_p(n)$ for a prime p. Then there is a character $y \in B$ such that*

1. *$\hat{y} = a\hat{x}$ in $(\mathbb{Z}/n\mathbb{Z})\hat{x}$,*
2. *$v_p(\operatorname{rank}(We^y)) = v_p(n) - v_p(a)$,*
3. *$v_p(c_2(We^y)) = v_p(a)$.*

Proof. Write $a = p^u v$ for an integer v prime to p and $u = v_p(a)$. Consider the character $z = x_1 + x_2 + \cdots + x_{p^u} \in B$. By [2, Section 4.2], we have $v_p(c_2(We^z)) = v_p(a)$. If $y := vz$ then $\hat{y} = vp^u\hat{x} = a\hat{x}$ and $c_2(We^y) = v^2 c_2(We^z)$, hence $v_p(c_2(We^y)) = v_p(c_2(We^z)) = v_p(a)$. Finally, $\operatorname{rank}(We^x) = \binom{n}{p^u}$ and

$$v_p(\operatorname{rank}(We^y)) = v_p(\operatorname{rank}(We^x)) = v_p\binom{n}{p^u} = v_p(n) - u = v_p(n) - v_p(a). \quad □$$

5.2. **Case $G = \mathbf{SL}_n/\boldsymbol{\mu}_m$.** Let m be a divisor of n and set $G = \mathbf{SL}_n/\boldsymbol{\mu}_m$. Let $A \subset B = \mathbb{Z}^n/\mathbb{Z} = \sum_i \mathbb{Z}x_i$ be the character group of the maximal torus S of classes of diagonal matrices. Thus A is the subgroup of B containing the root lattice Λ_r. The factor group $C = B/A = (\boldsymbol{\mu}_m)^*$ is equal to $(\mathbb{Z}/m\mathbb{Z})\bar{x}$, where $\bar{x}$ is the coset $x_i + A$ in C. The Weyl group W trivially on C, hence A is a W-submodule of B. We have the following groups:

$$\operatorname{Dec}(A) \subset S^2(A)^W_{\mathrm{red}} \subset S^2(A)^W \subset S^2(B)^W = \mathbb{Z}q,$$

where $q = -\sum_{i<j} x_i x_j \in S^2(B)^W$.

Lemma 5.4. *If $kq \in S^2(A)^W_{\mathrm{red}}$, then $k \in m\mathbb{Z}$.*

Proof. The class $\bar{x}$ in $C = (\mathbb{Z}/m\mathbb{Z})\bar{x}$ of first fundamental weight x_1 of G has order m. By [9, Proposition 10.6] or [7, Proposition 7.1], k is divisible by m. □

Lemma 5.5. *We have $2nq \in \mathrm{Dec}(A)$.*

Proof. Consider the character $x = x_1 - x_2 \in A$. By [8, Section 4b], $c_2(We^x) = -2nq \in \mathrm{Dec}(A)$. □

Lemma 5.6. *1. For every odd prime p, there is an integer k prime to p such that $kmq \in \mathrm{Dec}(A)$.*

2. Suppose that either n is odd or $v_2(m) < v_2(n)$. Then there is an odd integer k such that $kmq \in \mathrm{Dec}(A)$.

Proof. Let p be a prime integer. Suppose first that $v_p(m) < v_p(n)$. Let $r = v_p(m)$. By Lemma 5.3 applied to the integer $a = m$, there is a character $y \in B$ such that $v_p(N(We^y)) = v_p(m)$ and $\bar{y} = m\bar{x} = 0$ in $(\mathbb{Z}/m\mathbb{Z})\bar{x}$. In particular, $y \in A$ and $c_2(We^y) = kmq$ with k prime to p.

Now let p be an odd prime with $v_p(m) = v_p(n)$. By Lemma 5.5, $(2n/m)mq \in \mathrm{Dec}(A)$ and $2n/m$ is prime to p.

Finally, let n be odd. We have $mx_1 \in A$ and $c_2(We^{mx_1}) = m^2 q \in \mathrm{Dec}(A)$ and m is odd as it divides n. □

Now we are going to use the invariant ε defined in Section 2.

Lemma 5.7. *If k is divisible by m and $v_2(m) = v_2(n) > 0$, we have $\varepsilon(kq) = \left[\frac{k}{2}\bar{x}, 2, \frac{k}{2}\bar{x}\right]$ in $\Sigma^2(C)$.*

Proof. Since n/m is odd and $m\bar{x} = 0$, we have by Example 2.3:

$$\varepsilon(mq) = -\frac{m}{2}\left[\bar{x}, n, \bar{x}\right] = -\left[\frac{m}{2}\bar{x}, 2, \frac{n}{2}\bar{x}\right] = -\left[\frac{m}{2}\bar{x}, 2, \frac{m}{2}\bar{x}\right] = \left[\frac{m}{2}\bar{x}, 2, \frac{m}{2}\bar{x}\right].$$

It follows that $\varepsilon(kq) = \left[\frac{k}{2}\bar{x}, 2, \frac{k}{2}\bar{x}\right]$ since both sides are equal to $\varepsilon(mq) = \left[\frac{m}{2}\bar{x}, 2, \frac{m}{2}\bar{x}\right]$ if k/m is odd and is equal to zero if k/m is even. □

Proposition 5.8. *Let $G = \mathbf{SL}_n/\boldsymbol{\mu}_m$ and S a maximal split torus of G. Then*

$$\mathrm{Dec}(S^*) = S^2(S^*)^W_{\mathrm{red}} = \begin{cases} 2m\mathbb{Z}q, & \text{if } v_2(m) = v_2(n) > 0; \\ m\mathbb{Z}q, & \text{otherwise.} \end{cases}$$

Proof. The second case follows from Lemmas 5.4 and 5.6. Suppose $v_2(m) = v_2(n) > 0$. It follows from Lemmas 5.5 and 5.6 that $2mq \in \mathrm{Dec}(A)$. It suffices to show that if $kq \in S^2(A)^W_{\mathrm{red}}$, then $k \in 2m\mathbb{Z}$. By Lemma 5.4, k is divisible by m. Recall that $\bar{x}$ has order m in $C = B/A$. In view of Lemma 5.7, $\varepsilon(kq) = \left[\frac{k}{2}\bar{x}, 2, \frac{k}{2}\bar{x}\right]$ in $\Sigma^2(C)$. By Lemma 2.1, $\frac{k}{2}\bar{x} = 0$ in C, i.e., $k \in 2m\mathbb{Z}$. □

It follows from Proposition 5.8 that every reductive invariant of $\mathbf{SL}_n/\boldsymbol{\mu}_m$ is trivial (see [7, §7]) or, equivalently, central simple algebras of degree n and exponent dividing m have no indecomposable degree 3 invariants.

6. Semisimple groups of type A

Let $n_1, n_2, \dots, n_k$ be positive integers and D a subgroup of relations in $\coprod_{j=1}^{k}(\mathbb{Z}/n_j\mathbb{Z})$. Let $Z \subset \boldsymbol{\mu}$ be the subgroup such that $Z^* = \boldsymbol{\mu}^*/D$ and $G = (\prod_{j=1}^{k} \mathbf{SL}_{n_j})/Z$ as in Section 4.

Let $B = B_1 \oplus B_2 \cdots \oplus B_k$ denote the character group of a split maximal torus of G with the B_j's as in Section 5.2. Write A for the kernel of the natural surjection $B \to C =: Z^*$, so A is the character lattice of a split maximal torus of G. For every j, the image of the projection $Z \to \boldsymbol{\mu}_{n_j}$ is the subgroup $\boldsymbol{\mu}_{m_j}$ of $\boldsymbol{\mu}_{n_j}$ for a divisor m_j of n_j. We have then natural homomorphisms $G \to \mathbf{SL}_{n_j}/\boldsymbol{\mu}_{m_j}$. Write $\bar{x}_j$ for the canonical generator of the cyclic group $(\boldsymbol{\mu}_{m_j})^* \subset B/A = C$ of order m_j. Thus, C is generated by the $\bar{x}_j$'s.

The group D is the kernel of the natural surjection $B/\Lambda_r \to C$, so D is the character group of the center of G. We have the following diagram with the exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & D & \longrightarrow & B/\Lambda_r & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

Note that $B/\Lambda_r = \coprod_j (\mathbb{Z}/n_j\mathbb{Z})\hat{x}_j$, where $\hat{x}_j$ is the class of a canonical generator of B_j in B/Λ_r. The image of $\hat{x}_j$ under the homomorphism $B/\Lambda_r \to C$ is equal to $\bar{x}_j$.

The Weyl group W of G is the product of symmetric groups $W_j = S_{n_j}$. Write $q_j \in S^2(B_j)^{W_j} \subset S^2(B)^W$ for the canonical quadratic forms (see Section 5.2). Then $\{q_1, q_2, \dots, q_k\}$ is a $\mathbb{Z}$-basis for $S^2(B)^W$.

Below is a generalization of Lemma 5.4.

Lemma 6.1. *If $\sum_j k_j q_j \in S^2(A)^W_{\mathrm{red}}$, then $k_j \in m_j\mathbb{Z}$ for all j.*

Proof. The class in C of first fundamental weight of the jth component of G has order m_j. By [9, Proposition 10.6] or [7, Proposition 7.1], k_j is divisible by m_j. □

Consider the subset $J \subset \{1, 2, \dots, k\}$ of all j such that $v_2(m_j) = v_2(n_j) > 0$. Write D' for the subgroup of D of all elements having zero components outside J, i.e.,

$$D' = D \cap \coprod_{j\in J} (\mathbb{Z}/n_j\mathbb{Z})\hat{x}_j.$$

Let $q = \sum_{j\in J} k_j q_j \in S^2(B)^W$ be such that $k_j \in m_j\mathbb{Z}$ for every j. By Lemma 5.7,

$$\varepsilon(q) = \sum_{j\in J} [\tilde{k}_j \bar{x}_j, 2, \tilde{k}_j \bar{x}_j] \quad \text{in} \quad \Sigma^2(C),$$

where $\tilde{k}_j = k_j/2$. Let $x \in B$ be a character with $\hat{x} := \sum_{j\in J} \tilde{k}_j\hat{x}_j \in B/\Lambda_r$. Since

$$[\bar{x}, 2, \bar{x}] = \sum_{j\in J}[\tilde{k}_j\bar{x}_j, 2, \tilde{k}_j\bar{x}_j] + \sum_{j\neq i}[\tilde{k}_j\bar{x}_j, 2, \tilde{k}_i\bar{x}_i]$$

and

$$[\tilde{k}_j\bar{x}_j, 2, \tilde{k}_i\bar{x}_i] + [\tilde{k}_i\bar{x}_i, 2, \tilde{k}_j\bar{x}_j] = [\tilde{k}_j\bar{x}_j, 2, \tilde{k}_i\bar{x}_i] - [\tilde{k}_i\bar{x}_i, 2, \tilde{k}_j\bar{x}_j] \in \mathrm{Im}(1-\tau)$$

for $j \neq i$, we have

$$\varepsilon(q) = [\bar{x}, 2, \bar{x}] \quad \text{in} \quad \Sigma^2(C). \tag{3}$$

Proposition 6.2. *Let $q = \sum_j k_j q_j \in S^2(B)^W$. The following conditions are equivalent:*

1. *$q \in S^2(A)^W_{\mathrm{red}}$,*
2. *$q' := \sum_{j\in J} k_j q_j \in S^2(A)^W_{\mathrm{red}}$ and $k_j \in m_j\mathbb{Z}$ for every j,*
3. *k_j is even for every $j \in J$ and $\sum_{j\in J} \frac{k_j}{2}\hat{x}_j \in D'$ and $k_j \in m_j\mathbb{Z}$ for all j.*

Proof. Set $\hat{x} := \sum_{j\in J} \tilde{k}_j\hat{x}_j \in B/\Lambda_r$.

(1) ⇒ (2): By Lemma 6.1, $k_j \in m_j\mathbb{Z}$ for all j. If $j \notin J$, then by Proposition 5.8, $k_j q_j \in S^2(A_j)^{W_j}_{\mathrm{red}} \subset S^2(A)^W_{\mathrm{red}}$. It follows that $q' \in S^2(A)^W_{\mathrm{red}}$.

(2) ⇒ (3): By (3), $0 = \varepsilon(q') = [\bar{x}, 2, \bar{x}]$ in $\Sigma^2(C)$. In view of Lemma 2.1, $\bar{x} = 0$ in C, i.e, $\hat{x} \in D$. Then $\hat{x} \in D'$.

(3) ⇒ (1): We have $\hat{x} \in D'$ and $k_j \in m_j\mathbb{Z}$ for all $j \in J$. In particular, k_j is even. It follows from (3) that $\varepsilon(q') = [\bar{x}, 2, \bar{x}] = 0$ in $\Sigma^2(C)$, hence $q' \in S^2(A)^W_{\mathrm{red}}$. If $j \notin J$, then by Proposition 5.8, $k_j q_j \in S^2(A)^W_{\mathrm{red}}$. Thus, $q \in S^2(A)^W_{\mathrm{red}}$. □

Consider a homomorphism

$$\alpha : {}_2D' \to S^2(A)^W_{\mathrm{red}}/\operatorname{Dec}(A),$$

where ${}_2D'$ is the subgroup of exponent 2 elements in D', defined as follows. Let $x \in {}_2D'$, i.e., $\hat{x} := \sum_{j\in J} \frac{k_j}{2}\hat{x}_j$ with $k_j \in n_j\mathbb{Z}$. Set

$$\alpha(\hat{x}) = \sum_{j\in J} k_j q_j + \operatorname{Dec}(A).$$

We have α well defined by Proposition 6.2.

Lemma 6.3. *There are no elements in ${}_2D'$ with exactly one nonzero component.*

Proof. Suppose that $\frac{n_j}{2}\hat{x}_j \in {}_2D'$ for some $j \in J$. Then $\frac{n_j}{2}\bar{x}_j = 0$ in C. It follows that m_j divides $\frac{n_j}{2}$ since the order of $\bar{x}_j$ in C is equal to m_j. This is a contradiction since $v_2(m_j) = v_2(n_j)$ for j in J. □

Let E be the subgroup of ${}_2D'$ generated by all elements with exactly two nonzero components.

Lemma 6.4. *We have $\alpha(E) = 0$.*

Proof. Every generator of E is of the form $\frac{n_j}{2}\hat{x}_j + \frac{n_k}{2}\hat{x}_k$ with $j \neq k$ in J. We want to show that $n_j q_j + n_k q_k \in \operatorname{Dec}(A)$. By Lemma 5.3 applied to the integers $\frac{n_j}{2}$ and $\frac{n_k}{2}$, respectively, there are characters $y_j \in B_j$ and $y_k \in B_k$ such that
(1) $\hat{y}_j = \frac{n_j}{2}\hat{x}_j$ in $(\mathbb{Z}/n_j\mathbb{Z})\hat{x}_j$, $\hat{y}_k = \frac{n_k}{2}\hat{x}_k$ in $(\mathbb{Z}/n_k\mathbb{Z})\hat{x}_k$,
(2) $v_2(\operatorname{rank}(W_j e^{y_j})) = 1$, $v_2(\operatorname{rank}(W_k e^{y_k})) = 1$,
(3) $v_2(N(W_j e^{y_j})) = v_2(n_j) - 1$, $v_2(N(W_k e^{y_k})) = v_2(n_k) - 1$.

Set $y := y_j + y_k$. As $\hat{y} = \hat{y}_j + \hat{y}_k \in E \subset {}_2D'$, we have $y \in A$. It follows from the equality

$$We^y = W_j e^{y_j} \cdot W_k e^{y_k}$$

and Lemma 2.4 that

$$\begin{aligned} c_2(We^y) =& c_2(W_j e^{y_j} \cdot W_k e^{y_k}) \\ =& c_2(W_j e^{y_j}) \operatorname{rank}(W_k e^{y_k}) + c_2(W_k e^{y_k}) \operatorname{rank}(W_j e^{y_j}) \\ =& N(W_j e^{y_j}) \operatorname{rank}(W_k e^{y_k}) q_j + N(W_k e^{y_k}) \operatorname{rank}(W_j e^{y_j}) q_k \\ =& t_j q_j + t_k q_k \end{aligned}$$

for the integers t_j and t_k with $v_2(t_j) = v_2(n_j)$ and $v_2(t_k) = v_2(n_k)$. Recall that $2n_j q_j$ and $2n_k q_k$ belong to $\operatorname{Dec}(A)$ by Lemma 5.5. It follows that $n_j q_j + n_k q_k \in \operatorname{Dec}(A)$. □

It follows from Lemma 6.4 that α factors through a homomorphism

$$\alpha' : ({}_2D')/E \to S^2(A)^W_{\text{red}} / \operatorname{Dec}(A).$$

We prove that α' is an isomorphism by constructing the inverse map. Define a homomorphism

$$\beta : S^2(A)^W_{\text{red}} \to {}_2D'$$

as follows. Let $q = \sum_j k_j q_j \in S^2(A)^W_{\text{red}}$. By Lemma 6.1, $k_j \in m_j\mathbb{Z}$ for all j. Set

$$\beta(q) = \sum_{j \in J} \frac{k_j n_j}{2m_j} \hat{x}_j.$$

By Proposition 6.2, $\sum_{j \in J} \frac{k_j}{2}\hat{x}_j \in D'$. Since $m_j \hat{x}_j \in D'$ and n_j/m_j is odd, we have $\beta(q) \in D'$. Also, $2\beta(q) = 0$ since $n_j \hat{x}_j = 0$, hence $\beta(q) \in {}_2D'$.

Lemma 6.5. *We have $\beta(\operatorname{Dec}(A)) \subset E$.*

Proof. We shall show that $\beta(c_2(We^y)) \in E$ for every $y \in A$. Write $\hat{y} = \sum_j a_j \hat{x}_j$ for some $a_j \in \mathbb{Z}$ (unique modulo n_j). Since $c_2(We^{ty}) = t^2 c_2(We^y)$ for every integer t, we may replace y by ty for every odd integer t. In particular, we may assume that either $a_j = 0$ or $v_2(a_j) < v_2(n_j)$ for every j.

Let s be the number of indices j such that $a_j \neq 0$.

Case 1: $s \leq 2$. In this case $c_2(We^y)$ has at most 2 nonzero j-components, hence $\beta(c_2(We^y)) \in E$.

Case 2: $s \geq 3$. We show that $\beta(c_2(We^y)) = 0$. Fix a $k \in J$. It suffices to prove that v_2 of the q_k-coefficient $N_k(We^y)$ of $c_2(We^y)$ is strictly larger than $v_2(m_k) = v_2(n_k)$. Set $t_j := v_2(n_j) - v_2(a_j)$ for all j such that $a_j \neq 0$.

We claim that there is an i different from k such that

$$t_i \geq t_k. \tag{4}$$

Suppose that $t_k > t_i$ for all i different from k. Then there is an odd integer s such that $s2^{t_k-1}\hat{y} = s2^{t_k-1}\hat{x}_k$ is a nonzero element in ${}_2D'$ with only one nonzero component, a contradiction by Lemma 6.3. The claim is proved.

Write $y = \sum_j y_j$, where $y_j \in B_j$. We have $\hat{y}_j = a_j\hat{x}_j$ for all j and

$$We^y = \prod_j W_j e^{y_j} = W_k e^{y_k} \cdot z, \tag{5}$$

where z is the product of all $W_j e^{y_j}$ but $W_k e^{y_k}$. Hence by Lemma 2.4,

$$c_2(We^y) = N_k(W_k e^{y_k}) \operatorname{rank}(z) q_k + (\text{linear combination of } q_j\text{'s with } j \neq k).$$

By Lemma 5.2,

$$v_2(N_k(W_k e^{y_k})) \geq v_2(a_k). \tag{6}$$

Also, z is divisible by $W_i e^{y_i} \cdot W_j e^{y_j}$ for i as in (4) and some j such that $a_j \neq 0$ (such exists since $s \geq 3$). We have then

$$\operatorname{rank}(z) \in \operatorname{rank}(W_i e^{y_i}) \operatorname{rank}(W_j e^{y_j})\mathbb{Z}. \tag{7}$$

By Lemma 5.1,

$$v_2(\operatorname{rank}(W_i e^{y_i})) \geq v_2(n_i) - v_2(a_i) = t_i \tag{8}$$

and

$$v_2(\operatorname{rank}(W_j e^{y_j})) \geq v_2(n_j) - v_2(a_j) > 0. \tag{9}$$

It follows from (4)−(9) that

$$\begin{aligned} v_2(N_k(We^y)) =& v_2(N_k(W_k e^{y_k})) + v_2(\operatorname{rank}(z) \\ \geq& v_2(N_k(W_k e^{y_k})) + v_2(\operatorname{rank}(c_2(W_i e^{y_i})) + v_2(\operatorname{rank}(c_2(W_j e^{y_j})) \\ >& v_2(a_k) + t_i \\ \geq& v_2(a_k) + t_k \\ =& v_2(n_k). \end{aligned}$$

□

It follows from Lemma 6.5 that β factors through a homomorphism

$$\beta' : S^2(A)^W_{\text{red}} / \operatorname{Dec}(A) \to ({}_2D')/E.$$

Proposition 6.6. *Let S be a maximal split torus of the group $G = (\prod_{j=1}^k \mathbf{SL}_{n_j})/Z$. Then the map $\alpha' : ({}_2D')/E \to S^2(S^*)^W_{\text{red}} / \operatorname{Dec}(S^*)$ is an isomorphism.*

Proof. We show that β' is the inverse of α'. The composition $\beta' \circ \alpha'$ is the identity since n_j/m_j is odd for all $j \in J$. Let $q = \sum_j k_j q_j \in S^2(A)^W_{\text{red}}$. By Lemma 6.1, $k_j \in m_j\mathbb{Z}$ for all j. We have $\alpha' \circ \beta'(q) = \sum_{j \in J} \frac{k_j n_j}{m_j} q_j$. It follows from Proposition 5.8 that $2k_j q_j \in \operatorname{Dec}(A)$ for $j \in J$, therefore, $\frac{k_j n_j}{m_j} q_j$ is congruent to $k_j q_j$ modulo $\operatorname{Dec}(A)$ since n_j/m_j is odd.

If $j \notin J$, then by Proposition 5.8, $k_j q_j \in \mathrm{Dec}(A)$. It follows that $\alpha' \circ \beta'(q)$ is equal to q modulo $\mathrm{Dec}(A)$. □

7. Main theorem

Let $n_1, n_2, \dots, n_k$ be a sequence of positive integers, $D \subset \coprod_{j=1}^{k} (\mathbb{Z}/n_j\mathbb{Z})$ a subgroup of relations. Let CSA_D be the functor that takes a field extension K/F to the set of k-tuples of central simple K-algebras $(A_1, A_2, \dots, A_k)$ with $\deg(A_j) = n_j$ such that $\sum_j d_j[A_j] = 0$ in the Brauer group $\mathrm{Br}(K)$ for all tuples $(d_j + n_j\mathbb{Z}) \in D$.

For every j, write $m_j\mathbb{Z}/n_j\mathbb{Z} = D \cap (\mathbb{Z}/n_j\mathbb{Z})$ for a unique positive divisor m_j of n_j. Consider the set J of all indices j such that $v_2(m_j) = v_2(n_j) > 0$ and let $D' = D \cap \coprod_{j\in J} (\mathbb{Z}/n_j\mathbb{Z})$. Let E be the subgroup of ${}_2D'$ generated by elements with exactly two nonzero components.

Combining Theorem 3.1 and Propositions 4.1 and 6.6, we get the following main theorem of the paper.

Theorem 7.1. *For every group of relations D, there is a natural isomorphism*

$$({}_2D')/E \xrightarrow{\sim} \mathrm{Inv}^3\big(\mathrm{CSA}_D, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{ind}}.$$

Example 7.2. Let $n_1 = n_2 = \cdots = n_k = 2$ for $k \geq 3$ and let D be the cyclic subgroup (of order 2) generated by $(1, 1, \dots, 1)$. Then $\mathrm{CSA}_D(K)$ is the set of k-tuples of quaternion K-algebras $(Q_1, Q_2, \dots, Q_k)$ such that

$$[Q_1] + [Q_2] + \cdots + [Q_k] = 0$$

in $\mathrm{Br}(K)$. We have ${}_2D' = D = \mathbb{Z}/2\mathbb{Z}$ and $E = 0$, i.e., there is exactly one indecomposable degree 3 invariant of CSA_D. It is defined as follows (see [9, Example 11.2]). Let φ_j be the reduced norm quadratic form of Q_j. The sum φ of the forms φ_j in the Witt group $W(K)$ of K belongs to the cube of the fundamental ideal of $W(K)$ (this also makes sense when $\mathrm{char}(F) = 2$), i.e., φ is the sum of 3-fold Prister forms $\rho_1, \rho_2, \dots, \rho_s$. The Arason invariant $\sum_i e_3(\rho_i)$ of φ in $H^3(K, \mathbb{Q}/\mathbb{Z}(2))$, where $e_3(\rho_i)$ is the class of ρ_i in $H^3(K, \mathbb{Q}/\mathbb{Z}(2))$, yields the only nontrivial degree 3 nontrivial invariant Ar_k of CSA_D (see also [1]).

We can make explicit the isomorphism in Theorem 7.1. Let $d \in {}_2D'$. Write $d = \sum_j d_j \hat{x}_j$ for integers d_j such that $2d_j \in n_j\mathbb{Z}$. The map of $(\mathbb{Z}/2\mathbb{Z})^k$ to $\coprod_{j=1}^{k} (\mathbb{Z}/n_j\mathbb{Z})\hat{x}_j$ taking a tuple (b_j) to $\sum_j b_j d_j \hat{x}_j$ sends the generator $(1, 1, \dots, 1)$ from Example 7.2 to d. This describes the invariant P_d of CSA_D corresponding to d by Theorem 7.1 as follows. Let $A = (A_1, A_2, \dots, A_k)$ be a tuple of central simple algebras in $\mathrm{CSA}_D(K)$. In particular, $\sum_j d_j[A_j] = 0$ in $\mathrm{Br}(K)$. As $\deg(A_j) = n_j$, the class $d_j[A_j]$ is represented by a quaternion algebra Q_j, and we have $\sum_j [Q_j] = 0$. The invariant P_d is given by $P_d(A) := \mathrm{Ar}_k(Q)$, where $Q = (Q_j)$ with Ar_k from Example 7.2.

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