OPERATIONS IN CONNECTIVE K-THEORY

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Abstract. In this article we classify additive operations in connective K-theory with various torsion-free coefficients. We discover that the answer for the integral case requires understanding of the $\widehat{\mathbb{Z}}$ one. Moreover, although integral additive operations are topologically generated by Adams operations, these are not reduced to infinite linear combinations of the latter ones. We describe a topological basis for stable operations and relate it to a basis of stable operations in graded K-theory. We classify multiplicative operations in both theories and show that homogeneous additive stable operations with $\widehat{\mathbb{Z}}$-coefficients are topologically generated by stable multiplicative operations. This is not true for integral operations.

1. Introduction

Let k be a field of characteristic 0. An *oriented cohomology theory* A^* over k is a functor from the category $\mathbf{Sm}_k^{op}$ of smooth quasi-projective varieties over k to the category of $\mathbb{Z}$-graded commutative rings equipped with a push-forward structure and satisfying certain axioms. In this article, we study the, so-called, *small theories*. For these, the appropriate choice is [11, Definition 2.1] which employs a strong form of the localisation axiom and is some breed of the axioms of Panin-Smirnov [8, 9] and that of Levine-Morel [7, Definition 1.1.2]. In particular, every oriented cohomology theory A^* admits a theory of Chern classes c_n^A of vector bundles. Among such theories there is the universal one - the algebraic cobordism of Levine-Morel Ω^* [7]. We will work with the *free* theories, i.e. theories obtained from Ω^* by change of coefficients. These are exactly the theories of *rational type* for which the results of [11] apply.

Examples of free oriented cohomology theories are:

- *Chow theory* CH^* that assigns to a smooth variety X over k the Chow ring $\mathrm{CH}^*(X)$;
- *Graded K-theory* K^*_{gr} taking X to the Laurent polynomial ring $K_0(X)[t,t^{-1}]$ (graded by the powers of the *Bott element* t of degree -1) over the Grothendieck ring $K_0(X)$;
- *Connective K-theory* taking a smooth variety X to the ring $\mathrm{CK}^*(X)$ of X (see [3] and [5]).

The connective K-theory is the "smallest" oriented cohomology theory "living" above Chow theory and graded K-theory: there are natural graded morphisms

$$\begin{array}{ccc} & \mathrm{CK}^*(X) & \\ \swarrow & & \searrow \\ \mathrm{CH}^*(X) & & K^*_{gr}(X) \end{array}$$

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that yield graded isomorphisms

$$\mathrm{CK}^*(X)/t\,\mathrm{CK}^{*+1}(X) \xrightarrow{\sim} \mathrm{CH}^*(X) \quad \text{and} \quad \mathrm{CK}^*(X)[t^{-1}] \xrightarrow{\sim} K^*_{gr}(X).$$

Moreover, multiplication $\mathrm{CK}^{n+1}(X) \xrightarrow{t} \mathrm{CK}^n(X)$ by the Bott element $t \in \mathrm{CK}^{-1}(k)$ is an isomorphism if $n < 0$. The map $\mathrm{CK}^0(X) \to K^0_{gr}(X) = K_0(X)$ is also an isomorphism, so we can identify $\mathrm{CK}^n(X)$ with $K_0(X)$ for all $n \leqslant 0$.

For any $n \geqslant 0$ the image of $\mathrm{CK}^n(X) \xrightarrow{t^n} \mathrm{CK}^0(X) = K_0(X)$ is the subgroup $K_0^{(n)}(X) \subset K_0(X)$ generated by the classes of coherent $\mathcal{O}_X$-modules with codimension of support at least n. Note that the map t^n may not be injective in general if $n > 1$.

Let A^* and B^* be two oriented cohomology theories. An *additive operation* $G : A^* \to B^*$ is a morphism between functors A^* and B^* considered as contravariant functors from $\mathbf{Sm}_k$ to the category of abelian groups. Examples of additive operations are Adams operations in algebraic K-theory and Steenrod operations in the Chow groups modulo a prime integer.

If A^* is an oriented cohomology theory and R is a commutative ring, we write $A^n_R(X)$ for $A^n(X) \otimes_{\mathbb{Z}} R$ and $\mathbf{OP}^{n,m}_R(A)$ for the R-module of R-linear operations $A^n_R \to A^m_R$.

It is proved in [11, §6.3] that every free oriented cohomology theory A^* admits the *Adams operations* $\Psi^A_m \in \mathbf{OP}^{n,n}_R(A)$ for all n and m. The operation Ψ^A_m in $\mathbf{OP}^{1,1}_R(A)$ satisfies

$$\Psi^A_m(c^A_1(L)) = c^A_1(L^{\otimes m})$$

for a line bundle L. Moreover, there is an R-linear map

$$\mathrm{Ad}_n : R[[x]] \to \mathbf{OP}^{n,n}_R(A)$$

taking the power series $(1-x)^m$ to the Adams operation Ψ^A_m for all $m \in \mathbb{Z}$.

In general, the map Ad_n is neither injective not surjective. But it is shown in [11, §6.1] that Ad_n is an isomorphism if A^* is the graded K-theory, thus,

$$\mathbf{OP}^{n,n}_R(K_{gr}) \simeq R[[x]].$$

Since the power series $(1-x)^m$ generate $R[[x]]$ as topological group in the x-adic topology, we can say that the R-module $\mathbf{OP}^{n,n}_R(K_{gr})$ is topologically generated by the Adams operations in the graded K-theory. Moreover, since multiplication by the Bott element is an isomorphism in K^*_{gr}, we have $\mathbf{OP}^{n,m}_R(K_{gr}) = R[[x]] \cdot t^{n-m}$.

In the present paper we study the groups $\mathbf{OP}^{n,m}_R := \mathbf{OP}^{n,m}_R(\mathrm{CK})$ of operations in the connective K-theory over R. We write for simplicity $\mathbf{OP}^{n,m}$ for $\mathbf{OP}^{n,m}_{\mathbb{Z}}$.

The groups $\mathrm{CK}^n(X)$ for $n \leqslant 0$ are identified with $K_0(X)$, hence translating the above result on the operations in graded K-theory, we see that $\mathrm{Ad}_n : R[[x]] \to \mathbf{OP}^{n,n}_R$ is an isomorphism for $n \leqslant 0$.

The Adams operation Ψ_0 is trivial on CK^n_R for $n \geqslant 1$, i.e. $\mathrm{Ad}_n(1) = 0$, so we consider the restriction $\mathrm{Ad}'_n : xR[[x]] \to \mathbf{OP}^{n,n}_R$ of the map Ad_n. The R-module $\mathrm{CK}^1_R(X)$ is a canonical direct summand of $\mathrm{CK}^0_R(X) = K_0(X)_R$ with the complement $R \cdot 1$. This leads to a ring isomorphism $\mathbf{OP}^{0,0}_R \simeq R \times \mathbf{OP}^{1,1}_R$. Moreover, the map $\mathrm{Ad}'_1 : xR[[x]] \to \mathbf{OP}^{1,1}_R$ is an isomorphism.

The structure of the groups $\mathbf{OP}^{n,n}_R$ with $n > 1$ is much more delicate and depends on the base ring R. The homomorphisms $\mathrm{Ad}'_n : xR[[x]] \to \mathbf{OP}^{n,n}_R$ for $n \geqslant 2$ are not surjective in general.

It came as a surprise to us that the structure of $\mathbf{OP}^{n,n}_R$ is very simple over the ring of *profinite integers* $\widehat{\mathbb{Z}} = \lim(\mathbb{Z}/n\mathbb{Z})$:

Theorem. *The map* $\mathrm{Ad}'_n : xR[[x]] \to \mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}}}$ *is an isomorphism if* $n \geqslant 1$. *In particular, the* $\widehat{\mathbb{Z}}$*-module* $\mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}}}$ *is topologically generated by the Adams operations.*

Over $\mathbb{Z}$ the map Ad'_n is not surjective if $n \geqslant 2$.

Theorem. *The group* $\mathbf{OP}^{n,n}$ *of integral operations is isomorphic canonically to a subgroup of* $\mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}}}$. *Moreover, there is an exact sequence*

$$0 \to x\mathbb{Z}[[x]] \xrightarrow{\mathrm{Ad}'_n} \mathbf{OP}^{n,n} \to (\widehat{\mathbb{Z}}/\mathbb{Z})^{n-1} \to 0$$

if $n \geqslant 1$.

Thus, the group $\widehat{\mathbb{Z}}$ also shows up in the computation of $\mathbf{OP}^{n,n}$ over $\mathbb{Z}$. For example, $\mathbf{OP}^{2,2}$ as a subgroup of $\mathbf{OP}^{2,2}_{\widehat{\mathbb{Z}}} = x\widehat{\mathbb{Z}}[[x]]$ is generated by $x\mathbb{Z}[[x]]$ and the power series $\sum_{i>0} \frac{c-c_i}{i}\, x^i$ for all $c \in \widehat{\mathbb{Z}}$ and integers c_i such that $c - c_i$ is divisible by i for all $i > 0$, i.e., c_i in $\mathbb{Z}$ represents congruence class of c modulo i.

We prove that the rings $\mathbf{OP}^{n,n}$ and $\mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}}}$ are commutative. Moreover, the rings $\mathbf{OP}^{n,n}$ are "almost" integral domains: the only zero divisors are the multiples of $\Psi_1 \pm \Psi_{-1}$.

An operation $G : A^* \to B^*$ is called *multiplicative* if G is a morphism of functors $\mathbf{Sm}_k \to \mathit{Rings}$. Examples are *twisted* Adams operations Ψ^c_b defined as follows. Let $b \in \widehat{\mathbb{Z}}$ and $c \in \widehat{\mathbb{Z}}^\times$. Then the operation Ψ^c_b is homogeneous and equal to $c^{-n} \cdot \Psi_{bc}$ on $\mathrm{CK}^n_{\widehat{\mathbb{Z}}}$, where Ψ_{bc} is the (generalized) Adams operation with the power series $(1-x)^{bc}$. We classify all multiplicative operations on $\mathrm{CK}^*_{\widehat{\mathbb{Z}}}$ in Section 5.

The notion of "stability" in topology can be considered in algebraic setting as follows (see [11, §3.1]). Let $\mathbf{SmOp}$ be a category whose objects are pairs (X, U), where $X \in \mathbf{Sm}_k$ and U is an open subvariety of X. Any theory A^* extends from $\mathbf{Sm}_k$ to $\mathbf{SmOp}$ by the rule:

$$A^*((X,U)) := \mathrm{Ker}(A^*(X) \to A^*(U)).$$

and every additive operation $A^* \to B^*$ on $\mathbf{Sm}_k$ extends uniquely to an operation on $\mathbf{SmOp}$. There is an identification

$$\sigma^A_T : A^*((X,U)) \xrightarrow{\cong} A^{*+1}(\Sigma_T(X,U)),$$

where $\Sigma_T(X,U) := (X,U) \wedge (\mathbb{P}^1, \mathbb{P}^1 \backslash 0)$.

For any additive operation $G : A^* \to B^*$ we define its *desuspension* as the unique operation $\Sigma^{-1}G : A^* \to B^*$ such that

$$G \circ \sigma^A_T = \sigma^B_T \circ \Sigma^{-1}G.$$

A *stable* additive operation $G : A^* \to B^*$ is the collection $\{G^{(n)} | n \geqslant 0\}$ of operations $A^* \to B^*$ such that $G^{(n)} = \Sigma^{-1}G^{(n+1)}$.

In Section 6 we classify stable operations in connective K-theory over $\widehat{\mathbb{Z}}$. We prove that under the identification

$$\mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}}} = \begin{cases} \widehat{\mathbb{Z}}[[x]], & \text{if } n \leqslant 0; \\ x\widehat{\mathbb{Z}}[[x]], & \text{if } n \geqslant 1 \end{cases}$$

the desuspension map is given by the formula

$$\Sigma^{-1}(G) = \begin{cases} \Phi(G), & \text{if } n \leqslant 1; \\ \Phi(G) - \Phi(G)(0), & \text{if } n > 1. \end{cases}$$

where $G \in \mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}}}$ and $\Phi(G) = (x-1)\dfrac{\mathrm{d}G}{\mathrm{d}x}$. Thus, the desuspension map Σ^{-1} yields a tower of injective maps

$$\widehat{\mathbb{Z}}[[x]] = \mathbf{OP}^{0,0}_{\widehat{\mathbb{Z}}} \hookleftarrow \mathbf{OP}^{1,1}_{\widehat{\mathbb{Z}}} \hookleftarrow \ldots \hookleftarrow \mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}}} \hookleftarrow \ldots.$$

The group of homogeneous degree 0 stable operations $\mathrm{CK}^*_{\widehat{\mathbb{Z}}} \to \mathrm{CK}^*_{\widehat{\mathbb{Z}}}$ is canonically isomorphic to the group

$$S := \cap_n \operatorname{Im}(\Phi^n) \subset \widehat{\mathbb{Z}}[[x]].$$

We identify this group in Section 6. In particular we prove that S is the closure in the x-adic topology of $\widehat{\mathbb{Z}}[[x]]$ of the set of all (finite) $\widehat{\mathbb{Z}}$-linear combinations of the Adams power series A_r for $r \in \widehat{\mathbb{Z}}^\times$. The $\widehat{\mathbb{Z}}$-module S and its integral version S_0 appear to be of an uncountable rank. We describe a topological basis for them.

We call a multiplicative operation G *stable* if the constant sequence $(G, G, G, \ldots)$ is stable. We prove that stable multiplicative operations $\mathrm{CK}^*_{\widehat{\mathbb{Z}}} \to \mathrm{CK}^*_{\widehat{\mathbb{Z}}}$ are exactly operations Ψ^c_1, for $c \in \widehat{\mathbb{Z}}^\times$. Thus, we obtain:

Theorem. *Homogeneous degree* 0 *stable additive operations on* $\mathrm{CK}^*_{\widehat{\mathbb{Z}}}$ *are topologically generated by the stable multiplicative operations there.*

Similarly, stable multiplicative operations on CK^* are $\Psi_1^{\pm 1}$. This time though, they don't generate the group of stable additive operations which is of uncountable rank.

Recall that operations in (graded) K-theory were determined in [11, §6.1]. In the present paper we determine stable and multiplicative operations in K_{gr}. We describe a basis of the group of stable K_{gr}-operations and relate it to the basis of stable CK-operations. The ring of stable operations is dual to the Hopf algebra of co-operations defined over $\mathbb{Z}$ and therefore has a structure of (topological) Hopf algebra. The Hopf algebra of co-operations coincides with $K_0(K)$ in topology and has been studied in [1], [2], [4], [6] and [10].

The main tool used in our proofs is the general result of the second author [11, Theorem 6.2] that asserts, when applied to the connective K-theory, that an operation $G \in \mathbf{OP}^{n,m}_R$ for $n \geqslant 1$ is given by a sequence of symmetric power series $G_l \in R[[x_1, \ldots, x_l]]$ for all $l \geqslant n$ satisfying certain conditions. In particular, G_l divisible by $x_1 \cdot \ldots \cdot x_l$ and $-G_{l+1} = \partial(G_l)$, the *partial derivative* of G_l (see Definition 2.1) for all $l \geqslant n$, i.e., all power series G_l are determined by G_n. We show that if R is torsion free, then G_n can be integrated over $K = R \otimes \mathbb{Q}$: there is a unique power series $H \in xK[[x]]$ such that $G_n = \partial^{n-1}(H)$. Thus, the operation G is determined by a power series H in one variable over K such that $\partial^{n-1}(H) \in R[[x_1, \ldots, x_n]]$.

The article is organized as follows. In Section 2 we prove general results which will permit us to integrate the multivariate symmetric power series and reduce the classification of operations to the description of power series in one variable with certain integrality properties. These properties are then studied and the respective power series are classified in Section 3. In Section 4 we apply the obtained results in combination with [11, Theorem 6.2] to produce a description of additive operations in CK with integral and $\widehat{\mathbb{Z}}$-coefficients. We describe the ring structure on the set of homogeneous operations. The description of operations in K_{gr} comes as an easy by-product. In the latter case, we also describe the dual bi-algebra of co-operations. Multiplicative operations in CK and K_{gr} are studied in Section 5. Finally, Section 6 is devoted to the computation of stable operations.

2. Symmetric power series

2.1. **Partial derivatives.** Let $F(x,y)$ be a (commutative) formal group law over a commutative ring R. We write $x*y := F(x,y)$.

Let $G(x_1,\dots,x_n) \in R[[x_1,\dots,x_n]]$ be a power series in $n \geqslant 1$ variables.

Definition 2.1. The *partial derivative* of G (with respect to F) is the power series

$$\begin{aligned}(\partial G)(x_1,x_2,\dots,x_{n+1}) = {} & G(x_1{*}x_2,x_3,\dots,x_{n+1}) - G(x_1,x_3,\dots,x_{n+1}) \\ & - G(x_2,x_3,\dots,x_{n+1}) + G(0,x_3,\dots,x_{n+1}) \in R[[x_1,\dots,x_{n+1}]].\end{aligned}$$

Note that the partial derivative is always taken with respect to the first variable (in this case x_1) in the list of variables. Write ∂^m for the iterated partial derivative. We also set $(\partial^0 G)(x_1,\dots,x_n) = G(x_1,\dots,x_n) - G(0,x_2,\dots,x_n)$.

For a subset $I \subset [1,m+1] := \{1,\dots,m+1\}$ write x_I for the $*$-sum of all x_i with $i \in I$. In particular, $x_\emptyset = 0$. Then

$$(\partial^m G)(x_1,\dots x_{m+n}) = \sum (-1)^{|I|} G(x_I, x_{m+2},\dots,x_{m+n}) \in R[[x_1,x_2,\dots,x_{m+n}]],$$

where the sum is taken over all 2^{m+1} subsets $I \subset [1,m+1]$. In particular, $\partial^m G$ is symmetric with respect to the first $m+1$ variables.

Observation 2.2. If $G \in R[[x_1,\dots,x_n]]$ is so that ∂G is symmetric power series, then $\partial^m G$ is symmetric for all $m \geqslant 1$.

Indeed, since ∂G is symmetric, $\partial^m G = \partial^{m-1}(\partial G)$ is symmetric with respect to the last n variables. But $\partial^m G$ is symmetric with respect to the first $m+1$ variables, hence it is symmetric.

Notation 2.3. For any commutative $\mathbb{Q}$-algebra K write

$$\lg_1(x) := \log(1-x) = -\sum_{i \geqslant 1} \frac{x^i}{i} \in K[[x]]$$

and for any $n \geqslant 0$,

$$\lg_n(x) := \frac{1}{n!} \left(\lg_1(x)\right)^n \in K[[x]].$$

In particular, $\lg_0(x) = 1$.

For the rest of this section $*$ denotes the multiplicative formal group law, i.e., $x*y = x+y-xy$.

The power series $\lg_1(x)$ belongs to the kernel of ∂. Moreover, we have the following statement.

Proposition 2.4. *For any commutative $\mathbb{Q}$-algebra K and any $n>0$, the kernel of $\partial^{n-1}: K[[x]] \to K[[x_1,\dots,x_n]]$ is equal to*

$$\sum_{0\leqslant r<n} K\cdot \lg_r(x).$$

Proof. We change the variables: $y_i = \lg_1(x_i) = \log(1-x_i)$, where $x_1 = x$. The multiplicative group law $*$ translates to the additive one. In the new variables the partial derivative is homogeneous and lowers the degree in y_1 by 1. Therefore, the kernel of ∂^n is spanned by $1, y_1, \dots, y_1^{n-1}$. □

The following formula is very useful.

Proposition 2.5. *Let K be a $\mathbb{Q}$-algebra, $G\in K[[x]]$ and n a positive integer. Then*

$$(\partial^n G)(x_1,x_2,\dots x_{n+1}) = \sum_{k=1}^{\infty} \frac{1}{k!}\ \partial^{n-1}\Big((1-x)^k \frac{\mathrm{d}^k G}{\mathrm{d}x^k}\Big)(x_1,x_2,\dots,x_n)\cdot x_{n+1}^k.$$

Proof. Note that both sides don't contain monomials $\overline{x}^\alpha := x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_{n+1}^{\alpha_{n+1}}$ if at least one α_i is zero. We prove that for every multi-index α with $\alpha_i>0$ for all i, the $\overline{x}^\alpha$- coefficients of both sides are equal. Set $k=\alpha_{n+1}$.

The $\overline{x}^\alpha$-coefficient of the left hand side is the same as the $\overline{x}^\alpha$-coefficient of $G(x_1*x_2*\cdots*x_{n+1})$. To determine this coefficient, we differentiate (in the standard way) k times the series $G(x_1*x_2*\cdots*x_{n+1})$ by x_{n+1}, plug in $x_{n+1}=0$ and divide by $k!$. Note that

$$\frac{\mathrm{d}}{\mathrm{d}x_{n+1}}(x_1*x_2*\cdots*x_{n+1}) = (1-x_1)(1-x_2)\cdots(1-x_n).$$

It follows that the $\underline{x}^\alpha$-coefficient in the left hand side is equal to the $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$-coefficient of

$$\frac{1}{k!}(1-x_1)^k(1-x_2)^k\cdots(1-x_n)^k\frac{\mathrm{d}^k G}{\mathrm{d}x^k}(x_1*x_2*\cdots*x_n).$$

On the other hand, note that the $\underline{x}^\alpha$-coefficient of the right hand side is equal to the $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$-coefficient of $\frac{1}{k!}\partial^{n-1}\Big((1-x)^k\frac{\mathrm{d}^k G}{\mathrm{d}x^k}\Big)(x_1,x_2,\dots,x_n)$. This is the same as the $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$-coefficient of

$$\frac{1}{k!}(1-x_1*x_2*\cdots*x_n)^k G^{(k)}(x_1*x_2*\cdots*x_n) = \frac{1}{k!}(1-x_1)^k(1-x_2)^k\cdots(1-x_n)^k\frac{\mathrm{d}^k G}{\mathrm{d}x^k}(x_1*x_2*\cdots*x_n).$$

□

For a nonzero power series $H\in R[[x_1,\dots,x_n]]$ denote by $v(H)$ the smallest degree of monomials in H. Set also $v(0)=\infty$.

Observation 2.6. Suppose that a commutative ring R is torsion free. A direct calculation shows that for positive integers n and m, we have $v(\partial^{n-1}(x^m)) = m$ if $m\geqslant n$. It follows that $v(\partial^{n-1}(G)) = v(G)$ for every $G\in R[[x]]$ such that $v(G)\geqslant n$.

2.2. **Integration of symmetric power series.**

Definition 2.7. A power series $G \in R[[x_1, \dots, x_n]]$ is called *double-symmetric* if G itself and ∂G are both symmetric.

In the following proposition we prove that double-symmetric power series can be symmetrically integrated over any commutative $\mathbb{Q}$-algebra.

Proposition 2.8. *Let K be a commutative $\mathbb{Q}$-algebra and $G \in K[[x_1, \dots, x_n]]$, $n \geqslant 2$, be a symmetric power series divisible by $x_1 \cdot \ldots \cdot x_n$. The following are equivalent:*

1. *G is double-symmetric;*
2. *All derivatives $\partial^m(G)$, $m \geqslant 0$, are symmetric power series;*
3. *There is a power series $L \in K[[x]]$ such that $G = \partial^{n-1}(L)$;*
4. *There is $H \in K[[x_1, \dots, x_{n-1}]]$ such that $\partial(H) = G$;*
5. *There is a unique symmetric $H \in K[[x_1, \dots, x_{n-1}]]$, divisible by $x_1 \cdot \ldots \cdot x_{n-1}$, with zero coefficient at $x_1 \cdot \ldots \cdot x_{n-1}$ and such that $\partial(H) = G$.*

Proof. Note that (1) $\Leftrightarrow$ (2) by Observation 2.2. We will prove the equivalence of all statements by induction on n. The implication (3) $\Rightarrow$ (2) is clear, (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (4) are trivial.

(5) $\Rightarrow$ (3) follows by induction applied to H.

(1) or (4) $\Rightarrow$ (5) Over a commutative $\mathbb{Q}$-algebra every formal group law is isomorphic to the additive one. So we may assume that the group law is additive, i.e., the derivative is defined by

$$(\partial G)(x, y, \bar{t}) = G(x + y, \bar{t}) - G(x, \bar{t}) - G(y, \bar{t}) + G(0, \bar{t}).$$

We first prove uniqueness. Indeed if $\partial H = 0$, then H is linear in x_1, and since H is symmetric and divisible by $x_1 \cdot \ldots \cdot x_{n-1}$, we must have $H = 0$.

Case $n = 2$: The implication (4) $\Rightarrow$ (5) is obvious. We prove (1) $\Rightarrow$ (5). We may assume that G is a homogeneous polynomial of degree $d > 1$. The symmetry of the derivative of $G(x, y)$ results in the following cocycle condition:

$$G(x + y, z) + G(x, y) = G(x + z, y) + G(x, z).$$

In particular, we have the following equalities:

$$\begin{aligned} G(x + y, x + y) + G(x, y) &= G(2x + y, y) + G(x, x + y), \\ G(2x + y, y) + G(2x, y) &= G(2x, 2y) + G(y, y), \\ G(x, x + y) + G(x, y) &= G(2x, y) + G(x, x). \end{aligned}$$

It follows that

$$\begin{aligned} \partial(G(x, x))(x, y) &= G(x + y, x + y) - G(x, x) - G(y, y) \\ &= G(2x, 2y) - 2G(x, y) \\ &= (2^d - 2)(G(x, y)), \end{aligned}$$

hence $G(x, y) = \partial(H)$, where $H(x) = G(x, x)/(2^d - 2)$.

Case $n = 3$: Write $G(x, y, z) = \sum_{i\geqslant 1} G_i(x, y)z^i$. Clearly, all $G_i(x, y)$ satisfy (1) or (4) and hence (5) by induction. Integrating each $G_i(x, y)$, we get a power series $H = \sum_{i,j\geqslant 1} a_{i,j}x^i y^j$ in two variables such that $\partial H = G$.

Note that we can change H by any series $\sum_i c_i xy^i$ without changing ∂H. This way, we can make $H = \sum_{i,j\geqslant 1} a_{i,j}x^i y^j$ with $a_{i,1} = a_{1,i}$ and $a_{1,1} = 0$. We claim that H is symmetric. Indeed, from the symmetry of ∂H, we have:

$$\binom{i+k}{i}a_{i+k,j} = \binom{j+k}{j}a_{j+k,i},$$

for any $i, j, k \geqslant 1$. This implies that

$$\tfrac{1}{i+l}\binom{i+l}{i}a_{i+l-1,1} = a_{l,i},$$

and so, $a_{i,l} = a_{l,i}$, for any $i, l \geqslant 2$. This shows that H is symmetric. Observe that such symmetric integration is unique provided $a_{1,1} = 0$.

Case $n > 3$: Write $G = \sum_{i\geqslant 1} G_i \cdot x_n^i$ with $G_i \in K[[x_1, \dots, x_{n-1}]]$. The slices G_i of G are double-symmetric. By the inductive assumption, these can be uniquely integrated to symmetric power series $H_i \in K[[x_1, ..., x_{n-2}]]$ as in (5). Putting these power series together, we obtain

$$H = \sum_{i\geqslant 1} H_i \cdot x_{n-1}^i \in K[[x_1, ..., x_{n-1}]]$$

such that $\partial H = G$. Write

$$H = \sum_{i_1,\dots,i_{n-1}} a_{i_1,\dots,i_{n-1}} x_1^{i_1} \dots x_{n-1}^{i_{n-1}}.$$

Modifying H by $x_1 \dots x_{n-1}L(x_{n-1})$ for an appropriate power series L, we may assume that $a_{i,1,\dots,1} = a_{1,1,\dots,i}$ for all i.

We claim that H is symmetric. The $x_1^{i_1} \dots x_n^{i_n}$-coefficient of $G = \partial H$ is equal to $\binom{i_1+i_2}{i_1}a_{i_1+i_2,i_3,\dots,i_n}$. Therefore, since G is symmetric, H is symmetric with respect to $x_2, ..., x_{n-1}$, if $i_1 > 1$. Recall than H is also symmetric in $x_1, \dots, x_{n-2}$. Therefore, it suffices to show that the coefficient $a_{1,i_2,\dots,i_{n-1}}$ does not change if we interchange i_{n-1} with i_k for some $k = 2, \dots, n-2$.

Suppose all indices $i_1, \dots, i_{n-1}$ but one are equal to 1. Then the statement follows from the equality $a_{1,1,\dots,i} = a_{i,1,\dots,1} = a_{1,i,\dots,1}$ for all i. Otherwise, at least two indices, say $i_k = u$ and $i_l = v$ with $k < l$ are greater than 1.

If $l < n-1$, set $w = i_{n-1}$. We have (here and below we indicate only the indices which are permuted, hidden indices remain unchanged):

$$a_{1,u,v,w} = a_{v,u,1,w} = a_{v,w,1,u} = a_{1,w,v,u},$$

so we interchanged i_k and i_{n-1}. If $l = n-1$, we can write

$$a_{1,u,v} = a_{u,1,v} = a_{u,v,1} = a_{v,u,1} = a_{v,1,u} = a_{1,v,u},$$

i.e., we again interchanged i_k and i_{n-1}. □

3. The groups $\mathcal{Q}_R^n$

The formal group law is multiplicative in this section. Let R be a commutative ring and $K = R \otimes_{\mathbb{Z}} \mathbb{Q}$. We assume that R is torsion free (as abelian group), i.e., R can be identified with a subring of K.

Definition 3.1. For any integer $n \geqslant 1$, let us denote by $\mathcal{Q}_R^n$ the R-module of power series G in $xK[[x]]$, for which $\partial^{n-1}(G) \in R[[x_1, ..., x_n]]$. For example, $\mathcal{Q}_R^1 = xR[[x]]$. We also set $\mathcal{Q}_R^n = R[[x]]$ if $n \leqslant 0$.

Note that $xR[[x]]$ and $\sum_{0<r<n} K \cdot \lg_r(x)$ are contained in $\mathcal{Q}_R^n$ in view of Proposition 2.4.

Lemma 3.2. *Suppose R has no nontrivial $\mathbb{Z}$-divisible elements. Then*

$$xR[[x]] \cap \Big(\sum_{0<r<n} K \cdot \lg_r(x) \Big) = 0.$$

Proof. Observe that

$$\Phi(\lg_r(x)) := (x-1) \cdot \frac{\mathrm{d}}{\mathrm{d}x} (\lg_r(x)) = \lg_{r-1}(x).$$

Suppose $\sum_{0<r<n} q_r \cdot \lg_r(x) \in xR[[x]]$, where $q_r \in K$ and let r be the largest index such that $q_r \neq 0$. Applying Φ^{r-1} to the sum we see that $q_{r-1} + q_r \lg_1(x) \in R[[x]]$. Let $n \in \mathbb{N}$ be a natural number such that $nq_{r-1} \in R$ and $nq_r \in R$. It follows that $nq_r \in iR$ for every integer $i > 0$, i.e., nq_r is a nonzero $\mathbb{Z}$-divisible element in R, a contradiction. □

Definition 3.3. Let n and m be integers. If $n > 0$ denote by $\mathcal{Q}_R^{n,m}$ the submodule of $\mathcal{Q}_R^n$ consisting of all power series G such that $v(\partial^{n-1}G) \geqslant m$. If $n \leqslant 0$, set $\mathcal{Q}_R^{n,m} = x^{\max(0,m)} \cdot R[[x]]$.

Since $v(\partial^{n-1}G) \geqslant n$ for every $G \in \mathcal{Q}_R^n$ with $n > 0$, we have $\mathcal{Q}_R^{n,m} = \mathcal{Q}_R^{n,n} = \mathcal{Q}_R^n$ if $n \geqslant m$. Note also that $\mathcal{Q}_R^{1,m} = x^{\max(1,m)} \cdot R[[x]]$.

3.1. **The groups $\mathcal{Q}_{\widehat{\mathbb{Z}}}^n$.** In this section we determine the structure of the modules $\mathcal{Q}_{\widehat{\mathbb{Z}}}^n$ over the ring $\widehat{\mathbb{Z}} = \lim(\mathbb{Z}/n\mathbb{Z})$. We write $\widehat{\mathbb{Q}}$ for $\widehat{\mathbb{Z}} \otimes \mathbb{Q}$. Note that $\widehat{\mathbb{Q}} = \widehat{\mathbb{Z}} + \mathbb{Q}$ and $\mathbb{Z} = \widehat{\mathbb{Z}} \cap \mathbb{Q}$ in $\widehat{\mathbb{Q}}$.

Lemma 3.4. *Let $b_1, b_2, \ldots, b_m \in \widehat{\mathbb{Z}}$ be such that $b_i \equiv b_j \ (mod\ j)$ for every i divisible by j. Then there is $b \in \mathbb{Z}$ such that $b \equiv b_i \ (mod\ i)$ for all $i = 1, \ldots, m$.*

Proof. Let $p_1, p_2, \ldots, p_s$ be all primes at most m. For every k, let $q_k = p_k^{r_k}$ be the largest power of p_k such that $q_k \leqslant m$. By Chinese Remainder Theorem, we can find $b \in \mathbb{Z}$ such that $b \equiv b_{q_k} \ (mod\ q_k)$ for all k. We claim that b works. Take any $i \leqslant m$. We prove that $b \equiv b_i \ (mod\ i)$. Write i as the product $i = \prod q_k'$, where q_k' is a power of p_k. Clearly, q_k' divides q_k. We have

$$\begin{aligned} b_{q_k'} &\equiv b_i \quad (mod\ \ q_k') \quad \text{by assumption,} \\ b_{q_k} &\equiv b_{q_k'} \quad (mod\ \ q_k') \quad \text{by assumption,} \\ b &\equiv b_{q_k} \quad (mod\ \ q_k) \quad \text{by construction.} \end{aligned}$$

It follows that $b \equiv b_i \ (mod\ q_k')$ for all k, hence $b \equiv b_i \ (mod\ i)$. □

Let $G(x) = \sum_{i=1}^{\infty} a_i x^i$ with $a_i \in \widehat{\mathbb{Q}}$.

Lemma 3.5. *For positive integers $j \leqslant s$, the $x^j y^s$-coefficient of ∂G is equal to*

$$\sum_{i=0}^{j} (-1)^{j-i} \binom{s+i}{s} \binom{s}{j-i} a_{s+i}.$$

Proof. We have

$$\frac{1}{s!} \frac{\mathrm{d}^s G}{\mathrm{d}x^s} = \sum_{i=0}^{\infty} \binom{s+i}{s} a_{s+i} x^i.$$

The statement follows from Proposition 2.5. □

Set $b_i = i a_i$ for all $i \geqslant 1$.

Corollary 3.6. *If $\partial G \in \widehat{\mathbb{Z}}[[x, y]]$ then $b_i - b_1 \in \widehat{\mathbb{Z}}$ for all $i \geqslant 1$. In particular, if $a_1 \in \widehat{\mathbb{Z}}$, then all b_i are in $\widehat{\mathbb{Z}}$.*

Proof. The xy^j-coefficient of ∂G is equal to $b_{j+1} - b_j$. □

Proposition 3.7. *Let $G \in \mathcal{Q}^2_{\widehat{\mathbb{Z}}}$ and let $n > 1$ be an integer such that $a_i \in \widehat{\mathbb{Z}}$ for all $i < n$. Let $p^t < n$ be power of a prime integer p such that p^t divides n. Then p^t divides b_n.*

Proof. Take $j = p^t$ and $s = n - p^t \geqslant p^t$. By Lemma 3.5, the $x^j y^s$-coefficient of ∂G is equal to

$$\sum_{i=0}^{j} (-1)^{j-i} \binom{s+i}{s} \binom{s}{j-i} a_{s+i} \in \widehat{\mathbb{Z}}.$$

By assumption, all terms in the sum but the last one belong to $\widehat{\mathbb{Z}}$, hence so does the last one: $\binom{n}{p^t} a_n \in \widehat{\mathbb{Z}}$. But $\binom{n}{p^t} a_n = \binom{n-1}{p^t-1} b_n / p^t$, hence $\binom{n-1}{p^t-1} b_n$ is divisible by p^t. As $\binom{n-1}{p^t-1}$ is prime to p, the coefficient b_n is divisible by p^t. □

Proposition 3.8. *We have*

$$\mathcal{Q}^2_{\widehat{\mathbb{Z}}} = \widehat{\mathbb{Q}} \cdot \lg_1(x) \oplus x\widehat{\mathbb{Z}}[[x]].$$

Proof. Let $G(x) = \sum_{i=1}^{\infty} a_i x^i \in \mathcal{Q}^2_{\widehat{\mathbb{Z}}}$ and set as before $b_i = i a_i$. Adding $a_1 \lg_1(x)$ to $G(x)$ we may assume that $a_1 = 0$. By Corollary 3.6, $b_i \in \widehat{\mathbb{Z}}$ for all i.

We claim that for every positive integers $i < n$ such that i divides n we have $b_n \equiv b_i$ modulo i. We prove this by induction on n. By Lemma 3.4 applied to $m = n - 1$, there is $b \in \mathbb{Z}$ such that $b \equiv b_i$ modulo i for all $i < n$. Subtracting $b \lg_1(x)$ from $G(x)$, we may assume that b_i is divisible by i for all $i < n$, or equivalently, $a_i \in \widehat{\mathbb{Z}}$ for all $i < n$. We prove that b_n is divisible by i.

Case 1: $n = p^k$ is a power of a prime p. Then $i = p^t$ is a smaller power of p. By Proposition 3.7, i divides b_n.

Case 2: n is not power of a prime. Write n as a product of powers of distinct primes: $n = q_1 q_2 \cdots q_s$. By Proposition 3.7, q_k divides b_n for every k, hence n divides b_n. In particular, i divides b_n. The claim is proved.

Let $b \in \widehat{\mathbb{Z}}$ be such that $b \equiv b_n \ (mod\ n)$ for all n. We have

$$G = b \lg_1(x) + \sum_{n \geqslant 1} \frac{b_n - b}{n} x^n \in \widehat{\mathbb{Z}} \cdot \lg_1(x) + x\widehat{\mathbb{Z}}[[x]].$$

□

Corollary 3.9. *Let $G(x) = ax + \ldots \in \mathcal{Q}^2_{\widehat{\mathbb{Z}}}$ be a power series with $a \in \widehat{\mathbb{Z}}$. Then $G(x) \in \widehat{\mathbb{Z}} \cdot \lg_1(x) + x\widehat{\mathbb{Z}}[[x]]$.*

Lemma 3.10. *Let $H(x,y) = \sum_{i,j\geqslant 1} a_{i,j}x^i y^j \in \widehat{\mathbb{Q}}[[x,y]]$ be such power series that both ∂-partial derivatives of H have coefficients in $\widehat{\mathbb{Z}}$ and $a_{i,1}$ as well as $a_{1,i}$ are in $\widehat{\mathbb{Z}}$, for all i. Then $H(x,y) \in \widehat{\mathbb{Z}}[[x,y]]$.*

Proof. Consider some j-th row of H: $y^j \cdot \sum_{i\geqslant 1} a_{i,j}x^i$. We know that $\sum_{i\geqslant 1} a_{i,j}x^i \in \mathcal{Q}^2_{\widehat{\mathbb{Z}}}$. By Corollary 3.9, $\sum_{i\geqslant 1} a_{i,j}x^i$ is equal to $c_j \cdot \lg_1(x)$ modulo $x\widehat{\mathbb{Z}}[[x]]$ for some $c_j \in \widehat{\mathbb{Z}}$. Hence $\frac{c_j}{i} \equiv a_{i,j}\ (mod\ \widehat{\mathbb{Z}})$ for all i. Applying the same considerations to the i-th column, we obtain:

$$\frac{c_j}{i} \equiv \frac{d_i}{j}\ (mod\ \widehat{\mathbb{Z}}),$$

for certain $d_i \in \widehat{\mathbb{Z}}$. Let us show that all c_i's (and d_j's) are zeros. Indeed, we have:

$$jc_j \equiv id_i\ (mod\, ij).$$

Hence, jc_j is divisible by i, for any i and, hence $c_j = 0$. This implies that $a_{i,j} \in \widehat{\mathbb{Z}}$ for any i, j. □

Lemma 3.11. *Suppose, $H(x_1,...,x_n) = \sum_{i_1,\ldots,i_n>0} a_{i_1,\ldots,i_n}x_1^{i_1} \cdot \ldots \cdot x_n^{i_n} \in \widehat{\mathbb{Q}}[[x_1,\ldots,x_n]]$ be such a power series that all ∂-partial derivatives of H with respect to all variables have coefficients in $\widehat{\mathbb{Z}}$ and $a_{i_1,\ldots,i_n} \in \widehat{\mathbb{Z}}$ as long as all i_j's but one are equal to 1. Then H has coefficients in $\widehat{\mathbb{Z}}$.*

Proof. Induction on n. For $n = 1$ there is nothing to prove. For $n = 2$ this is Lemma 3.10. We can assume that $n \geqslant 3$. Suppose we know the statement for $r < n$. Note, that all the cells of H (where we set certain i_j's to be 1) also satisfy the conditions of the Lemma. By our assumption, these have all coefficients in $\widehat{\mathbb{Z}}$. That is, $a_{i_1,\ldots,i_n} \in \widehat{\mathbb{Z}}$ provided, at least, one of i_j's is 1. Consider a hyper-slice H_{i_1} of H (we fix i_1). Then H_{i_1} satisfies the conditions of the Lemma (note that $n \geqslant 3$). Thus, H_{i_1} has coefficients in $\widehat{\mathbb{Z}}$ and so does H. □

The following theorem is a generalization of Proposition 3.8.

Theorem 3.12. *For every $n \geqslant 1$,*

$$\mathcal{Q}^n_{\widehat{\mathbb{Z}}} = \coprod_{0<r<n} \widehat{\mathbb{Q}} \cdot \lg_r(x) \oplus x\widehat{\mathbb{Z}}[[x]].$$

Proof. The statement is clear if $n \leqslant 0$. Now assume that $n \geqslant 1$. It follows from Lemma 3.2 that $\coprod_{0<r<n} \widehat{\mathbb{Q}} \cdot \lg_r(x) \cap \widehat{\mathbb{Z}}[[x]] = 0$.

We prove the rest by induction on n. For $n = 1$ this is so by definition and for $n = 2$ this is given by Proposition 3.8.

$n \Rightarrow n+1$: Let $G \in \mathcal{Q}^{n+1}_{\widehat{\mathbb{Z}}}$. Consider the power series $H(x_1,...,x_n) = \partial^{n-1}(G)$. Let

$$H(x_1,...,x_n) = \sum_{i_1,\ldots,i_n\geqslant 1} a_{i_1,\ldots,i_n}x_1^{i_1} \cdot \ldots \cdot x_n^{i_n}.$$

Note that the smallest term of $\partial^{n-1}(\lg_n(x))$ is $(-1)^n x_1 \cdot ... \cdot x_n$. By subtracting an appropriate $\widehat{\mathbb{Q}}$-multiple of $\lg_n(x)$ from G, we may assume that $a_{1,...,1} = 0$.

As $\partial(H)$ has coefficients in $\widehat{\mathbb{Z}}$, the "ray" $\sum_{i\geqslant 1} a_{i,1,...,1}x_1^i$ is a power series with terms of degree $\geqslant 2$ whose ∂-derivative is integral. By Corollary 3.9, up to a power series in $\widehat{\mathbb{Z}}[[x_1]]$, it is equal to $c \cdot \lg_1(x_1)$, for some $c \in \hat{\mathbb{Z}}$.

Since

$$\partial^{n-1}(\lg_n)(x_1, ..., x_n) = \lg_1(x_1) \cdot \ldots \cdot \lg_1(x_n),$$

subtracting from $G(x)$ an appropriate multiple of $\lg_n(x)$, we may assume that the coefficients $a_{i,1,...,1}$ are in $\widehat{\mathbb{Z}}$, for all $i \geqslant 1$. Since H is symmetric, by Lemma 3.11, all coefficients of the power series H are in $\widehat{\mathbb{Z}}$. By the induction hypothesis, $G(x) \in \mathcal{Q}^n = \coprod_{0<r<n} \widehat{\mathbb{Q}} \cdot \lg_r(x) + x\widehat{\mathbb{Z}}[[x]]$. □

3.2. **The groups $\mathcal{Q}^n$.** Write $\mathcal{Q}^n$ for $\mathcal{Q}^n_{\mathbb{Z}} \subset \mathcal{Q}^n_{\widehat{\mathbb{Z}}}$.

We define a homomorphism

$$\rho_n : \mathcal{Q}^n \to \widehat{\mathbb{Q}}^{n-1}$$

for $n \geqslant 1$ as the composition (see Theorem 3.12)

$$\mathcal{Q}^n \hookrightarrow \mathcal{Q}^n_{\widehat{\mathbb{Z}}} = \coprod_{0<r<n} \widehat{\mathbb{Q}} \cdot \lg_r(x) \oplus x\widehat{\mathbb{Z}}[[x]] \xrightarrow{\text{proj}} \coprod_{0<r<n} \widehat{\mathbb{Q}} \cdot \lg_r(x) \simeq \widehat{\mathbb{Q}}^{n-1}.$$

We will show that the map ρ_n is surjective.

As in the proof of Lemma 3.13 below we deduce that

$$\lg_r(x) = (-1)^r \sum_{0<i_1<...<i_r} \frac{x^{i_r}}{i_1 \cdot ... \cdot i_r} \equiv (-1)^r \frac{x^r}{r!} \quad (mod\ x^{r+1}).$$

For a sequence of $a = (a_i)_{i\geqslant 1}$ in $\widehat{\mathbb{Z}}$ let us denote by $a \cdot \lg_r(x) \in \widehat{\mathbb{Q}}[[x]]$ the power series

$$(-1)^r \sum_{0<i_1<...<i_r} \frac{a_{i_1} \cdot x^{i_r}}{i_1 \cdot ... \cdot i_r} \equiv (-1)^r \frac{a_1}{r!} x^r \quad (mod\ x^{r+1}).$$

If all $a_i \in \mathbb{Z}$, we have $a \cdot \lg_r(x) \in \mathbb{Q}[[x]]$.

Lemma 3.13. *For every sequence a, we have*

$$(x-1) \cdot \frac{\mathrm{d}}{\mathrm{d}x} (a \cdot \lg_r(x)) = a \cdot \lg_{r-1}(x).$$

Proof. Write $(-1)^r a \cdot \lg_r(x) = \sum b_i x^i$ and $(-1)^{r-1} a \cdot \lg_{r-1}(x) = \sum c_i x^i$. We need to prove that $(m+1)b_{m+1} - mb_m = c_m$ for every m. We have

$$(m+1)b_{m+1} = \sum_{0<i_1<...<i_r<m+1} \frac{a_{i_1}}{i_1 \cdot ... \cdot i_r}.$$

The sum of the terms with $i_r < m$ is equal to mb_m. The sum of the terms with $i_r = m$ coincides with c_m. □

Lemma 3.14. *For every $c \in \widehat{\mathbb{Z}}$ and every integer $r > 0$ there is a sequence $\tilde{c} = (c_i)_{i\geqslant 1}$ of integers $c_i \in \mathbb{Z}$ such that $c_i \equiv c\ (mod\ i)$ for all i and*

$$(c - \tilde{c}) \cdot \lg_i(x) \in \widehat{\mathbb{Z}}[[x]]$$

for all $i = 1, \ldots, r$, where $c - \tilde{c}$ is the sequence $(c - c_i)_{i\geqslant 1}$.

Proof. Take any collection $\tilde{c} = (c_i)_{i\geqslant 1}$ of integers. Note that for every $i \geqslant 1$ and $k = 1, \dots, r$, the x^{i+k-1}-coefficient of $\tilde{c}\cdot \lg_k(x)$ is a linear combination of $c_1, \dots, c_i$ with rational coefficients where the c_i-coefficient is equal to $(-1)^k/(i(i+1)\dots(i+k-1))$.

We will modify $c_1, c_2, \dots$ inductively to make all coefficients of the power series

$$G_k = (c - \tilde{c}) \cdot \lg_k(x)$$

integral for all $k = 1, \dots, r$. Let c_1 be an integer congruent to c modulo $r!$, so the x^k-coefficient of G_k is integer for every $k = 1, \dots, r$. Suppose we have modified $c_1, \dots c_n$ so that the x^j-coefficient of G_k is integral for all $k = 1, \dots, r$ and $j \leqslant n + k - 1$.

By induction on $k = 1, \dots, r$, we will modify c_{n+1} to make integral the x^{n+k}-coefficient of G_k. Note that the integral x^j-coefficients of G_k for $j \leqslant n + k - 1$ will not change. If $k = 1$ we don't modify c_{n+1}: the power series G_1 is already integral.

$k \Rightarrow k + 1$: By Lemma 3.13,

$$(x-1)\cdot \frac{\mathrm{d}G_{k+1}}{\mathrm{d}x} = G_k.$$

Hence, if $G_k = \sum_{i\geqslant k} b_i x^i$ and $G_{k+1} = \sum_{i\geqslant k+1} a_i x^i$, then

$$a_{n+l+1} = \tfrac{-1}{n+l+1}(b_k + \ldots + b_{n+l})$$

for all l.

By induction, $b_k, \dots, b_{n+k}$ are integral. Recall that these are linear combinations of the c_i''s, where $c_i' = c - c_i$ and c_{n+1}' appears only in b_{n+k}. We modify c_{n+1} by adding to c_{n+1} the integer $t(n+1)(n+2)\dots(n+k)$ with some $t \in \mathbb{Z}$. Note that $b_k, \dots, b_{n+k-1}$ remain unchanged and b_{n+k} changes to $b_{n+k} + t$, so it stays integral. Choose t to make a_{n+k+1} integral.

Note that c_{n+1}' comes with coefficient $(-1)^l/((n+1)\dots(n+l))$ in the x^{n+l}-coefficient of G_l. Since $(n+1)\dots(n+l)$ divides $(n+1)\dots(n+k)$ when $l \leqslant k$, the x^{n+l}-coefficient of G_l remains integral for $l \leqslant k$. □

Now we prove that the map $\rho_n : \mathcal{Q}^n \to \widehat{\mathbb{Q}}^{n-1}$ is surjective. Since $q\cdot\lg_r \in \mathcal{Q}^n$ for all $q \in \mathbb{Q}$ and $r = 1, \dots, n-1$, we have $\mathbb{Q}^{n-1} \subset \mathrm{Im}(\rho_n)$. It suffices to show that $\widehat{\mathbb{Z}}^{n-1} \subset \mathrm{Im}(\rho_n)$. Choose $c_r \in \widehat{\mathbb{Z}}$ for $r = 1, \dots, n-1$. By Lemma 3.14, there are sequences of integers $\tilde{c}_r$ such that $(c_r - \tilde{c}_r)\cdot \lg_r(x) \in \widehat{\mathbb{Z}}[[x]]$.

As

$$\tilde{c}_r \cdot \lg_r(x) = c_r \cdot \lg_r(x) - (c_r - \tilde{c}_r)\cdot \lg_r(x),$$

we have $\rho_n\big(\sum_{0<r<n} \tilde{c}_r \cdot \lg_r(x)\big) = (c_r)_{r=1,\dots,n-1}$ proving that ρ_n is surjective.

Note that the kernel of ρ_n is equal to $x\widehat{\mathbb{Z}}[[x]] \cap \mathbb{Q}[[x]] = x\mathbb{Z}[[x]]$. Thus, we have an exact sequence

$$0 \to x\mathbb{Z}[[x]] \to \mathcal{Q}^n \xrightarrow{\rho_n} \widehat{\mathbb{Q}}^{n-1} \to 0. \tag{3.15}$$

We have proved that if $n \geqslant 1$, the group $\mathcal{Q}^n$ is generated by $x\mathbb{Z}[[x]]$ and the power series $(c - \tilde{c})\cdot \lg_r(x)$ as in Lemma 3.14, where $c \in \widehat{\mathbb{Z}}$ and $r = 1, \dots, n-1$.

The power series in $\mathcal{Q}^n$ can be approximated by polynomials as follows:

Lemma 3.16. *For every $m > 0$ and n, we have*

$$\mathcal{Q}^n \subset \mathbb{Z}[x]_{\leqslant m-1} + \sum_{0<r<n} \mathbb{Q} \cdot \lg_r(x) + x^m \mathbb{Q}[[x]],$$

where $\mathbb{Z}[x]_{\leqslant m-1}$ is the group of integral polynomials of degree at most $m-1$.

Proof. We may assume that $n > 1$. In view of (3.15), the group $\mathcal{Q}^n$ modulo

$$x\mathbb{Z}[[x]] + \sum_{0<r<n} \mathbb{Q} \cdot \lg_r(x)$$

is generated by power series of the form $\tilde{c} \cdot \lg_r(x)$, where $r = 1, \dots, n-1$ and $\tilde{c}$ is the collection of integers such that $c_i \equiv c \ (mod\ i)$ for all i for an element $c \in \widehat{\mathbb{Z}}$ as in Lemma 3.14.

Let d be an integer congruent to c modulo the least common multiple of the denominators of the x^i-coefficients of $\lg_r(x)$ for all $i = 1, \dots, m-1$. Then the x^m-truncation F of $(\tilde{c} - d) \cdot \lg_r(x)$ is contained in $\mathbb{Z}[x]_{\leqslant m-1}$ and $\tilde{c} \cdot \lg_r(x)$ is congruent to F modulo $\mathbb{Z} \cdot \lg_r(x) + x^m \mathbb{Q}[[x]]$. □

4. Operations

Let k be a field of characteristic 0 and write $\mathbf{Sm}_k$ for the category of smooth quasi-projective varieties over k. An *oriented cohomology theory* A^* over k is a functor from $\mathbf{Sm}_k^{op}$ to the category of $\mathbb{Z}$-graded commutative rings equipped with a push-forward structure and satisfying certain axioms (see [11, Definition 2.1]). We write

$$A^*(X) = \coprod_{n \in \mathbb{Z}} A^n(X)$$

for a variety X in $\mathbf{Sm}_k$ and let $A^*(k)$ denote the *coefficient ring* $A^*(\operatorname{Spec} k)$.

Let A^* be an oriented cohomology theory. There is a (unique) associated formal group law

$$F_A(x, y) = \sum_{i,j \geqslant 1} a^A_{i,j} x^i y^j = x + y + a_{1,1} \cdot xy + \text{higher terms} \in A^*(k)[[x, y]]$$

that computes the first Chern class of the tensor product of two line bundles L and L' (see, for example, [7, §1.1] or [11, §2.3]):

$$c_1^A(L \otimes L') = F_A(c_1^A(L), c_1^A(L')).$$

Example 4.1. The *Chow theory* CH^* takes a smooth variety X to the Chow ring $\mathrm{CH}^*(X)$ of X. We have $\mathrm{CH}^*(k) = \mathbb{Z}$ and $F_{\mathrm{CH}}(x, y) = x + y$ is the *additive* group law.

Example 4.2. (see [7, Example 1.15]) The *graded K-theory* K^*_{gr} takes X to the Laurent polynomial ring $K_0(X)[t, t^{-1}]$ (graded by the powers of the *Bott element* t of degree -1) over the Grothendieck ring $K_0(X)$ of X. We have $K^*_{gr}(k) = \mathbb{Z}[t, t^{-1}]$ and $F_{K_{gr}}(x, y) = x + y - txy$ is the *multiplicative* group law.

Example 4.3. (see [3] and [5]) The *connective K-theory* takes X to the ring $\mathrm{CK}^*(X)$ of X. We have $\mathrm{CK}^*(k) = \mathbb{Z}[t]$ and $F_{CK}(x, y) = x + y - txy$.

All cohomology theories in these examples are of *rational type* (see [11, §4.1]).

If A^* is an oriented cohomology theory and R a commutative ring, the functor A^*_R defined by $A^*_R(X) = A^*(X) \otimes_{\mathbb{Z}} R$ is also an oriented cohomology theory with values in the category of graded R-algebras.

Definition 4.4. Let A^* and B^* be two oriented cohomology theories. An *R-linear operation* $G : A^*_R \to B^*_R$ is a morphism between functors A^*_R and B^*_R considered as contravariant functors from $\mathbf{Sm}_k$ to the category of R-modules (cf. [11, Definition 3.3]). Note that G may not respect the gradings on A^*_R and B^*_R.

Let $n, m \in \mathbb{Z}$. A morphism $G : A^n_R \to B^m_R$ between contravariant functors from $\mathbf{Sm}_k$ to the category of R-modules can be viewed as an R-linear operation via the obvious composition $A^*_R \twoheadrightarrow A^n_R \to B^m_R \hookrightarrow B^*_R$. All such operations form an R-module $\mathbf{OP}^{n,m}_R(A^*, B^*)$. The composition of operations yields an R-linear pairing

$$\mathbf{OP}^{n,m}_R(A^*, B^*) \otimes_R \mathbf{OP}^{m,r}_R(B^*, C^*) \to \mathbf{OP}^{n,r}_R(A^*, C^*).$$

In particular, $\mathbf{OP}^{n,n}_R(A^*) := \mathbf{OP}^{n,n}_R(A^*, A^*)$ has a structure of an R-algebra.

Example 4.5. (see [3] and [5]) Multiplication by t yields an operation $\mathrm{CK}^{n+1}_R \to \mathrm{CK}^n_R$ that is an isomorphism if $n < 0$. There are graded R-linear operations

$$\mathrm{CK}^*_R \to \mathrm{CH}^*_R \quad \text{and} \quad \mathrm{CK}^*_R \to (K^*_{gr})_R.$$

The sequence

$$\mathrm{CK}^{n+1}(X) \xrightarrow{t} \mathrm{CK}^n(X) \to \mathrm{CH}^n(X) \to 0$$

is exact for every n and X.

If $n \geqslant 0$ the image of the homomorphism $\mathrm{CK}^n(X) \to K^n_{gr}(X) = K_0(X)t^{-n} \simeq K_0(X)$ is generated by the classes of coherent $\mathcal{O}_X$-modules with codimension of support at least n. If $n \leqslant 0$ this map is an isomorphism.

The following fundamental theorem was proved in [11, Theorem 6.2].

Theorem 4.6. *Let A^* be a cohomology theory of rational type and B^* be any oriented cohomology theory over k. Let R be a commutative ring. Then there is a 1-to-1 correspondence between the set $\mathbf{OP}^{n,m}_R(A^*, B^*)$ of R-linear operations $G : A^n_R \to B^m_R$ and the set consisting of the following data $\{G_l, l \in \mathbb{Z}_{\geqslant 0}\}$:*

$$G_l \in \mathrm{Hom}_R\left(A^{n-l}(k) \otimes R, B^*(k)[[x_1, \ldots, x_n]]_{(m)} \otimes R\right) \quad \textit{satisfying}$$

1. *$G_l(\alpha)$ is a symmetric power series for all l and $\alpha \in A^{n-l}(k) \otimes R$,*
2. *$G_l(\alpha)$ is divisible by $x_1 \cdot \ldots \cdot x_l$ for all l and α,*
3. *$G_l(\alpha)(y +_B z, x_2, \ldots, x_l) = \sum_{i,j} G_{i+j+l-1}(\alpha \cdot a^A_{i,j})(y^{\times i}, z^{\times j}, x_2, \ldots, x_l)$, where $a^A_{i,j}$ are the coefficients of the formal group law of A^* and the sum $y +_B z$ is taken with respect to the formal group law of B^*.*

Here $B^*(k)[[x_1, \ldots, x_n]]_{(m)}$ is the subgroup in $B^*(k)[[x_1, \ldots, x_n]]$ consisting of all homogeneous degree m power series (all the x_i's have degree 1).

The functions G_l are determined by the operation G as follows (see [11, §5]). Write L_i for the pull-back of the canonical line bundle on $\mathbb{P}^\infty$ with respect to the i-th projection

$(\mathbb{P}^\infty)^l \to \mathbb{P}^\infty$. Then

$$G_l(\alpha)\big(c_1^B(L_1),\dots,c_1^B(L_l)\big) = G\big(\alpha\cdot c_1^A(L_1)\cdot\ldots\cdot c_1^A(L_l)\big), \tag{4.7}$$

where c_1 is the first Chern class.

Remark 4.8. Theorem 4.6 was proved in [11, Theorem 6.2] in the case $R=\mathbb{Z}$. The general case readily follows. Indeed, multiplication by an element $r\in R$ yields operations $r: A_R^n\to A_R^n$ and $r: B_R^m\to B_R^m$. An additive operation $G: A_R^n\to B_R^m$ is R-linear if and only if $G\circ r = r\circ G$ for all $r\in R$. The latter is equivalent to the equality $G_l\circ r = r\circ G_l$ for all l, i.e., that all G_l are R-linear.

Example 4.9. (see [11, §6.3]) Let A^* be a cohomology theory of rational type and $m\in\mathbb{Z}$. Consider the power series $[m](x) := x +_A \ldots +_A x \in A^*(k)[[x]]$ (m times). The *Adams operation* $\Psi_m^A\in \mathbf{OP}_R^{*,*}$ is determined by $(G_l)_{l\geqslant 0}$, where G_l is multiplication by the power series $[m](x_1)\cdot\ldots\cdot[m](x_l)$. The Adams operations satisfy the relations

$$\Psi_k^A\circ\Psi_m^A = \Psi_{km}^A = \Psi_m^A\circ\Psi_k^A$$

for all k and m.

4.1. **Operations in connective K-theory.** We would like to determine the R-module $\mathbf{OP}_R^{n,m}$ of all R-linear operations $G:\mathrm{CK}_R^n\to\mathrm{CK}_R^m$ for any pair of integers n and m. By Theorem 4.6, G is given by a collection of power series $G_l(\alpha)\in R[t][[x_1,\dots,x_n]]_{(m)}$, where $\alpha\in\mathrm{CK}_R^{n-l}(k)$ and $l\geqslant 0$, satisfying conditions of the theorem. The group $\mathrm{CK}_R^{n-l}(k)$ is trivial if $l<n$ and $\mathrm{CK}_R^{n-l}(k) = R\cdot t^{l-n}$ otherwise. (Recall that t has degree -1.) In the first case $G_l(\alpha)=0$ and in the latter case the power series $G_l(\alpha)$ are uniquely determined by $G_l(t^{l-n})$. We will simply write G_l for $G_l(t^{l-n})$.

If $l\geqslant\max(1,n)$, condition (3) in Theorem 4.6 reads as follows:

$$G_l(x+y-txy,\bar z) = G_l(x,\bar z) + G_l(y,\bar z) - G_{l+1}(x,y,\bar z).$$

In other words,

$$G_{l+1} = -\partial_t G_l, \tag{4.10}$$

where the derivative ∂_t is taken with respect to $F_{\mathrm{CK}}(x,y) = x+y-txy$. Thus, G_{l+1} is uniquely determined by G_l.

If $n>0$, the operation G yields the double-symmetric power series $G_n\in R[t][[x_1,\dots,x_n]]_{(m)}$ that is divisible by $x_1\cdot\ldots\cdot x_n$. Conversely, if $H\in R[t][[x_1,\dots,x_n]]_{(m)}$ is a double-symmetric power series divisible by $x_1\cdot\ldots\cdot x_n$, then setting $G_{n+i} := (-1)^i\partial_t^i(H)$ for all $i\geqslant 0$, we get a sequence of power series that determines an R-linear operation G (see Observation 2.2).

If $n\leqslant 0$ the operation G is determined by $G_0\in R[t]_m$ and power series $G_1\in R[t][[x]]_m$ that is uniquely determined by $(G_1)|_{t=1}\in x^{\max(1,m)}R[[x]]$. If $m>0$ then $G_0=0$, otherwise $G_0\in R\cdot t^{-m}$ and we can combine G_0 and G_1 together into the power series $H = (G_0-G_1)|_{t=1}\in R[[x]]$.

If $L\in R[t][[x_1,\dots,x_n]]_{(m)}$, then $v(L|_{t=1})\geqslant m$. Conversely, for every $J\in R[[x_1,\dots,x_n]]$ with $v(J)\geqslant m$, there is a unique homogeneous power series $L\in R[t][[x_1,\dots,x_n]]$ of degree m such that $L|_{t=1}=J$. If L is double-symmetric and divisible by $x_1\cdot\ldots\cdot x_n$, then

so is $L|_{t=1}$ (with respect to the derivative ∂ given by the formal group law $x+y-xy$) and conversely.

We have proved the following statement.

Proposition 4.11. *Let R be a commutative ring and let n and m be two integers. An R-linear operation $G : \mathrm{CK}^n_R \to \mathrm{CK}^m_R$ is determined by*

1. *A power series $H \in x^{\max(0,m)}R[[x]]$ if $\underline{n \leqslant 0}$. In this case $G_0 = H(0)\cdot t^{-m}$ and $G_1 \in xR[t][[x]]_{(m)}$ is a unique homogeneous power series such that $H = (G_0 - G_1)|_{t=1}$ and $G_l = (-1)^{l-1}\partial_t^{l-1}(G_1)$ for $l > 1$,*
2. *A double-symmetric power series $J \in R[[x_1, \dots, x_n]]$ divisible by $x_1 \cdot \ldots \cdot x_n$ such that $v(J) \geqslant m$ if $\underline{n > 0}$. In this case $G_l = 0$ for $l = 0, \dots, n-1$ and $G_n \in R[t][[x_1, \dots, x_n]]_{(m)}$ is a unique homogeneous power series such that $G_n|_{t=1} = J$ and $G_l = (-1)^{l-n}\partial_t^{l-n}(G_n)$ for $l > n$.*

Let R be a commutative ring that is torsion free as abelian group. Define an R-module homomorphism

$$\lambda_{n,m} : \mathcal{Q}^{n,m}_R \to \mathbf{OP}^{n,m}_R$$

as follows. If $n \leqslant 0$, $\lambda_{n,m}(H)$ for $H \in \mathcal{Q}^{n,m} = x^{\max(0,m)} \cdot R[[x]]$ is the operation given by Proposition 4.11(1). If $n > 0$, $\lambda_{n,m}(H)$ for $H \in \mathcal{Q}^{n,m}$ is the operation given by the polynomial $J = (-1)^n\partial^{n-1}(H)$ as in Proposition 4.11(2).

The following theorem determines the R-module of operations $\mathbf{OP}^{n,m}_R$ in terms of the modules $\mathcal{Q}^{n,m}_R$ of power series in one variable.

Theorem 4.12. *Let R be a commutative ring that is torsion free as abelian group and $K = R \otimes \mathbb{Q}$. The homomorphisms $\lambda_{n,m}$ yield an R-linear isomorphisms between $\mathbf{OP}^{n,m}_R$ and the factor module of $\mathcal{Q}^{n,m}_R$ by the K-subspace spanned by $\lg_i(x)$, $i = 1, \dots, n-1$. In particular, $\mathbf{OP}^{n,m}_R \simeq x^{\max(0,m)} \cdot R[[x]]$ if $n \leqslant 0$ and $\mathbf{OP}^{1,m}_R \simeq x^{\max(1,m)} \cdot R[[x]]$.*

Proof. The surjectivity of $\lambda_{n,m}$ follows from Propositions 2.8 and 4.11. The kernel of $\lambda_{n,m}$ is determined in Proposition 2.4. □

Corollary 4.13. *The map $\lambda_{n,m}$ yields an isomorphism*

$$\mathcal{Q}^n_R \cap x^{\max(0,n,m)} \cdot K[[x]] \xrightarrow{\sim} \mathbf{OP}^{n,m}_R .$$

Proof. The case $m \leqslant n$ follows from the theorem. Otherwise, by Observation 2.6, $v(\partial^{n-1}x^i) = i$ for all $i \geqslant n$. □

Let $n, m \in \mathbb{Z}$ and i, j non-negative integers. We define an R-linear homomorphism

$$\mathcal{Q}^{n,m}_R \to \mathcal{Q}^{n+i,m-j}_R$$

as follows. If $n \leqslant 0, m \leqslant 0$ and $n + i > 0$ the map

$$\mathcal{Q}^{n,m}_R = R[[x]] \to xR[[x]] \hookrightarrow \mathcal{Q}^{n+i,m-j}_R$$

takes H to $\partial^0(H) = H - H(0)$. Otherwise, $\mathcal{Q}^{n,m}_R \subset \mathcal{Q}^{n+i,m-j}_R$, and the map we define is the inclusion.

Multiplication by t^k yields an operation $\mathrm{CK}^{*+k}_R \to \mathrm{CK}^*_R$ and therefore, the homomorphisms $\mathbf{OP}^{n,m}_R \to \mathbf{OP}^{n+i,m-j}_R$ for all $i, j \geqslant 0$.

Proposition 4.14. *The diagram*

$$\begin{array}{ccc} \mathcal{Q}_R^{n,m} & \longrightarrow & \mathcal{Q}_R^{n+i,m-j} \\ {\scriptstyle\lambda_{n,m}}\downarrow & & \downarrow{\scriptstyle\lambda_{n+i,m-j}} \\ \mathbf{OP}_R^{n,m} & \longrightarrow & \mathbf{OP}_R^{n+i,m-j}, \end{array}$$

is commutative.

Proof. The case $i = 0$ follows directly from the definition. It remains to consider the case $i = 1$ and $j = 0$.

Suppose first that $n > 0$. Let $H \in \mathcal{Q}_R^{n,m} \subset \mathcal{Q}_R^{n+i,m-j}$ and $G = \lambda_{n,m}(H) \in \mathbf{OP}_R^{n,m}$. In particular, $G_n|_{t=1} = (-1)^{n-1}\partial^{n-1}(H)$. Denote by G' the image of G in $\mathbf{OP}_R^{n+1,m}$. Write L_i for the pull-back of the canonical line bundle on $\mathbb{P}^\infty$ with respect to the i-th projection $(\mathbb{P}^\infty)^{n+1} \to \mathbb{P}^\infty$. The power series G'_{n+1} is determined by the equality (see (4.7))

$$\begin{aligned} G'_{n+1}(c_1(L_1), \dots, c_1(L_{n+1})) &= G'(c_1(L_1) \cdot \ldots \cdot c_1(L_{n+1})) \\ &= G(tc_1(L_1) \cdot \ldots \cdot c_1(L_{n+1})) \\ &= G_{n+1}(t)(c_1(L_1), \dots, c_1(L_{n+1})) \\ &= G_{n+1}(c_1(L_1), \dots, c_1(L_{n+1})), \end{aligned}$$

hence $G'_{n+1} = G_{n+1}$. It follows from (4.10) that

$$G'_{n+1}|_{t=1} = G_{n+1}|_{t=1} = -(\partial_t G_n)|_{t=1} = -\partial(G_n|_{t=1}) = -\partial((-1)^n\partial^{n-1}(H)) = (-1)^{n+1}\partial^n(H),$$

and therefore, $G' = \lambda_{n+1,m}(H)$.

If $n < 0$ or if $n = 0$ and $m > 0$ we have $\mathcal{Q}_R^{n,m} \subset \mathcal{Q}_R^{n+1,m}$ and the statement follows immediately from the definitions. It remains to consider the case $n = 0$ and $m \leqslant 0$. Let $H \in \mathcal{Q}_R^{0,m} = R[[x]]$ and $G = \lambda_{0,m}(H) \in \mathbf{OP}_R^{0,m}$. In particular, $H = (G_0 - G_1)|_{t=1}$. Denote by G' the image of G in $\mathbf{OP}_R^{1,m}$. A computation as above shows that $G'_1 = G_1$. Hence

$$G'_1|_{t=1} = G_1|_{t=1} = -(H - H(0))$$

Therefore, $G' = \lambda_{1,m}(H - H(0))$ and $H - H(0)$ is the image of H in $\mathcal{Q}_R^{1,m}$. □

Corollary 4.13 and Proposition 4.14 yield:

Corollary 4.15. *If $m \leqslant n$ then the map $\mathbf{OP}_R^{n,n} \to \mathbf{OP}_R^{n,m}$ is an isomorphism.*

In particular, there is a canonical ring homomorphism

$$\mathbf{OP}_R^{n,n} \to \mathbf{OP}_R^{n+1,n} \xrightarrow{\sim} \mathbf{OP}_R^{n+1,n+1}.$$

Example 4.16. Note that the identification $\mathbf{OP}_R^{0,0} = R[[x]]$ is not a ring isomorphism. The corresponding ring structure on $R[[x]]$ will be described in Section 4.5. The natural surjective homomorphism

$$R[[x]] = \mathbf{OP}_R^{0,0} \to \mathbf{OP}_R^{1,1} = xR[[x]]$$

takes a power series $G(x)$ to $G(x) - G(0)$. Its kernel is generated by 1. The complementary operation $G(x) \mapsto G(0)$ on $\mathrm{CK}^0 = K_0$ is an idempotent that takes the class of a vector

bundle E to $\operatorname{rank}(E)\cdot 1$, where 1 is the identity in K_0. In particular, we get a natural R-algebra isomorphism $\mathbf{OP}^{0,0}_R \simeq R \times \mathbf{OP}^{1,1}_R$.

4.2. **Adams operations.** Let R be a torsion free ring. We define the composition

$$\mathrm{Ad}_n : R[[x]] \to \mathcal{Q}^n_R \xrightarrow{\lambda_{n,n}} \mathbf{OP}^{n,n}_R,$$

where the first map is the identity if $n \leqslant 0$ and it is the composition of the projection $\partial^0 : R[[x]] \to xR[[x]]$ and the inclusion of $xR[[x]]$ into $\mathcal{Q}^n_R$. The image of Ad_n is denoted $\mathbf{OP}^{n,n}_{R,cl}$ and called the submodule of *classical operations*.

If $n \leqslant 0$, we have $\mathbf{OP}^{n,n}_{R,cl} = \mathbf{OP}^{n,n}_R = R[[x]]$. If $n \geqslant 1$ it follows from Lemma 3.2 and Theorem 4.12 that in the case R has no nontrivial $\mathbb{Z}$-divisible elements (for example. $R = \mathbb{Z}$ or $\widehat{\mathbb{Z}}$), the restriction of Ad_n on $xR[[x]]$ is injective and therefore, $\mathbf{OP}^{n,n}_{R,cl} \simeq xR[[x]]$.

Let m be an integer. In the notation of the Example 4.9, $[m](x) = (1-(1-tx)^m)/t$. In view of Proposition 4.11, the Adams operations $\Psi_m \in \mathbf{OP}^{n,n}_{R,cl}$ are defined by

$$\Psi_k = \mathrm{Ad}_n((1-x)^m). \tag{4.17}$$

Since the power series $(1-x)^m$ generate $R[[x]]$ as topological group in the x-adic topology, the group of classical operations $\mathbf{OP}^{n,n}_{R,cl}$ is topologically generated by the Adams operations.

By Proposition 4.14, the operations Ψ_k are compatible with the canonical homomorphisms $\mathbf{OP}^{n,n}_R \to \mathbf{OP}^{n+1,n+1}_R$.

For every $k \geqslant 0$ consider additive operations $\Upsilon_k = \sum_{i=0}^k (-1)^i \binom{k}{i} \Psi_i$. Then $\Upsilon_k = \lambda_{n,n}(x^k)$ if $k \geqslant 0$. Recall that $\Upsilon_0 = 0$ if $n \geqslant 1$. It follows that the R-module $\mathbf{OP}^{n,n}_{R,cl}$ consists of all linear combinations $\sum_{k\geqslant 0} \alpha_k \cdot \Upsilon_k$ with $\alpha_k \in R$ (cf. [11, Theorem 6.8]). If R has no nontrivial $\mathbb{Z}$-divisible elements, the coefficients α_k, (where $k \geqslant 0$ if $n \leqslant 0$ and $k \geqslant 1$ if $n \geqslant 1$) are uniquely determined by the operation.

4.3. **Operations over $\widehat{\mathbb{Z}}$.** In Section 3 we determined the modules $\mathcal{Q}^n_R$ over the ring $R = \widehat{\mathbb{Z}}$. Theorems 3.12 and 4.12 yield:

Theorem 4.18. *There are canonical isomorphisms*

$$\mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}}} = \mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}},cl} \simeq \begin{cases} \widehat{\mathbb{Z}}[[x]], & \text{if } n \leqslant 0; \\ x\widehat{\mathbb{Z}}[[x]], & \text{if } n \geqslant 1. \end{cases}$$

In particular, the natural map $\mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}}} \to \mathbf{OP}^{n+1,n+1}_{\widehat{\mathbb{Z}}}$ is an isomorphism for all $n \geqslant 1$.

It follows from Theorem 4.12 that for any two integers n and m,

$$\mathbf{OP}^{n,m}_{\widehat{\mathbb{Z}}} \simeq \begin{cases} x^{\max(0,m)} \cdot \widehat{\mathbb{Z}}[[x]], & \text{if } n \leqslant 0; \\ \{G \in x\widehat{\mathbb{Z}}[[x]] \mid v(\partial^{n-1}(G)) \geqslant m\}, & \text{if } n \geqslant 1. \end{cases}$$

4.4. **Operations over $\mathbb{Z}$.** Now we turn to the case $R = \mathbb{Z}$ and for simplicity write $\mathbf{OP}^{n,m}$ for $\mathbf{OP}^{n,m}_{\mathbb{Z}}$.

Corollary 4.13 implies that the natural homomorphism $\mathbf{OP}^{n,m} \to \mathbf{OP}^{n,m}_{\widehat{\mathbb{Z}}}$ is injective. In particular, we can identify $\mathbf{OP}^{n,n}$ with a subgroup of $\mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}}} = x\widehat{\mathbb{Z}}[[x]]$ for all $n \geqslant 1$, so

we have a sequence of subgroups

$$\mathbf{OP}^{1,1} \subset \mathbf{OP}^{2,2} \subset \ldots \subset \mathbf{OP}^{n,n} \subset \ldots \subset x\widehat{\mathbb{Z}}[[x]].$$

Recall (Theorem 4.12) that $\mathbf{OP}^{n,m} \simeq x^{\max(0,m)} \cdot \mathbb{Z}[[x]]$ if $n \leqslant 0$ and $\mathbf{OP}^{n,m} \simeq \mathbf{OP}^{n,n}$ if $m \leqslant n$ by Corollary 4.15.

Let $m \geqslant n \geqslant 1$. By Theorem 4.12, we can identify $\mathbf{OP}^{n,m}$ with the factor group of $\mathcal{Q}^{n,m}$ by the subgroup $\sum_{r=1}^{n-1} \mathbb{Q} \cdot \lg_r(x)$. It follows that the map ρ_n in (3.15) yields a homomorphism

$$\mathbf{OP}^{n,m} \to (\widehat{\mathbb{Q}}/\mathbb{Q})^{n-1} = (\widehat{\mathbb{Z}}/\mathbb{Z})^{n-1}.$$

By the proof of Lemma 3.16, this map is surjective. Its kernel is denoted $\mathbf{OP}^{n,m}_{cl}$ and called the subgroup of *classical operations*. In the case $n = m$ this group coincides with the group of classical operation defined earlier. In view of Corollary 4.13, $\mathbf{OP}^{n,m}_{cl}$ is identified with the group $\left(\coprod_{r=1}^{n-1} \mathbb{Q} \cdot \lg_r(x) + x\mathbb{Z}[[x]]\right) \cap x^m\mathbb{Q}[[x]]$.

We view the group $x\mathbb{Z}[x]_{\leqslant m-1}$ of integral polynomials of degree at most $m-1$ as a lattice in the $\mathbb{Q}$-space $x\mathbb{Q}[[x]]/(x^m)$. Denote by $\mathcal{L}^{n,m}$ the intersection of $x\mathbb{Z}[x]_{\leqslant m-1}$ with the image in $x\mathbb{Q}[[x]]/(x^m)$ of the space $\coprod_{r=1}^{n-1} \mathbb{Q} \cdot \lg_r(x)$. Then $\mathcal{L}^{n,m}$ is a subgroup of $x\mathbb{Z}[x]_{\leqslant m-1}$ of rank $n-1$.

We get the following description of the group of classical operations:

$$\mathbf{OP}^{n,m}_{cl} = \mathcal{L}^{n,m} \oplus x^m\mathbb{Z}[[x]].$$

If $m = n \geqslant 1$, the map of $\mathbb{Q}$-spaces is an isomorphism and $\mathcal{L}^{n,n} = x\mathbb{Z}[x]_{\leqslant n-1}$. It follows that

$$\mathbf{OP}^{n,n}_{cl} = x\mathbb{Z}[[x]].$$

Recall that $\mathbf{OP}^{n,n}_{cl} = \mathbf{OP}^{n,n} = \mathbb{Z}[[x]]$ if $n \leqslant 0$ and $\mathbf{OP}^{n,m} = \mathbf{OP}^{n,n}$ if $m \leqslant n$.

We summarize our results in the following statement.

Theorem 4.19. *The natural homomorphism $\mathbf{OP}^{n,m} \to \mathbf{OP}^{n,m}_{\widehat{\mathbb{Z}}}$ is injective. For any integers $m \geqslant n \geqslant 1$ there is an exact sequence*

$$0 \to \mathbf{OP}^{n,m}_{cl} \to \mathbf{OP}^{n,m} \to (\widehat{\mathbb{Z}}/\mathbb{Z})^{n-1} \to 0,$$

where $\mathbf{OP}^{n,m}_{cl} = \mathcal{L}^{n,m} \oplus x^m\mathbb{Z}[[x]]$. Moreover, $\mathbf{OP}^{n,n}_{cl} = x\mathbb{Z}[[x]]$.

Remark 4.20. Similar arguments yield the following formula for $m \geqslant n \geqslant 1$:

$$\mathbf{OP}^{n,m}_{\widehat{\mathbb{Z}}} = \mathcal{L}^{n,m}_{\widehat{\mathbb{Z}}} \oplus x^m\widehat{\mathbb{Z}}[[x]],$$

where $\mathcal{L}^{n,m}_{\widehat{\mathbb{Z}}} = \mathcal{L}^{n,m} \otimes \widehat{\mathbb{Z}}$.

4.5. **Composition.** The R-module homomorphism $\mathrm{Ad}_n : R[[x]] \to \mathbf{OP}^{n,n}_R$ is not a ring homomorphism. In this section we introduce a new product on $R[[x]]$ so that Ad_n becomes an R-algebra homomorphism.

Let $H, H' \in R[[x]]$, write $H' = \sum_{i \geqslant 0} a_i x^i$ and define the *composition* in H and H' by the formula

$$H \circ H' = a_0 \cdot H(0) + \sum_{i \geqslant 1} (-1)^i a_i \cdot (\partial^{i-1}H)(x^{\times i}).$$

The composition $\circ$ is distributive in H and H' with respect to addition. (Note that the usual substitution of power series is only one-sided distributive.) The polynomial $1 - x$

is the identity for the composition: $(1-x)\circ H = H = H\circ(1-x)$ for all H. We view $R[[x]]$ as an R-algebra with product given by the composition.

Lemma 4.21. *The maps* $\mathrm{Ad}_n : R[[x]] \to \mathbf{OP}_R^{n,n}$ *are R-algebra homomorphisms.*

Proof. In view of Proposition 4.14 it suffices to consider the case $n=0$. Let $H, H' \in R[[x]]$ and write $H' = \sum_{i\geqslant 0} a_i x^i$. If $G_0, G_1, \ldots \in R[t][[x]]$ is the sequence of power series corresponding to $\mathrm{Ad}_0(H)$ (see Proposition 4.11), then $G_0 = H(0) \in R$, $H = (G_0 - G_1)|_{t=1}$ and $G_i = (-1)^{i-1}\partial_t^{i-1}(G_1)$ for $i>1$. Note that $G_1(t,x) = -H(tx) + H(0)$.

Write L for the canonical line bundle on $\mathbb{P}^\infty$. By (4.7) and (4.10),

$$\mathrm{Ad}_0(H)(c_1(L)^i) = G_i(c_1(L)^{\times i}) = (-1)^{i-1}(\partial_t^{i-1}G_1)(c_1(L)^{\times i}) = (-1)^i(\partial^{i-1}H)(tc_1(L)^{\times i}).$$

Therefore, we have

$$\begin{aligned}(\mathrm{Ad}_0(H)\circ\mathrm{Ad}_0(H'))(c_1(L)) &= -\mathrm{Ad}_0(H)(\sum_{i\geqslant 1} a_i c_1(L)^i)\\ &= -\sum_{i\geqslant 1} a_i(\mathrm{Ad}_0(H))(c_1(L)^i)\\ &= \sum_{i\geqslant 1}(-1)^{i-1}a_i\cdot(\partial^{i-1}H)(tc_1(L)^{\times i}).\end{aligned}$$

On the other hand, write $H\circ H' = (G_0'' - G_1'')|_{t=1}$, where $G_0'' = a_0\cdot H(0)$ and

$$G_1'' = \sum_{i\geqslant 1}(-1)^{i-1}a_i\cdot(\partial^{i-1}H)(tx^{\times i}).$$

It follow that

$$\mathrm{Ad}_0(H\circ H')(c_1(L)) = G_1''(c_1(L)) = \sum_{i\geqslant 1}(-1)^{i-1}a_i\cdot(\partial^{i-1}H)(tc_1(L)^{\times i}) = (\mathrm{Ad}_0(H)\circ\mathrm{Ad}_0(H'))(c_1(L)).$$

If $r\in R = \mathrm{CK}_R^0(k)$, then

$$\mathrm{Ad}_0(H\circ H')(r) = G_0''\cdot r = a_0\cdot H(0)\cdot r = \mathrm{Ad}_0(H)(a_0\cdot r) = (\mathrm{Ad}_0(H)\circ\mathrm{Ad}_0(H'))(r).$$

Overall, $\mathrm{Ad}_0(H\circ H') = \mathrm{Ad}_0(H)\circ\mathrm{Ad}_0(H')$. □

The polynomials $A_m := (1-x)^m$ satisfy $\mathrm{Ad}_n(A_m) = \Psi_m$ in $\mathbf{OP}_R^{n,n}$. It follows from Lemma 4.21 and Example 4.9 that

$$A_k\circ A_m = A_{km} = A_m\circ A_k$$

for all k and m.

Proposition 4.22. *Let R be a commutative ring and K a $\mathbb{Q}$-algebra. Then*

1. *The composition $\circ$ in $R[[x]]$ is commutative.*
2. *The power series $\lg_r(x)\in K[[x]]$, $r\geqslant 0$, are orthogonal idempotents that partition the identity, that is, $\lg_n(x)\circ\lg_m(x) = \delta_{n,m}\cdot\lg_n(x)$ and $1-x = \sum_{r\geqslant 0}\lg_r(x)$.*

Proof. (1) It follows from the definition that the power series $x^n\circ G$ and $G\circ x^n$ are contained in $x^nR[[x]]$ for all n and G. Let $H, G\in R[[x]]$. Fix an integer $n>0$ and write $H = H_1 + H_2$ and $G = G_1 + G_2$, where H_1 and G_1 are linear combinations of the Adams polynomials A_i and $H_2, G_2 \in x^nR[[x]]$. As H_1 and G_1 commute, the remark above yields $H\circ G - G\circ H \in x^nR[[x]]$. Since this holds for all n, we have $H\circ G = G\circ H$.

(2) The iterated derivative $\partial^i(\lg_n(x))$ is zero if $i \geqslant n$ and

$$(\partial^{n-1} \lg_n)(x_1, \dots, x_n) = \prod_{i=1}^{n} \log(1 - x_i).$$

It follows that $\lg_n(x) \circ x^m = 0$ if $m > n$ and

$$\lg_n(x) \circ x^n = (-1)^n(\partial^{n-1} \lg_n)(x^{\times n}) = (-1)^n(\log(1-x))^n = (-1)^n n! \lg_n(x).$$

This calculation together with the first part of the proposition show that the power series $\lg_r(x)$ are orthogonal idempotents.

Write $1 - x = \sum_{i \geqslant 0} a_i \lg_i(x)$ for $a_i \in K$. Composing with $\lg_n(x)$ we get $a_n = 1$ for all n. □

Theorems 4.18 and 4.19 together with Proposition 4.22 yield the following corollary.

Corollary 4.23. *The rings* $\mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}}}$ *and* $\mathbf{OP}^{n,n}$ *are commutative.*

Let K be a $\mathbb{Q}$-algebra. We view $K[[x]]$ as a ring with respect to addition and composition. Let $G \in K[[x]]$ and write $G = \sum_{i \geqslant 0} a_i \lg_i$ for (unique) $a_i \in K$. It follows from Proposition 4.22 that the map

$$b : K[[x]] \to K^{[0,\infty)}, \tag{4.24}$$

taking G to the sequence $(a_i)_{i \geqslant 0}$ is a ring isomorphism. It takes $x^n K[[x]]$ onto $K^{[n,\infty)}$ for every n.

Example 4.25. The image of the polynomial $A_m(x) = (1-x)^m$ is equal to $(1, m, m^2, \dots)$. Indeed substituting $y = \log(1 - x)$ in the equality $e^{my} = \sum_{i \geqslant 0} \frac{m^i y^i}{i!}$ yields $A_m(x) = \sum_{i \geqslant 0} m^i \lg_i(x)$.

4.6. **Topology.** In this section we introduce three topologies on $\widehat{\mathbb{Z}}[[x]]$.

Proposition 4.26. *Let* $G \in \mathbf{OP}^{n,n}_R$ *and* $m \geqslant n$. *The following conditions are equivalent:*

(1) $G \in \operatorname{Im}(\mathbf{OP}^{n,m}_R \to \mathbf{OP}^{n,n}_R)$;
(2) G *is zero on every smooth variety of dimension* $< m$.

Proof. (1) $\Rightarrow$ (2) Since $\mathrm{CK}^m_R(X) = 0$, for any variety X of dimension $< m$, the operation G is zero on X.

(2) $\Rightarrow$ (1) Let $n \geqslant 1$. By Proposition 4.11(2), the operation G is given by a double-symmetric power series $H(x_1, ..., x_n) \in R[[x_1, ..., x_n]]_{(n)}$ such that $H = (G_n)|_{t=1}$. We need to prove that $v(H) \geqslant m$. We will show that any monomial $\overline{x}^{\overline{r}} = x_1^{r_1} \cdot \ldots \cdot x_n^{r_n}$ of H with $\sum_i r_i < m$ is zero.

Consider $X_{\overline{r}} := \prod_i \mathbb{P}^{r_i}$. This is a variety of dimension $< m$. Write x_i for the first Chern class in $\mathrm{CK}^1_R(X_{\overline{r}})$ of the pull-back of the canonical line bundle on $\mathbb{P}^{r_i}$ with respect to the i-th projection $X_{\overline{r}} \to \mathbb{P}^{r_i}$. By formula (4.7),

$$0 = G(x_1 \cdot \ldots \cdot x_n) = G_n(x_1, \dots, x_n) \in \mathrm{CK}^n_R(X_{\overline{r}}).$$

By Projective Bundle Theorem,

$$\mathrm{CK}^n_R(X_{\overline{r}}) = R[[x_1, \dots, x_n]]/(x_1^{r_1+1}, \dots, x_n^{r_n+1}).$$

Therefore the monomial $\overline{x}^{\overline{r}}$ of H is trivial.

The case $n \leqslant 0$ follows similarly (and easier) from Proposition 4.11(1). □

Corollary 4.27. *Let $d \geqslant 0$ be an integer and $G \in \mathbf{OP}^{n,n}$. Then there is a $\mathbb{Z}$-linear combination $G' \in \mathbf{OP}^{n,n}$ of the Adams operations Ψ_k with $k = 0, \dots, d$ such that G and G' agree on $\mathrm{CK}^n(X)$ for all smooth varieties X of dimension $\leqslant d$.*

Proof. By Lemma 3.16, applied to $m = d + 1$, there is a polynomial $G' \in \mathbb{Z}[x]$ of degree at most d such that $G - G' \in \sum_{0<r<n} \mathbb{Q} \cdot \lg_r(x) + x^{d+1}\mathbb{Q}[[x]]$. Let X be a smooth variety of dimension $\leqslant d$. As $v(\partial^{n-1}(G - G')) \geqslant d + 1$, in view of Theorem 4.12, $G - G' \in \mathrm{Im}(\mathbf{OP}^{n,m} \to \mathbf{OP}^{n,n})$. Therefore, by Proposition 4.26, $G - G'$ is trivial on X. Finally, G' is a linear combination of the Adams polynomials A_k with $k = 0, \dots, d$. □

Definition 4.28. We introduce three topologies on $\widehat{\mathbb{Z}}[[x]]$:

- τ_s is generated by the neighborhoods of zero U_m consisting of power series divisible by x^m, for some $m \geqslant 0$, i.e., τ_s is the x-adic topology.
- τ_w is generated by the neighborhoods of zero $U_m + V_N$, where V_N consists of all power series divisible by some $N \in \mathbb{N}$.
- τ_o is generated by the neighborhoods of zero W_m consisting of power series, where the respective operation acts trivially on varieties of dimension $< m$.

Recall, that a topology φ is *coarser* than the topology ψ, denoted $\varphi \leqslant \psi$, if any set open with respect to φ is also open with respect to ψ.

Proposition 4.29. $\tau_w \leqslant \tau_o \leqslant \tau_s$.

Proof. Since $v(G(x)) \geqslant m$ implies $v(\partial^{n-1}G(x)) \geqslant m$ and hence $G \in \mathrm{Im}(\mathbf{OP}^{n,m}_{\widehat{\mathbb{Z}}} \to \mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}}})$ by Theorem 4.12. Therefore, it follows from Proposition 4.26 that $\tau_o \leqslant \tau_s$.

The topology τ_w is generated by the neighborhoods of zero $U_{N,m} = (N, x^m) \subset \widehat{\mathbb{Z}}[[x]]$, and τ_o is generated by the neighborhoods of zero $W_k = \{G \in \widehat{\mathbb{Z}}[[x]] \mid v(\partial^{n-1}(G)) \geqslant k\}$ by Proposition 4.26. We need to show that for every N and m there is k with $W_k \subset U_{N,m}$.

We have similar compact (Hausdorff) topology τ_w on $\widehat{\mathbb{Z}}[[x_1, \dots, x_n]]$ so that the map ∂^{n-1} is continuous in τ_w. Note that the map $\partial^{n-1} : \widehat{\mathbb{Z}}[[x]] \to \widehat{\mathbb{Z}}[[x_1, \dots, x_n]]$ is injective and the induced map from $\widehat{\mathbb{Z}}[[x]]$ to the image of ∂^{n-1} is a homeomorphism (since the image of every closed subset is closed as $\widehat{\mathbb{Z}}[[x]]$ is compact and the target is Hausdorff). In particular, if $G_k \in \widehat{\mathbb{Z}}[[x]]$ is a sequence such that the sequence $\partial^{n-1}(G_k)$ converges to 0, then the sequence G_k converges to 0 in $\widehat{\mathbb{Z}}[[x]]$.

Now we prove that for every N and m there is k with $W_k \subset U_{N,m}$. Assume on the contrary that for every k we can find $G_k \in W_k$, but $G_k \notin U_{N,m}$. Then $\partial^{n-1}(G_k)$ converges to 0, but G_k does not converge to 0 in $\widehat{\mathbb{Z}}[[x]]$, a contradiction. □

Observation 4.30. 1) For $n = 1$, $\tau_o = \tau_s$;
2) For $n > 1$, $\tau_w \neq \tau_o \neq \tau_s$.

Proof. 1) This follows from Proposition 4.26, since $n = 1$.

2) For $n > 1$, W_m contains, in particular, all power series $\sum_i a_i x^i \in \widehat{\mathbb{Z}}[[x]]$, where $a_1 = ia_i$, for all $0 < i < m$, which is not contained in any U_l, for $l > 1$. Thus, $\tau_o \neq \tau_s$.

For $m > n \geqslant 1$, W_m/U_m is a free $\widehat{\mathbb{Z}}$-module of rank $(n - 1)$, while $(U_m + V_N)/U_m$ is a free $\widehat{\mathbb{Z}}$-module of rank $(m - 1)$. Hence, $\tau_w \neq \tau_o$. □

We view $\mathbf{OP}^{n,n}$ and $\mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}}}$ as the topological rings for the topologies τ_w, τ_o and τ_s respectively via the inclusions $\mathbf{OP}^{n,n} \hookrightarrow \mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}}} \hookrightarrow \widehat{\mathbb{Z}}[[x]]$.

Note that the x-adic topology τ_s can be defined on $R[[x]]$ for every R.

Consider the restriction $b : R[[x]] \to K^{[0,\infty)}$ of the map (4.24). We view $K^{[0,\infty)}$ as a topological ring with the basis of neighborhoods of zero given by the ideals $K^{[n,\infty)}$ for all $n > 0$, so that the map is continuous.

Proposition 4.31. *The image of the map $b : R[[x]] \to K^{[0,\infty)}$ is contained in $R^{[0,\infty)}$.*

Proof. By Example 4.25, the image of the Adams polynomial A_m under the map (4.24) is contained in $R^{[0,\infty)}$. But the set of all linear combinations of Adams polynomials is dense in $R[[x]]$ in the topology τ_s. The statement follows since $R^{[0,\infty)}$ is closed in $K^{[0,\infty)}$. □

Proposition 4.31 identifies the ring $\mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}}} \subset \widehat{\mathbb{Z}}[[x]]$ with a subring of $\widehat{\mathbb{Z}}^{[n,\infty)}$ and $\mathbf{OP}^{n,n}$ with a subring of $\mathbb{Z}^{[n,\infty)}$ if $n \geqslant 0$. Indeed, if $n \geqslant 1$ the kernel of the composition

$$\mathcal{Q}^n_{\widehat{\mathbb{Z}}} \xrightarrow{\lambda_{n,n}} \!\!\!\!\rightarrow \mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}}} \xrightarrow{b} \widehat{\mathbb{Z}}^{[0,\infty)} \to \widehat{\mathbb{Z}}^{[n,\infty)}$$

is generated by $\lg_r$ with $0 < r < n$ and all these logarithms are contained in the kernel of $\lambda_{n,n}$.

The ring $\mathbf{OP}^{n,n}$ is not a domain: we have $(\Psi_1 + \Psi_{-1})(\Psi_1 - \Psi_{-1}) = 0$. Let

$$e_\pm = \tfrac{1}{2}(\Psi_1 \pm \Psi_{-1}) \in \mathbf{OP}^{n,n}\left[\tfrac{1}{2}\right],$$

so e_+ and e_- are orthogonal idempotents and $e_+ + e_- = 1$. There is an embedding

$$\mathbf{OP}^{n,n} \hookrightarrow \mathbf{OP}^{n,n}\left[\tfrac{1}{2}\right] = \mathbf{OP}^{n,n}\left[\tfrac{1}{2}\right] e_+ \times \mathbf{OP}^{n,n}\left[\tfrac{1}{2}\right] e_-.$$

Proposition 4.32. *If $n \geqslant 1$ the rings $\mathbf{OP}^{n,n}\left[\frac{1}{2}\right]e_\pm$ are domains.*

Proof. Recall that there is an injective ring homomorphism

$$b : \mathbf{OP}^{n,n} \hookrightarrow \mathbb{Z}^{[1,\infty)}$$

such that $b(\Psi_m) = (m, m^2, m^3, \ldots)$ for all m. In particular,

$$b(e_+) = (0, 1, 0, 1, \ldots) \quad \text{and} \quad b(e_-) = (1, 0, 1, 0, \ldots).$$

Lemma 4.33. *Let $(a_1, a_2, \ldots) \in \mathrm{Im}(b)$. Then for any prime integer p, we have $a_i \equiv a_j$ modulo p if $i \equiv j$ modulo $p - 1$.*

Proof. It suffices to prove the statement for $b(\Psi_m)$. We have $a_i - a_j = m^i - m^j = m^j(m^{i-j} - 1)$. If m is not divisible by p, then $m^{i-j} - 1$ is divisible by p. □

Let $G \cdot H = 0$ in $\mathbf{OP}^{n,n}$. Set $(a_1, a_2, \ldots) = b(G)$ and $(b_1, b_2, \ldots) = b(H)$. We have $a_i b_i = 0$ for all i. To prove the statement it suffices to show that if $a_i \neq 0$ for some i, then $b_j = 0$ for all $j \equiv i$ modulo 2.

Choose an odd prime p that does not divide a_i. By Lemma 4.33, a_j is not divisible by p for all j such that $i \equiv j$ modulo $p - 1$. In particular $a_j \neq 0$, hence $b_j = 0$. Thus, we have proved that $b_j = 0$ for all $j \equiv i$ modulo $p - 1$.

Lemma 4.34. *There are infinitely many primes q such that* $\gcd(q - 1, p - 1) = 2$.

Proof. Let c be the odd part of $p-1$ (that is $(p-1)/c$ is a 2-power). By Dirichlet, there are infinitely many primes q such that $q \equiv 3$ modulo 4 and $q \equiv 2$ modulo c. Clearly, $\gcd(q-1, p-1) = 2$ for such q. □

Let j be such that $j \equiv i$ modulo 2. We need to prove that $b_j = 0$. Take any prime q as in Lemma 4.34. There are positive integers k and m such that $t := i+(p-1)k = j+(q-1)m$. We have proved that $b_t = 0$ since $t \equiv i$ modulo $p-1$. By Lemma 4.33, $0 = b_t \equiv b_j$ modulo q, i.e., b_j is divisible by q. We have proved that b_j is divisible by infinitely many primes q, hence $b_j = 0$. □

4.7. **Operations in graded K-theory.** In this section we determine the R-module of all R-linear operations $G : K^n_{grR} \to K^m_{grR}$ for any pair of integers n and m denoted by $\mathbf{OP}^{n,m}_R(K^*_{gr})$. Recall that $K^n_{grR} = K^0_{grR} \cdot t^{-n} = \mathrm{CK}^0_R \cdot t^{-n}$, hence by Theorem 4.12,

$$\mathbf{OP}^{n,m}_R(K^*_{gr}) = \mathbf{OP}^{0,0}_R(K^*_{gr}) \cdot t^{m-n} = \mathbf{OP}^{0,0}_R(\mathrm{CK}^*) \cdot t^{m-n} = R[[x]] \cdot t^{m-n}.$$

Recall that product operation in the ring $\mathbf{OP}^{n,m}_R(K^*_{gr}) = R[[x]]$ is the composition $\circ$ (see Section 4.5). Moreover, $R[[x]]$ is a (topological) bi-algebra over R with co-product defined by the rule $(1-x)^n \to (1-x)^n \otimes (1-x)^n$ for all $n \geqslant 0$ that reads $\Psi \mapsto \Psi \otimes \Psi$ in the language of operations.

Let us describe the dual bi-algebra A (over $\mathbb{Z}$) of *co-operations* as follows. Let A be the subring of the polynomial ring $\mathbb{Q}[s]$ consisting of all polynomials f such that $f(a) \in \mathbb{Z}$ for all $a \in \mathbb{Z}$. In particular, $\mathbb{Z}[s] \subset A$. The polynomials

$$e_n := \frac{1}{n!}\,(-s)(1-s)\dots(n-1-s) = (-1)^n\binom{s}{n} \in A$$

for all $n \geqslant 0$ form a basis of A as an abelian group. Consider a pairing

$$A \otimes R[[x]] \to R, \qquad a \otimes G \mapsto \langle a, G\rangle \in R,$$

such that $\langle e_n, x^m\rangle = \delta_{n,m}$. This pairing identifies $R[[x]]$ with the dual co-algebra for A via the isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(A, R) \xrightarrow{\sim} R[[x]],$$

taking a homomorphism $\alpha : A \to R$ to the power series $\sum_{n\geqslant 0} \alpha(e_n)x^n$.

Lemma 4.35. *For every polynomial $f \in A$, we have $\langle f, (1-x)^m\rangle = f(m)$.*

Proof. We may assume that $f = e_n$ for some n. Then

$$\langle f, (1-x)^m\rangle = \langle e_n, (1-x)^m\rangle = (-1)^n\binom{m}{n} = e_n(m) = f(m).$$ □

The lemma shows that a co-operation f evaluated at the Adams operation Ψ_m is equal to $f(m)$.

It follows from Lemma 4.35 that

$$\langle s^n, (1-x)^{km}\rangle = (km)^n = k^n \cdot m^n = \langle s^n, (1-x)^k\rangle \cdot \langle s^n, (1-x)^m\rangle.$$

As the composition in $R[[x]]$ satisfies $(1-x)^k \circ (1-x)^m = (1-x)^{km}$, the composition in $R[[x]]$ is dual to the co-product of A taking s^n to $s^n \otimes s^n$ in $A \otimes A$.

The equality

$$\langle s^{i+j}, (1-x)^m\rangle = m^{i+j} = m^i \cdot m^j = \langle s^i, (1-x)^m\rangle \cdot \langle s^j, (1-x)^m\rangle$$

shows that the product in A is dual to the co-product in $R[[x]]$. Thus, the bi-algebra $R[[x]]$ of operations is dual to the bi-algebra A of co-operations.

Remark 4.36. The polynomial ring $\mathbb{Z}[s]$ is a bi-algebra with respect to the co-product $s \to s \otimes s$. The dual bi-algebra over R is $R^{[0,\infty)}$. The dual of the embedding $\mathbb{Z}[s] \to A$ is the homomorphism $b : R[[x]] \to R^{[0,\infty)}$ defined in Proposition 4.31 since by Lemma 4.35,

$$\langle s^n, (1-x)^m \rangle = m^n = \langle s^n, b((1-x)^m) \rangle$$

as $b((1-x)^m) = (1, m, \dots, m^n, \dots)$.

5. Multiplicative operations

Definition 5.1. A *multiplicative operation* $G : A^* \to B^*$ is a morphism of functors from $\mathbf{Sm}_k$ to the category of rings. That is, the ring structure is respected. (We don't assume that G is a graded ring homomorphism.)

If A^* and B^* are cohomology theories over k, to any multiplicative operation $G : A^* \to B^*$ we can assign the morphism $(\varphi_G, \gamma_G) : (A^*(k), F_A) \to (B^*(k), F_B)$ of the respective formal group laws, where $\varphi_G : A^*(k) \to B^*(k)$ is the restriction of G to $\mathrm{Spec}(k)$ and $\gamma_G(x) \in xB^*(k)[[x]]$ is defined by the condition:

$$G(c_1^A(O(1))) = \gamma_G(c_1^B(O(1))) \in B^*(\mathbb{P}^\infty) = B^*(k)[[x]].$$

(The power series $\gamma_G(x)/x$ is called the *inverse Todd genus* of G.)

Theorem 5.2. ([11, Theorem 6.9]) *Let A^* be a theory of rational type and B^* any oriented cohomology theory. Then the assignment $G \mapsto (\varphi_G, \gamma_G)$ is a bijection between the set of multiplicative operations $G : A^* \to B^*$ and the set of morphisms of formal group laws.*

Example 5.3. Let R be either $\mathbb{Z}$, $\mathbb{Z}_p$ or $\widehat{\mathbb{Z}}$ and $b \in R$. The Adams operation $\Psi_b : \mathrm{CK}^*_R \to \mathrm{CK}^*_R$ is homogeneous and multiplicative. The corresponding map φ is the identity and $\gamma = \frac{1-(1-tx)^b}{t}$. If $c \in R^\times$, write Ψ^c_b for the homogeneous multiplicative *twisted* Adams operation with $\varphi(t) = ct$ and $\gamma = \frac{1-(1-tx)^{bc}}{ct}$ (in particular, $\Psi^1_b = \Psi_b$). It follows from the equality $\Psi^c_b(tx) = \Psi^c_b(t)\Psi^c_b(x) = ct \cdot \gamma(x) = 1-(1-tx)^{bc}$ that on CK^n_R the operation Ψ^c_b is equal to $c^{-n} \cdot \Psi_{bc}$. For any $c \in R$, let Ψ^c_0 be the homogeneous multiplicative operation with $\varphi(t) = ct$ and $\gamma = 0$. This operation is zero in positive degrees and is equal to $c^n \cdot \mathrm{rank}$ on $\mathrm{CK}^{-n}_R = (K_0)_R$ for $n \geqslant 0$.

Write Θ for the multiplicative operation $\mathrm{CK}^*_R \to \mathrm{CK}^*_R$ which is identity on CK^0_R, multiplication by $t^n : \mathrm{CK}^n_R \to \mathrm{CK}^0_R$ if $n \geqslant 0$ and the canonical isomorphism $\mathrm{CK}^n_R \to \mathrm{CK}^0_R$ (inverse to multiplication by t^{-n}) if $n \leqslant 0$. This operation is not homogeneous and its image is CK^0_R. Set $\widetilde{\Psi}^c_b := \Theta \circ \Psi^c_b$. This is a multiplicative operation with image in CK^0_R. The corresponding function $\varphi(t) = c$ and $\gamma = \frac{1-(1-tx)^{bc}}{c}$.

Introduced operations satisfy the following relations: $\Psi^0_0 = \widetilde{\Psi}^0_0$ and

$$\Psi^c_b \circ \Psi^e_d = \Psi^{ce}_{bd}; \quad \Psi^c_b \circ \widetilde{\Psi}^e_d = \widetilde{\Psi}^e_{cbd}; \quad \widetilde{\Psi}^c_b \circ \Psi^e_d = \widetilde{\Psi}^{ce}_{bd}; \quad \widetilde{\Psi}^c_b \circ \widetilde{\Psi}^e_d = \widetilde{\Psi}^e_{cbd}.$$

Over $\mathbb{Q}$ every formal group law is isomorphic to the additive one. Hence, for every theory C^*, we have isomorphisms of formal group laws.

$$(\mathrm{id}, \exp_C) : (C^* \otimes_{\mathbb{Z}} \mathbb{Q}, F_C) \rightleftarrows (C^* \otimes_{\mathbb{Z}} \mathbb{Q}, F_{add}) : (\mathrm{id}, \log_C)$$

Suppose that the coefficient ring $B^*(k)$ of the target theory has no torsion. Then the composition $(\mathrm{id}, \exp_B) \circ (\varphi_G, \gamma_G) \circ (\mathrm{id}, \log_A)$ identifies the set of multiplicative operations $A^* \to B^*$ with a subset of morphisms of formal group laws $(A^* \otimes_{\mathbb{Z}} \mathbb{Q}, F_{add}) \to (B^* \otimes_{\mathbb{Z}} \mathbb{Q}, F_{add})$. The latter morphism is defined by (ψ, γ), where, in our case, $\psi = \varphi \otimes_{\mathbb{Z}} \mathbb{Q}$, for some ring homomorphism $\varphi = \varphi_G : A^*(k) \to B^*(k)$ and $\gamma(x) = b \cdot x$, for some $b \in B^*(k)$. In other words, $(\varphi_G, \gamma_G) = (\mathrm{id}, \log_B) \circ (\varphi_G, \gamma) \circ (\mathrm{id}, \exp_A)$. Then

$$\gamma_G(x) = \varphi_G(\exp_A)(b \cdot \log_B(x)).$$

5.1. **Multiplicative operations in** CK. For $A^* = B^* = \mathrm{CK}^*_{\widehat{\mathbb{Z}}}$ we have: $A = B = \widehat{\mathbb{Z}}[t]$, $F_A = F_B = x + y - txy$ and

$$\log_{\mathrm{CK}}(x) = \frac{\log(1-tx)}{t}, \quad \exp_{\mathrm{CK}}(z) = \frac{1-e^{zt}}{t}.$$

Note that a ring homomorphism φ from $\widehat{\mathbb{Z}}[t]$ to a ring T such that $\bigcap_{n>0} nT = 0$ is uniquely determined by $\varphi(t)$ in T. Indeed, suppose that φ and ψ satisfy $\varphi(t) = \psi(t)$. For any $f \in \widehat{\mathbb{Z}}[t]$ and $n > 0$ write $f = g + nh$ for some $g \in \mathbb{Z}[t]$ and $h \in \widehat{\mathbb{Z}}[t]$. Then $\varphi(g) = \psi(g)$ and hence $\varphi(f) - \psi(f) \in nT$. Since this holds for all $n > 0$, we have $\varphi(f) - \psi(f) = 0$ for all f.

Thus, the map $\varphi_G : \widehat{\mathbb{Z}}[t] \to \widehat{\mathbb{Z}}[t]$ is determined by $\varphi_G(t) = c(t) \in \widehat{\mathbb{Z}}[t]$. Let $b = b(t) \in \widehat{\mathbb{Z}}[t]$. Note that any choice of $b(t)$ and $c(t)$ gives a morphism of rational formal group laws and so, a multiplicative operation $G : \mathrm{CK}^*_{\widehat{\mathbb{Z}}} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{CK}^*_{\widehat{\mathbb{Z}}} \otimes_{\mathbb{Z}} \mathbb{Q}$ with

$$\gamma_G(t,x) = \frac{1-(1-tx)^{\frac{b(t)c(t)}{t}}}{c(t)} = \sum_{n \geqslant 1} (-1)^{n-1} (tx)^n \frac{\binom{\frac{b(t)c(t)}{t}}{n}}{c(t)},$$

which lifts to an operation $\mathrm{CK}^*_{\widehat{\mathbb{Z}}} \to \mathrm{CK}^*_{\widehat{\mathbb{Z}}}$ if and only if the coefficients of our power series belong to $\widehat{\mathbb{Z}}$. The coefficient at x^n is

$$a_n = (-1)^{n-1} \frac{b(t) \prod_{k=1}^{n-1} (b(t)c(t) - kt)}{n!}. \tag{5.4}$$

Denote as $b_p(t), c_p(t)$ the $\mathbb{Z}_p$-components of our polynomials. If $\deg(b_p(t)c_p(t)) > 1$ for some p, the leading term of our t-polynomial will be clearly non-integral (for some n). Similarly, if for some p, the constant term of $b_p(t)c_p(t)$ is non-zero, then the smallest term of the p-component of our t-polynomial will be non-integral, for some n. Hence, the polynomial $b(t)c(t)$ is linear. Then, for a given prime p, either $b_p(t) = b_p$ and $c_p(t) = c_p t$, or $b_p(t) = b_p t$ and $c_p(t) = c_p$, for some $b_p, c_p \in \mathbb{Z}_p$. Then the $\mathbb{Z}_p$- component of our coefficient is:

$$(a_n)_p = (-1)^{n-1} t^m b_p \frac{\binom{b_p c_p - 1}{n-1}}{n}, \quad \text{where } m = n-1, \text{ or } m = n.$$

If $b_p \neq 0$, then this will be integral for all n if and only if $c_p \in \mathbb{Z}_p^\times$, while if $b_p = 0$, then c_p can be an arbitrary element from $\mathbb{Z}_p$. Let us denote the ($\mathbb{Z}_p$-components of) operations with $m = n-1$ as $\Psi^{c_p}_{b_p}$, while the ones with $m = n$ as $\widetilde{\Psi}^{c_p}_{b_p}$ (see Example 5.3; we suppress p from notations). Here $\Psi^{c_p}_{b_p}$ respects the grading on $\mathrm{CK}^*_{\mathbb{Z}_p}$, while $\widetilde{\Psi}^{c_p}_{b_p}$ maps $\mathrm{CK}^*_{\mathbb{Z}_p}$ to $\mathrm{CK}^0_{\mathbb{Z}_p}$. The pairs (b_p, c_p) run over the set $(\mathbb{Z}_p \backslash 0) \times \mathbb{Z}_p^\times \cup \{0\} \times \mathbb{Z}_p$ and, in addition, $\Psi^0_0 = \widetilde{\Psi}^0_0$.

Thus, any multiplicative operation G on $\mathrm{CK}^*_{\widehat{\mathbb{Z}}} = \times_p \mathrm{CK}^*_{\mathbb{Z}_p}$ splits into the product $\times_p G_{(p)}$ of operations on $\mathrm{CK}^*_{\mathbb{Z}_p}$, where each $G_{(p)}$ is one of the $\Psi^{c_p}_{b_p}$ or $\widetilde{\Psi}^{c_p}_{b_p}$. Let $\mathcal{P}$ will be the set of prime numbers and $J \subset \mathcal{P}$ be the subset of those primes, for which $(b_p, c_p) \neq (0,0)$ and $G_{(p)}$ is $\widetilde{\Psi}$. Then the data (J, b, c), where the p-components of $b, c \in \widehat{\mathbb{Z}}$ are b_p and c_p, determines our operation G. Let us call it ${}^J\Psi^c_b$. Here (J, b, c) runs over all possible triples satisfying: 1) $b_p \neq 0 \Rightarrow c_p \in \mathbb{Z}_p^\times$ and 2) $(b_p, c_p) = (0,0) \Rightarrow p \notin J$.

The operations ${}^\emptyset\Psi^1_b$ are (non-twisted) Adams operations with $\varphi_G = \mathrm{id}$, which naturally form a ring isomorphic to $\widehat{\mathbb{Z}}$. These operations commute with every other operation. The operations ${}^\emptyset\Psi^c_1$ are invertible and form a group isomorphic to $\widehat{\mathbb{Z}}^\times$. Below we will suppress $J = \emptyset$ from notations and will denote the respective operations simply as Ψ^c_b.

The formulas in Example 5.3 show that the monoid of multiplicative operations is non-commutative.

5.2. **Multiplicative operations in K_{gr} over $\mathbb{Z}$.** For $A^* = B^* = K^*_{gr}$ we have: $A = B = \mathbb{Z}[t, t^{-1}]$, $F_A = F_B = x + y - txy$. Similar calculations as in the previous section show that the coefficient a_n in (5.4) will belong to $\mathbb{Z}[t, t^{-1}]$ for every n, if and only if $b(t)c(t)$ is linear in t. Thus, $c(t) = ct^l$, for $c = \pm 1$ and $l \in \mathbb{Z}$, and $b(t) = bt^{1-l}$, for some $b \in \mathbb{Z}$.

Then the coefficient a_n is

$$(-1)^{n-1} t^{n-l} \frac{\binom{bc}{n}}{c}.$$

Denote this operation as ${}^l\Psi^c_b$. It scales the grading on K^*_{gr} by the coefficient l. So, only the operations ${}^1\Psi^c_b$ are homogeneous.

The case $c(t) = t$ and $b(t) = b$, that is, ${}^1\Psi^1_b$ corresponds to the Adams operation Ψ_b - see [11, Sect. 6.3]. In this case $\varphi_G = id$. The operation ${}^{-1}\Psi^1_1$ is an automorphism of order 2 acting identically on K^0_{gr} and mapping t to t^{-1}.

We will omit l and c from the notation ${}^l\Psi^c_b$ when these will be equal to 1.

6. Stable operations

To be able to discuss stability of operations, we need the notion of a suspension. Following Voevodsky and Panin-Smirnov we can introduce the category of pairs **SmOp** whose objects are pairs (X, U), where $X \in \mathbf{Sm}_k$ and U is an open subvariety of X - see [11, Def. 3.1], with the smash product:

$$(X, U) \wedge (Y, V) := (X \times Y, X \times V \cup U \times Y),$$

and the natural functor $\mathbf{Sm}_k \to \mathbf{SmOp}$ given by $X \mapsto (X, \emptyset)$. Then suspension can be defined as:

$$\Sigma_T(X, U) := (X, U) \wedge (\mathbb{P}^1, \mathbb{P}^1 \backslash 0).$$

Any theory A^* extends from $\mathbf{Sm}_k$ to **SmOp** by the rule:

$$A^*((X, U)) := \mathrm{Ker}(A^*(X) \to A^*(U)).$$

Any additive operation $A^* \to B^*$ on $\mathbf{Sm}_k$ extends uniquely to an operation on **SmOp**.

An element $\varepsilon^A = c^A_1(O(1)) \in A^*((\mathbb{P}^1, \mathbb{P}^1 \backslash 0))$ defines an identification:

$$\sigma^A_T : A^*((X, U)) \xrightarrow{\cong} A^{*+1}(\Sigma_T(X, U)),$$

given by $x \mapsto x \wedge \varepsilon^A$.

Definition 6.1. For any additive operation $G : A^* \to B^*$ we define its *desuspension* as the unique operation $\Sigma^{-1}G : A^* \to B^*$ such that

$$G \circ \sigma_T^A = \sigma_T^B \circ \Sigma^{-1}G.$$

Definition 6.2. A *stable* additive operation $G : A^* \to B^*$ is the collection $\{G^{(n)} | n \geqslant 0\}$ of operations $A^* \to B^*$ such that $G^{(n)} = \Sigma^{-1}G^{(n+1)}$.

Proposition 6.3. *Suppose, $G : A^* \to B^*$ is a multiplicative operation with $\gamma_G(x) \equiv bx$ modulo x^2 for some $b \in B^*(k)$. Then $\Sigma^{-1}G = b \cdot G$.*

Proof. We have: $G(\sigma_T^A(u)) = G(u \wedge \varepsilon^A) = G(u) \wedge G(\varepsilon^A) = G(u) \wedge (b \cdot \varepsilon^B) = \sigma_T^B(b \cdot G(u))$. □

We call a multiplicative operation G *stable* if the constant sequence $(G, G, G, \ldots)$ is stable. By Proposition 6.3, G is stable if and only if the linear coefficient of γ_G is equal to 1 - cf. [11, Proposition 3.8].

For a commutative ring R define the operator

$$\Phi = \Phi_R : R[[x]] \to R[[x]], \quad \Phi(G) = (x-1)\frac{\mathrm{d}G}{\mathrm{d}x},$$

6.1. **Stable operations in** CK **over** $\widehat{\mathbb{Z}}$**.** Recall that in the case $A^* = B^* = \mathrm{CK}^*_{\widehat{\mathbb{Z}}}$, the group $\mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}}}$ of additive operations for $n \leqslant 0$ and $n \geqslant 1$ can be identified with $\widehat{\mathbb{Z}}[[x]]$, respectively, $x\widehat{\mathbb{Z}}[[x]]$.

Proposition 6.4. *The desuspension operator $\Sigma^{-1} : \mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}}} \to \mathbf{OP}^{n-1,n-1}_{\widehat{\mathbb{Z}}}$ is given by the rule*

$$\Sigma^{-1}(G) = \begin{cases} \Phi(G), & \text{if } n \leqslant 1; \\ \partial^0(\Phi(G)) = \Phi(G) - \Phi(G)(0), & \text{if } n > 1. \end{cases}$$

Proof. The Adams operation Ψ_k is identified with the power series $A_k(x) = (1-x)^k$ if $n \leqslant 0$ and with $(1-x)^k - 1$ if $n > 0$. By Proposition 6.3, $\Sigma^{-1}\Psi_k = k\Psi_k$, so the formula holds for $G = \Psi_k$.

The map Σ^{-1} is continuous in τ_o and the map Φ is continuous in τ_s. Hence both maps are continuous as the maps $\tau_s \to \tau_o$. Since τ_o is Hausdorff (as τ_w is), it follows that the set of power series where Σ^{-1} and Φ coincide is closed in τ_s. But the set of linear combinations of Adams operations is everywhere dense in τ_s. □

It follows from Proposition 6.4 that the desuspension map Σ^{-1} is injective and yields a tower of injective maps in the other direction:

$$\widehat{\mathbb{Z}}[[x]] = \mathbf{OP}^{0,0}_{\widehat{\mathbb{Z}}} \xleftarrow{\Sigma^{-1}} \mathbf{OP}^{1,1}_{\widehat{\mathbb{Z}}} \xleftarrow{\Sigma^{-1}} \ldots \xleftarrow{\Sigma^{-1}} \mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}}} \xleftarrow{\Sigma^{-1}} \ldots. \tag{6.5}$$

Moreover, the group $\mathbf{OP}^{st}_{\widehat{\mathbb{Z}}}$ of homogeneous degree 0 stable operations $\mathrm{CK}^*_{\widehat{\mathbb{Z}}} \to \mathrm{CK}^*_{\widehat{\mathbb{Z}}}$ that is the limit of the sequence 6.5 is naturally isomorphic to the group

$$S := \cap_n \operatorname{Im}(\Phi^n) = \cap_n \operatorname{Im}((\Sigma^{-1})^n) \subset \widehat{\mathbb{Z}}[[x]].$$

Indeed, if $\{G^{(n)} | n \geqslant 0\}$ is a stable operation, then $G^{(0)} = \Phi^n(G^{(n)})$ for every n, hence $G^{(0)} \in S$. Conversely, given $G \in S$, write $G = \Phi^n(H^{(n)})$ for every n. Since $\operatorname{Ker}(\Phi^n)$ consists of constant power series only, the sequence $G^{(n)} = \Phi(H^{(n+1)})$ is a stable operation.

Lemma 6.6. *Let $G \in x\widehat{\mathbb{Z}}[[x]]$ and $n \geqslant 1$. Then*

(1) *$\partial^n(G)$ has coefficients in $\mathbb{Z}$ if and only if $\partial^{n-1}(\Phi(G))$ has coefficients in $\mathbb{Z}$.*
(2) *$v(\partial^n(G)) \geqslant m$ for some m if and only if $v(\partial^{n-1}(\Phi(G))) \geqslant m-1$.*

Proof. ($\Rightarrow$) Follows from Proposition 2.5 for both (1) and (2).

($\Leftarrow$) Simply write H_k for $(x-1)^k \dfrac{\mathrm{d}^k G}{\mathrm{d}x^k}$. We claim that $\partial^{n-1}(H_k)$ has coefficients in $\mathbb{Z}$ in case (1) and $v(\partial^{n-1}(H_k)) \geqslant m-k$ in case (2) for every $k \geqslant 1$. We prove the statements by induction on k.
($k \Rightarrow k+1$) We have $H_{k+1} = \Phi(H_k) - kH_k$, hence

$$\partial^{n-1}(H_{k+1}) = \partial^{n-1}\big(\Phi(H_k)\big) - k\partial^{n-1}(H_k).$$

Then $k\partial^{n-1}(H_k)$ has coefficients in $\mathbb{Z}$ in case (1) and $v(k\partial^{n-1}(H_k)) \geqslant m-k$ in case (2) by the induction hypothesis. As the derivative $\partial^n(H_k)$ has coefficients in $\mathbb{Z}$ in case (1) and $v(\partial^n(H_k)) \geqslant m-k$ in case (2), it follows from Proposition 2.5, applied to the power series H_k, that $\partial^{n-1}(\Phi(H_k))$ also has coefficients in $\mathbb{Z}$ in case (1) and $v\big(\partial^{n-1}(\Phi(H_k))\big) \geqslant m-k-1$ in case (2). It follows that $\partial^{n-1}(H_{k+1})$ has coefficients in $\mathbb{Z}$ in case (1) and $v(\partial^{n-1}(H_{k+1})) \geqslant m-k-1$ in case (2). The claim is proved.

Note that all coefficient of H_k are divisible by $k!$ in $\widehat{\mathbb{Z}}$. It follows that the power series $\frac{1}{k!}\partial^{n-1}(H_k)$ have coefficients in $\mathbb{Z}$ in case (1). By Proposition 2.5, $\partial^n(G)$ has coefficients in $\mathbb{Z}$ in case (1) and $v(\partial^n(G)) \geqslant m$ in case (2). □

In particular, we can describe the integral operations $\mathbf{OP}^{n,m}$ as follows.

Proposition 6.7. *Let $G \in x\widehat{\mathbb{Z}}[[x]]$ and $m \geqslant n \geqslant 1$. Then $G \in \mathbf{OP}^{n,m}$ if and only if $\Phi^n(G) \in \mathbb{Z}[[x]]$ and $v(\Phi^n(G)) \geqslant m-n$.*

Proof. Theorem 4.12 and iterated application of Lemma 6.6 show that $G \in \mathbf{OP}^{n,m}$ if and only if $\partial^0(\Phi^{n-1}(G)) \in \mathbb{Z}[[x]]$ and $v(\partial^0(\Phi^{n-1}(G))) \geqslant m-n+1$. Thus, it suffices to prove the following for a power series $H \in \widehat{\mathbb{Z}}[[x]]$ and integer $k \geqslant 0$:
1. $\partial^0(H) \in \mathbb{Z}[[x]] \iff \Phi(H) \in \mathbb{Z}[[x]]$,
2. $v(\partial^0(H)) \geqslant k+1 \iff v(\Phi(H)) \geqslant k$.

If $\partial^0(H) \in \mathbb{Z}[[x]]$, then clearly $\Phi(H) \in \mathbb{Z}[[x]]$. Conversely, if $\Phi(H) \in \mathbb{Z}[[x]]$, then $\partial^0(H) \in \mathbb{Q}[[x]] \cap \widehat{\mathbb{Z}}[[x]] = \mathbb{Z}[[x]]$. The second statement follows from the obvious equality $v(\partial^0(H)) = v(\Phi(H)) + 1$. □

Let m a positive integer. It follows from Lemma 6.6(2) that there is a tower of inclusions as in (6.5):

$$x^m\widehat{\mathbb{Z}}[[x]] = \mathbf{OP}^{0,m}_{\widehat{\mathbb{Z}}} \hookleftarrow \mathbf{OP}^{1,m+1}_{\widehat{\mathbb{Z}}} \hookleftarrow \ldots \hookleftarrow \mathbf{OP}^{n,n+m}_{\widehat{\mathbb{Z}}} \hookleftarrow \ldots. \tag{6.8}$$

and for every n the intersection of $\mathbf{OP}^{n,n+m}_{\widehat{\mathbb{Z}}}$ and $\mathbf{OP}^{n+1,n+1}_{\widehat{\mathbb{Z}}}$ in $\mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}}}$ coincides with $\mathbf{OP}^{n+1,n+m+1}_{\widehat{\mathbb{Z}}}$. Therefore, we obtain:

Proposition 6.9. *The group of homogeneous degree m stable operations $\mathrm{CK}^*_{\widehat{\mathbb{Z}}} \to \mathrm{CK}^{*+m}_{\widehat{\mathbb{Z}}}$ is naturally isomorphic to the intersection $x^{\max(0,m)}\widehat{\mathbb{Z}}[[x]] \cap S$.*

The map $\Phi : \widehat{\mathbb{Z}}[[x]] \to \widehat{\mathbb{Z}}[[x]]$ is continuous in τ_w and the space $\widehat{\mathbb{Z}}[[x]]$ is compact Hausdorff. Hence $\mathrm{Im}(\Phi^n)$ is closed in $\widehat{\mathbb{Z}}[[x]]$ for any n. It follows that the set S is also closed in $\widehat{\mathbb{Z}}[[x]]$ in the topology τ_w and hence in τ_o and τ_s.

It follow from Proposition 4.26 and Lemma 6.6 that the topology on $\mathbf{OP}^{st}_{\widehat{\mathbb{Z}}}$ induced by τ_o is generated by the neighborhoods of zero W_m consisting of all collections $\{G^{(n)} | n \geqslant 0\}$ such that $G^{(n)}$ acts trivially on varieties of dimension $< n + m$. We still denote this topology by τ_o.

Let $A_r(x) = (1-x)^r \in \widehat{\mathbb{Z}}[[x]]$ for $r \in \widehat{\mathbb{Z}}$. Note that $\Phi(A_r) = r \cdot A_r$. In particular, if r is invertible in $\widehat{\mathbb{Z}}$, then $A_r \in S$.

We can describe the set S via divisibility conditions on the coefficients of the power series.

Theorem 6.10. *The set $S = \cap_r \mathrm{Im}(\Phi^r) \subset \widehat{\mathbb{Z}}[[x]]$ consists of all power series $G = \sum_{i \geqslant 0} a_i x^i$ satisfying the following property: for every prime p and every positive integers n and m such that m is divisible by p^n, for every nonnegative $j < m$ divisible by p, the sum $\sum_{i=j}^{m-1} \binom{i}{j} a_i$ is divisible by p^n.*

Proof. Let n be a positive integer, $G \in S$ and write $G = \Phi^n(H)$ for some $H \in \widehat{\mathbb{Z}}[[x]]$. Consider the ideal $I = (p^n, x^m) \subset \widehat{\mathbb{Z}}[[x]]$, where m is divisible by p^n. Note that $\Phi(I) \subset I$ since p^n divides m.

Let G' be the x^m-truncation of G and H' the x^m-truncation of H. As $G - G' \in I$ and $H - H' \in I$, we have $G' - \Phi^n(H') \in I$. Since G' and $\Phi^n(H')$ are polynomials of degree less than m, we conclude that G' and $\Phi^n(H')$ are congruent modulo p^n.

We write G' and H' as polynomials in $y = x - 1$. Since $\Phi^n(y^i) = i^n y^i$, the y^i-coefficients of $\Phi^n(H')$ are divisible by p^n for all i divisible by p. It follows that the same property holds for G'. As

$$G' = \sum_{i=0}^{m-1} a_i x^i = \sum_{i=0}^{m-1} a_i (y+1)^i = \sum_{i=0}^{m-1} a_i \sum_{j=0}^{i} \binom{i}{j} y^j = \sum_{j=0}^{m-1} y^j \sum_{i=j}^{m-1} \binom{i}{j} a_i,$$

the divisibility condition holds.

Conversely, as $\widehat{\mathbb{Z}} = \prod \mathbb{Z}_p$, it suffices to prove the statement over $\mathbb{Z}_p$. Let $G \in \mathbb{Z}_p[[x]]$ satisfy the divisibility condition in the theorem. Choose n and m such that m is divisible by p^n and set $I = (p^n, x^m) \subset \mathbb{Z}_p[[x]]$ as above. Recall that $\Phi(I) \subset I$. Let F be the x^m-truncation of G. By assumption, we can write $F \equiv \sum b_i y^i$ modulo p^n, where the sum is taken over $i < m$ that are prime to p. In particular, $G \equiv \sum b_i y^i$ modulo I.

Choose $r > 0$ and set $F' = \sum \frac{b_i}{i^r} y^i$. Then $\Phi^r(F') = \sum b_i y^i \equiv G$ modulo I, i.e., G is in the image of Φ^r modulo I. As $\mathrm{Im}(\Phi^r)$ is closed in $\mathbb{Z}_p[[x]]$ in the topology τ_w, we have $G \in \mathrm{Im}(\Phi^r)$ for all r, i.e., $G \in S$. □

6.2. **Stable operations in** CK **over** $\mathbb{Z}$**.** Now we turn to the study of stable operations over $\mathbb{Z}$.

Proposition 6.11. *The pre-image of $\mathbf{OP}^{n,n}$ under $\Sigma^{-1} : \mathbf{OP}^{n+1,n+1}_{\widehat{\mathbb{Z}}} \to \mathbf{OP}^{n,n}_{\widehat{\mathbb{Z}}}$ is equal to $\mathbf{OP}^{n+1,n+1}$ for every $n \geqslant 0$.*

Proof. As $\Sigma^{-1} = \partial^0 \circ \Phi$, for $n \geqslant 1$, and $\Sigma^{-1} = \Phi$, for $n = 0$, this follows immediately from Proposition 6.7. □

Thus, we have a tower

$$\mathbb{Z}[[x]] = \mathbf{OP}^{0,0} \hookleftarrow \mathbf{OP}^{1,1} \hookleftarrow \ldots \hookleftarrow \mathbf{OP}^{n,n} \hookleftarrow \ldots,$$

given by the desuspension and the group $\mathbf{OP}^{st}$ of stable homogeneous degree 0 integral operations is identified with $S_0 := S \cap \mathbb{Z}[[x]]$, where S is described by Theorem 6.10. Applying Proposition 6.7 again we get:

Proposition 6.12. *The group of homogeneous degree m stable operations $\mathrm{CK}^* \to \mathrm{CK}^{*+m}$ is naturally isomorphic to the intersection $x^{\max(0,m)}\mathbb{Z}[[x]] \cap S_0$.*

We would like to determine the structure of S_0.

Lemma 6.13. *For every $n \geqslant 0$ there is a positive integer d such that $dx^n \in S + x^{n+1}\widehat{\mathbb{Z}}[[x]]$.*

Proof. Choose distinct elements $r_0, \ldots, r_n \in \widehat{\mathbb{Z}}^\times$ such that $r_i - r_j \in \mathbb{Z}$ for all i and j. The x_i-coefficients with $i = 0, 1, \ldots, n$ of the power series $A_{r_j}(x) = (1-x)^{r_j} \in S$ form an $(n+1) \times (n+1)$ Van der Monde type matrix $\left[(-1)^i \binom{r_j}{i}\right]$. Its determinant d is a nonzero integer since all $r_i - r_j$ are integers. It follows that there is a $\widehat{\mathbb{Z}}$-linear combination of the A_{r_j}'s that is equal to dx^n modulo x^{n+1}. □

Note that any ideal in $\widehat{\mathbb{Z}}$ that contains a non-zero integer is generated by a positive integer (the smallest positive integer in the ideal). It follows from Lemma 6.13 that for every $n \geqslant 0$ there exists a unique positive integer d_n such that the ideal of all $a \in \widehat{\mathbb{Z}}$ with the property $ax^n \in S + x^{n+1}\widehat{\mathbb{Z}}[[x]]$ is generated by d_n. We will determine the integers d_n below.

For every $n \geqslant 0$ choose a power series $G_n \in S$ such that $G_n \equiv d_n x^n$ modulo x^{n+1}.

Lemma 6.14. *Let $G = \sum_{i \geqslant 0} a_i x^i \in S$ be such that $a_0, \ldots, a_{n-1} \in \mathbb{Z}$. Then there exist $b_i \in \widehat{\mathbb{Z}}$ for all $i \geqslant n$ such that $G - \sum_{i \geqslant n} b_i G_i \in S_0$.*

Proof. Find an integer a'_n such that $a_n - a'_n$ is divisible by d_n, thus, $a_n = a'_n + d_n b_n$ for some $b_n \in \widehat{\mathbb{Z}}$. Then the x^i-coefficients of $G - b_n G_n$ are integer for $i = 0, \ldots, n$. Continuing this procedure, we determine all b_i for $i \geqslant n$, so that all coefficients of $G - \sum_{i \geqslant n} b_i G_i$ are integers. □

Theorem 6.15. *For all $n \geqslant 0$ there are power series $F_n \in S_0$ such that $F_n \equiv d_n x^n$ modulo x^{n+1}. Moreover,*

1. *The group S_0 consists of all infinite linear combinations $\sum_{n \geqslant 0} a_n F_n$ with $a_n \in \mathbb{Z}$.*
2. *The group of homogeneous degree m stable operations $\mathrm{CK}^* \to \mathrm{CK}^{*+m}$ is naturally isomorphic to the group of all infinite linear combinations $\sum_{n \geqslant \max(0,m)} a_n F_n$ with $a_n \in \mathbb{Z}$.*

Proof. Fix an $n \geqslant 0$. The coefficient d_n of G_n is integer. Applying Lemma 6.14, we find $b_i \in \widehat{\mathbb{Z}}$ for $i \geqslant n+1$ such that $F_n := G_n - \sum_{i \geqslant n+1} b_i G_i \in S_0$. Statements (1) and (2) are clear. □

6.3. **The integers d_n.** Our next goal is to determine the integers d_n. Let $n > 0$ be an integer. For an integer r write L_r for the n-tuple of binomial coefficients:

$$\left(\binom{r}{0}, \binom{r}{1}, \dots, \binom{r}{n-1}\right) = (1, r, \dots) \in \mathbb{Z}^n.$$

For a n-sequence $\bar{a} = (a_1, \dots, a_n)$ of positive integers let $d(\bar{a})$ be the determinant of the $n \times n$ matrix with columns $L_{a_1}, L_{a_2}, \dots, L_{a_n}$. We have

$$d(\bar{a}) = \big(\prod_{s>t}(a_s - a_t)\big) / \prod_{k=1}^{n-1} k! \in \mathbb{Z}. \tag{6.16}$$

Let p be a prime integer. An n-sequence $\bar{a}$ is called *p-prime* if all its terms are prime to p. Let $\bar{a}^{(n)}_{\min}$ be the "smallest" strictly increasing p-prime n-sequence

$$(1, 2, \dots, p-1, p+1, \dots).$$

Lemma 6.17. *Let $\bar{a}$ be a p-prime n-sequence that differs from $\bar{a}^{(n)}_{\min}$ at one term only. Then $d(\bar{a}^{(n)}_{\min})$ divides $d(\bar{a})$ in the ring of p-adic integers $\mathbb{Z}_p$.*

Proof. Suppose $\bar{a}$ is obtained from $\bar{a}^{(n)}_{\min}$ by replacing a term a by b. It follows from (6.16) that

$$d(\bar{a})/d(\bar{a}^{(n)}_{\min}) = \prod(b - a')/\prod(a - a'),$$

where the products are taken over all terms a' of $\bar{a}_{\min}$ but a. Since a is prime to p, the product $\prod(a - a')$ generates the same ideal in $\mathbb{Z}_p$ as $a!(c-a)!$, where c is the last term of $\bar{a}_{\min}$. Similarly, as b is prime to p, the product $\prod(b - a')$ generates the same ideal in $\mathbb{Z}_p$ as

$$b(b-1)\cdots(b-a+1)\cdot(b-a-1)\cdots(b-c+1)(b-c) = (-1)^{c-a}a!(c-a)!\binom{b}{a}\binom{c-b}{c-a}. \ \square$$

Corollary 6.18. *The integer $d(\bar{a}^{(n)}_{\min})$ divides $d(\bar{a}^{(n+1)}_{\min})$ in $\mathbb{Z}_p$.*

Proof. In the cofactor expansion (Laplace's formula) of the determinant $d(\bar{a}^{(n+1)}_{\min})$ along the last row all minors are divisible by $d(\bar{a}^{(n)}_{\min})$ in view of Lemma 6.17. $\square$

Write M_n for the $\mathbb{Z}_p$-submodule of $(\mathbb{Z}_p)^n$ generated by the tuples $L_{a_1}, L_{a_2}, \dots, L_{a_n}$, where $(a_1, a_2, \dots, a_n) = \bar{a}^{(n)}_{\min}$.

Lemma 6.19. *Let b be an integer prime to p. Then the n-tuple L_b is contained in M_n. In others words, the $\mathbb{Z}_p$-submodule of $(\mathbb{Z}_p)^n$ generated by L_b for all integers $b > 0$ prime to p coincides with M_n.*

Proof. By Cramer's rule, the solutions of the equation $L_b = x_1 L_{a_1} + \ldots + x_n L_{a_n}$ are given by the formula $x_i = d(\bar{a}_{(i)})/d(\bar{a}^{(n)}_{\min})$, where the sequence $\bar{a}_{(i)}$ is obtained from $\bar{a}^{(n)}_{\min}$ by replacing the i-th term with b. By Lemma 6.17, we have $x_i \in \mathbb{Z}_p$. $\square$

The following statement is a generalization of Lemma 6.17.

Corollary 6.20. *Let $\bar{a}$ be any p-prime n-sequence. Then $d(\bar{a}^{(n)}_{\min})$ divides $d(\bar{a})$ in $\mathbb{Z}_p$.* $\square$

Set

$$d_n = d_n^{(p)} := d(\bar{a}_{\min}^{(n+1)})/d(\bar{a}_{\min}^{(n)}).$$

By Corollary 6.18, $d_n \in \mathbb{Z}_p$.

Write n in the form $n = (p-1)k + i$, where $i = 0, 1 \dots, p-2$ and $k = \lfloor \frac{n}{p-1} \rfloor$. Then it follows from (6.16) that

$$d_n \mathbb{Z}_p = \frac{p^k \cdot k!}{n!} \mathbb{Z}_p, \quad \text{or, equivalently} \quad v_p(d_n) = k + v_p(k!) - v_p(n!), \tag{6.21}$$

where v_p is the p-adic discrete valuation.

Note that $v_p((n+k)!) = v_p((pk+i)!) = v_p((pk)!) = k + v_p(k!)$, hence $d_n \mathbb{Z}_p = \frac{(n+k)!}{n!} \mathbb{Z}_p$. Observe that the function $n \mapsto v_p(d_n)$ is not monotonic.

Proposition 6.22. *An $(n+1)$-tuple $(0, 0, \dots, 0, d)$ is contained in M_{n+1} if and only if d is divisible by d_n in $\mathbb{Z}_p$.*

Proof. As in the proof of Lemma 6.19, $(0, 0, \dots, 0, d) \in M_{n+1}$ if and only if $d \cdot d(\bar{a}_{(i)})$ is divisible by $d(\bar{a}_{\min}^{(n+1)})$ in $\mathbb{Z}_p$ for all i, where the sequence $\bar{a}_{(i)}$ is obtained from $\bar{a}_{\min}^{(n)}$ by deleting the i-th term in $\bar{a}_{\min}^{(n+1)}$. We have $d(\bar{a}_{(i)}) = d(\bar{a}_{\min}^{(n)})$ if $i = n+1$ and by Corollary 6.20, all $d(\bar{a}_{(i)})$ are divisible by $d(\bar{a}_{\min}^{(n)})$, whence the result. □

For an integer r, let as before $A_r(x) = (1-x)^r$. Note that the n-tuple L_r is the tuple of coefficients (after appropriate change of signs) of the x^n-truncation of the polynomial A_r. Denote by $N^{(p)}$ the $\mathbb{Z}_p$-submodule of $\mathbb{Z}_p[x]$ generated by A_r for all integers $r > 0$ prime to p. We get an immediate corollary from Lemma 6.19 and Proposition 6.22:

Proposition 6.23. *Let $d \in \mathbb{Z}_p$ and $n \geqslant 0$. Then $dx^n \in N^{(p)} + x^{n+1}\mathbb{Z}_p[x]$ if and only if d is divisible by d_n. Moreover, there is a $\mathbb{Z}_p$-linear combination G_n of the Adams polynomials $A_{a_1}, A_{a_2}, \dots, A_{a_{n+1}}$, where $(a_1, a_2, \dots, a_{n+1}) = \bar{a}_{\min}^{(n+1)}$, such that $G_n \equiv d_n x^n \ (mod\ x^{n+1})$.* □

Proposition 6.24. *The set $S_{\mathbb{Z}_p} = \cap_r \operatorname{Im}(\Phi^r_{\mathbb{Z}_p})$ contains a power series $\equiv dx^n \ (mod\ x^{n+1})$ if and only if d is divisible by d_n in $\mathbb{Z}_p$.*

Proof. Suppose that $G \in S_{\mathbb{Z}_p}$ and $G \equiv dx^n$ modulo x^{n+1}. Choose integers $k > 0$ such that p^k is divisible by $d(\bar{a}_{\min}^{(n+1)})$ and $m > n$ divisible by p^k and consider the ideal $I = (p^k, x^m) \subset \mathbb{Z}_p$. We have $G = \Phi^k(G')$ for some $G' \in \mathbb{Z}_p[[x]]$ and write

$$G' = \sum_{i=0}^{m-1} b_i A_i \quad \text{modulo} \quad x^m \mathbb{Z}_p[[x]]$$

for some $b_i \in \mathbb{Z}_p$. Applying Φ^k and taking into account the equality $\Phi^k(A_i) = i^k A_i$, we get $G = \Phi^k(G') \in N^{(p)} + I$. Taking the x^{n+1}-truncations, we see that

$$dx^n \in N^{(p)} + x^{n+1}\mathbb{Z}_p[x] + p^k \mathbb{Z}_p[x].$$

As p^k is divisible by $d(\bar{a}_{\min}^{(n+1)})$, we conclude that $p^k \mathbb{Z}_p[x] \subset N^{(p)} + x^{n+1}\mathbb{Z}_p[x]$, hence $dx^n \in N^{(p)} + x^{n+1}\mathbb{Z}_p[x]$. By Proposition 6.23, d is divisible by d_n. □

Now we turn to the ring $\widehat{\mathbb{Z}}$. The integers d_n defined before Lemma 6.14 are the products of primary parts of $d_n = d_n^{(p)}$ determined as above for every prime p. In view of (6.21) we have

$$v_p(d_n) = \lfloor \frac{n}{p-1} \rfloor + v_p(\lfloor \frac{n}{p-1} \rfloor !) - v_p(n!)$$

for every prime p. For example, $d_0 = 1$, $d_1 = 2$, $d_2 = 2^2 \cdot 3$, $d_3 = 2^3$, $d_4 = 2^4 \cdot 3 \cdot 5$, $d_5 = 2^5 \cdot 3$, $d_6 = 2^6 \cdot 3^2 \cdot 7$, $d_7 = 2^7 \cdot 3^2$.

Propositions 6.23 and 6.24 yield:

Theorem 6.25. *Let $n \geqslant 0$ be an integer. Then*

1. *There is a $\widehat{\mathbb{Z}}$-linear combination G_n of the Adams polynomials $A_{a_1}, A_{a_2}, \ldots, A_{a_{n+1}}$ for some $a_1, a_2, \ldots, a_{n+1} \in \widehat{\mathbb{Z}}^\times$ such that $G_n \equiv d_n x^n$ modulo x^{n+1}.*
2. *The set $S = \cap_r \operatorname{Im}(\Phi^r_{\widehat{\mathbb{Z}}})$ contains a power series $\equiv dx^n \ (mod\ x^{n+1})$ if and only if d is divisible by d_n in $\widehat{\mathbb{Z}}$.*

Remark 6.26. It follows from Proposition 6.23 that $a_1, a_2, \ldots, a_{n+1} \in \widehat{\mathbb{Z}}^\times$ can be chosen so that for every prime p, we have $((a_1)_p, (a_2)_p, \ldots, (a_{n+1})_p) = \bar{a}^{(n+1)}_{\min}$ with respect to p. In particular, $a_1 = 1$.

Proposition 6.27. *The set $S = \cap_r \operatorname{Im}(\Phi^r)$ is the closure in the topology τ_s, and hence, in the topologies τ_o and τ_w of the set of all (finite) $\widehat{\mathbb{Z}}$-linear combinations of the power series A_r for $r \in \widehat{\mathbb{Z}}^\times$.*

Proof. Denote as T_s, T_w, T_o the closures of the mentioned set of linear combinations in our three topologies. As S is closed in τ_w, we have $T_s \subset T_o \subset T_w \subset S$.

Let $G \in x^k\widehat{\mathbb{Z}}[[x]] \cap S$. Then by Theorem 6.25, $G \equiv dx^k \ (mod\, x^{k+1})$, where $d = d_k \cdot c$, for some $c \in \widehat{\mathbb{Z}}$. We know that there exists a $\widehat{\mathbb{Z}}$-linear combination G_k of the power series $A_{a_1}, A_{a_2}, \ldots, A_{a_k}$ (with invertible a_i's) such that $G_k \equiv d_k x^k \ (mod\, x^{k+1})$. Hence, $G - c \cdot G_k \in x^{k+1}\widehat{\mathbb{Z}}[[x]] \cap S$. Applying this inductively, we obtain that, for any $G \in S$ and any positive integer m, there exists a finite $\widehat{\mathbb{Z}}$-linear combination H of invertible A_r's, such that $G - H \in x^m\widehat{\mathbb{Z}}[[x]] \cap S$. Therefore, $T_s = S$ and hence $T_s = T_o = T_w = S$. □

6.4. **Stable operations in K_{gr}.** In section 4.7 we defined the bi-algebra A of co-operations in K_{gr}. Recall that for a commutative ring R, the bi-algebra of operations $\mathbf{OP}^{n,n}_R(K_{gr}) = \mathbf{OP}^{0,0}_R(\mathrm{CK}) = R[[x]]$ is dual to A. The same proof as in Proposition 6.4 shows that the desuspension operator

$$\Sigma^{-1} : R[[x]] = \mathbf{OP}^{n,n}_R(K_{gr}) \to \mathbf{OP}^{n-1,n-1}_R(K_{gr}) = R[[x]]$$

coincides with Φ. It follows that

$$\mathbf{OP}^{st}_R(K_{gr}) = \lim(R[[x]] \xleftarrow{\Phi} R[[x]] \xleftarrow{\Phi} R[[x]] \xleftarrow{\Phi} \ldots).$$

Lemma 6.28. *The desuspension operator Φ is dual to the multiplication by s in A.*

Proof. As $\Phi((1-x)^m) = m(1-x)^m$, in view of Lemma 4.35 we have

$$\langle e_n, \Phi((1-x)^m)\rangle = \langle e_n, m(1-x)^m\rangle = m \cdot e_n(m) = \langle se_n, (1-x)^m\rangle.$$ □

The localization $A[\frac{1}{s}]$ can be identified with $\operatorname{colim}(A \xrightarrow{s} A \xrightarrow{s} \dots)$. Therefore,

$$\mathbf{OP}^{st}_R(K_{gr}) \simeq \operatorname{Hom}(A\Big[\frac{1}{s}\Big], R),$$

i.e., the bi-algebra $\mathbf{OP}^{st}_R(K_{gr})$ of stable operations is dual to $A[\frac{1}{s}]$.

The bi-algebra $A[\frac{1}{s}]$ coincides with the algebra of degree 0 stable operations $K_0(K)$ in topology (see [6, Proposition 3] and [2]). Moreover, $A[\frac{1}{s}]$ is a free abelian group of countable rank [1, Theorem 2.2] and can be described as the set of all Laurent polynomials $f \in \mathbb{Q}[s, s^{-1}]$ such that $f(\frac{a}{b}) \in \mathbb{Z}[\frac{1}{ab}]$ for all integers a and $b \neq 0$.

It follows that the bi-algebra $A[\frac{1}{s}]$ admits an antipode $s \mapsto s^{-1}$ that makes $A[\frac{1}{s}]$ a Hopf algebra. It follows that $\mathbf{OP}^{st}_R(K_{gr})$ is a (topological) Hopf algebra.

Remark 6.29. We have a diagram of homomorphisms of bi-algebras and its dual:

$$\begin{array}{ccc} \mathbb{Z}[s] & \longrightarrow & A \\ \downarrow & & \downarrow \\ \mathbb{Z}[s, s^{-1}] & \longrightarrow & A[\frac{1}{s}] \end{array} \qquad \begin{array}{ccc} R^{[0,\infty)} & \xleftarrow{b} & R[[x]] \\ \uparrow & & \uparrow \\ R^{\mathbb{Z}} & \longleftarrow & \mathbf{OP}^{st}_R(K_{gr}) \end{array}$$

The bottom maps are homomorphisms of Hopf algebras. The antipode of $R^{\mathbb{Z}}$ takes a sequence r_i to r_{-i}.

The group of degree 0 stable operations $\mathbf{OP}^{st}_{\widehat{\mathbb{Z}}}(K_{gr})$ coincides with $\mathbf{OP}^{st}_{\widehat{\mathbb{Z}}}(\mathrm{CK}) = S$ whose structure is determined in Theorem 6.25. Our nearest goal is to prove similar result for $\mathbf{OP}^{st}_{\mathbb{Z}}(K_{gr})$. We remark that this group is different from $\mathbf{OP}^{st}_{\mathbb{Z}}(\mathrm{CK}) = S \cap \mathbb{Z}[[x]]$.

Let R be one of the following rings: $\mathbb{Z}$, $\mathbb{Z}_p$ or $\widehat{\mathbb{Z}}$. Recall that we have an injective homomorphism $b_R : R[[x]] \to R^{[0,\infty)}$ taking $(1-x)^m$ to the sequence $(1, m, m^2, \dots)$. The operation Φ on $R[[x]]$ corresponds to the shift operation Π on $R^{[0,\infty)}$ defined by $\Pi(a)_i = a_{i+1}$.

An *n-interval* of a sequence a in $R^{[0,\infty)}$ or $R^{\mathbb{Z}}$ is the n-tuple $(a_i, a_{i+1}, \dots, a_{i+n-1})$ for some i. We say that this interval *starts at i*.

For every $n \geqslant 1$, let M_n be the R-submodule of R^n generated by the n-tuples $\bar{r} := (1, r, r^2, \dots, r^{n-1})$ for all integers $r > 0$. Note that M_n is of finite index in R^n.

Lemma 6.30. *A sequence $a \in R^{[0,\infty)}$ belongs to the image of b_R if and only if for every $n > 0$, the n-interval of a starting at 0 is contained in M_n.*

Proof. The implication $\Rightarrow$ is clear. For the converse note that by assumption a is contained in the closure of $\operatorname{Im}(b_R)$. On the other hand, if $R = \mathbb{Z}_p$ or $\widehat{\mathbb{Z}}$, the space $R[[x]]$ is compact in τ_w and $R^{[0,\infty)}$ is Hausdorff, hence $\operatorname{Im}(b_R)$ is closed, i.e., $a \in \operatorname{Im}(b_R)$. If $R = \mathbb{Z}$, it follows from the case $R = \widehat{\mathbb{Z}}$ that $a = b_{\widehat{\mathbb{Z}}}(G)$ for some $G \in \widehat{\mathbb{Z}}[[x]]$. Since at the same time $G \in \mathbb{Q}[[x]]$, we have $G \in \mathbb{Z}[[x]]$. □

Let $T_R \subset R^{\mathbb{Z}}$ be the R-submodule of all sequences $a \in R^{\mathbb{Z}}$ such that every n-interval of a is contained in M_n for all $n \geqslant 1$. If $a \in T_R$, by Lemma 6.30, for every $n \geqslant 0$ there is $G_n \in R[[x]]$ such that $b_R(G_n) = (a_{-n}, a_{-n+1}, \dots)$. Since $\Phi(G_{n+1}) = G_n$, the

sequence $(G_n)_{n\geqslant 0}$ determines an element in $\mathbf{OP}^{st}_R(K_{gr})$. This construction establishes an isomorphism $\mathbf{OP}^{st}_R(K_{gr}) \simeq T_R$. Note that $T_{\widehat{\mathbb{Z}}} = \mathbf{OP}^{st}_{\widehat{\mathbb{Z}}}(\mathrm{CK}) = S = \cap_r \mathrm{Im}(\Phi^r)$.

For every $n \geqslant 1$, let N_n be the R-submodule of R^n generated by the n-tuples $\bar{r}$ for all $r \in R^\times$. Then N_n is of finite index in R^n if $R = \mathbb{Z}_p$ or $\widehat{\mathbb{Z}}$.

Note that every n-tuple $\bar{r}$ with $r \in R^\times$ extends to the sequence a with $a_i = r^i$ that is contained in T_R.

Lemma 6.31. *$N_n \subset M_n$ for all $n \geqslant 1$.*

Proof. It suffices to consider the case $R = \mathbb{Z}_p$. Choose an integer $m > 0$ such that $p^m \cdot \mathbb{Z}_p^n \subset M_n$. Let $r \in \mathbb{Z}_p^\times$. Find an integer $r' > 0$ congruent to r modulo p^m. Then the tuple $\bar{r} = (1, r, r^2, \ldots, r^{n-1})$ is congruent to $\bar{r}'$ modulo p^m. Hence $\bar{r} = \bar{r}' + (\bar{r} - \bar{r}') \in M_n + p^m\mathbb{Z}_p^n \subset M_n$. □

It follows from Lemma 6.31 that every element in N_n is an n-interval of a sequence in T_R.

Proposition 6.32. *If $R = \mathbb{Z}_p$ or $\widehat{\mathbb{Z}}$, the R-module T_R consists of all sequences $a \in R^{\mathbb{Z}}$ such that every n-interval of a is contained in N_n for all $n \geqslant 1$.*

Proof. We may assume that $R = \mathbb{Z}_p$. Let $a \in T_R$. In view of Lemma 6.31 it suffices to show that every n-interval v of a starting at i is contained in N_n for all $n \geqslant 1$. Take an integer $m > 0$ and consider the $(n+m)$-interval w of $\Pi^{-m}(a)$ starting at i, so that v is the part of w on the right. Write w as a (finite) linear combination $\sum t_r\bar{r}$ over positive integers r, where $t_r \in \mathbb{Z}_p$ and $\bar{r} = (1, r, r^2, \ldots, r^{n+m-1}) \in M_{n+m-1}$. Applying Π^m to $\Pi^{-m}(a)$ we see that $v = \sum t_r r^m \hat{r}$, where $\hat{r} = (1, r, r^2, \ldots, r^{n-1}) \in M_{n-1}$. As r^m is divisible by p^m if r is divisible by p, it follows from the definition of N_n that $v \in N_n + p^m M_n$. Since N_n is of finite index in M_n, we can choose m such that $p^m M_n \subset N_n$, hence $a \in N_n$. □

Denote by $\theta : R^{\mathbb{Z}} \to R^{\mathbb{Z}}$ the reflection operation taking a sequence a to the sequence $\theta(a)_i = a_{-i}$.

Corollary 6.33. *The module T_R is invariant under θ.*

Proof. In the case $R = \mathbb{Z}_p$ or $\widehat{\mathbb{Z}}$ it suffice to notice that if $r \in \mathbb{R}^\times$, the symmetric n-tuple $(r^{n-1}, r^{n-2}, \ldots, r, 1) = r^{n-1}(1, r^{-1}, (r^{-1})^2, \ldots, (r^{-1})^{n-1})$ is contained in N_n. If $R = \mathbb{Z}$ the statement follows from the equality $T_{\mathbb{Z}} = T_{\widehat{\mathbb{Z}}} \cap \mathbb{Z}^{\mathbb{Z}}$. □

Now let $R = \widehat{\mathbb{Z}}$ and $n \geqslant 0$. The ideal of all $t \in \widehat{\mathbb{Z}}$ such that $(0, \ldots, 0, t) \in N_n$ is generated by a (unique) positive integer $\tilde{d}_n = n! \cdot d_n$, where the integers d_n were introduced in Section 6.3. We know that

$$v_p(\tilde{d}_n) = v_p((n + k_p)!)$$

for all primes p, where $k_p = \lfloor \frac{n}{p-1} \rfloor$. By Theorem 6.15, there are power series $F_n \in S_0 = S \cap \mathbb{Z}[[x]]$ such that $F_n \equiv d_n x^n$ modulo x^{n+1}.

Let $f^{(n)} \in T_{\widehat{\mathbb{Z}}}$ be the image of F_n under the map $S = \mathbf{OP}^{st}_{\widehat{\mathbb{Z}}}(K_{gr}) \to \widehat{\mathbb{Z}}^{\mathbb{Z}}$. Thus, $(0, \ldots, 0, \tilde{d}_n)$ is the n-interval of $f^{(n)}$ starting at 0. For example, we can choose:

$$f^{(0)} = (\ldots, 1, 1, 1, 1, \ldots),$$

$$f^{(1)} = (\dots, 0, 2, 0, 2, \dots).$$

As in the proof of Theorem 6.15, modifying $f^{(n)}$ by adding multiples of the shifts of $f^{(m)}$ for $m > n$ and their reflections we can obtain $f^{(n)} \in \mathbb{Z}^{\mathbb{Z}}$ for all n.

Theorem 6.34. *Every sequence $a \in T_{\mathbb{Z}} \simeq \mathbf{OP}^{st}_{\mathbb{Z}}(K_{gr})$ can be written in the form*

$$a = \sum_{i=0}^{\infty} \left[b_{2i} \Pi^{-i} \theta(f^{(2i)}) + b_{2i+1} \Pi^{i}(f^{(2i+1)}) \right]$$

for unique $b_0, b_1, \dots \in \mathbb{Z}$.

Proof. We determine the integers $b_0, b_1, \dots$ inductively so that for every $m \geqslant 0$ the sum $\sum_{i=0}^{m}$ of the terms in the right hand side and the sequence a have the same $2m+2$-intervals starting at $-m$. □

Observe that $\{\Pi^{-i}\theta(f^{(2i)}),\ \Pi^{i}(f^{(2i+1)}) \,|\, i \in \mathbb{Z}_{\geqslant 0}\}$ is also a topological basis of $\mathbf{OP}^{st}_{\widehat{\mathbb{Z}}}(K_{gr})$. Note that, at the same time, $\{f^{(j)} \,|\, j \in \mathbb{Z}_{\geqslant 0}\}$ form a topological basis for $\mathbf{OP}^{st}_{\mathbb{Z}}(\mathrm{CK})$ and $\mathbf{OP}^{st}_{\widehat{\mathbb{Z}}}(\mathrm{CK})$.

6.5. **Stable multiplicative operations.** We first consider stable multiplicative operations on $\mathrm{CK}^*_{\widehat{\mathbb{Z}}}$. From Proposition 5.3 we obtain:

Proposition 6.35. *Stable multiplicative operations $\mathrm{CK}^*_{\widehat{\mathbb{Z}}} \to \mathrm{CK}^*_{\widehat{\mathbb{Z}}}$ are exactly operations Ψ^c_1, for $c \in \widehat{\mathbb{Z}}^{\times}$. These are invertible and form a group isomorphic to $\widehat{\mathbb{Z}}^{\times}$. Similarly, stable multiplicative operations on CK^* form a group isomorphic to $\mathbb{Z}^{\times}$.*

Restricted to $\mathrm{CK}^0_{\widehat{\mathbb{Z}}}$, the operation Ψ^c_1 is given by $G_0 = 1$ (as it is multiplicative and so, maps 1 to 1), while $G(tx) = G(t)G(x) = ct \cdot \gamma_G(x) = 1 - (1 - tx)^c$ and so, our operation corresponds to the power series $A_c = (1 - x)^c$. In other words, on $\mathrm{CK}^0_{\widehat{\mathbb{Z}}}$, operation Ψ^c_1 coincides with the Adams operation Ψ_c. Then on $\mathrm{CK}^n_{\widehat{\mathbb{Z}}}$ it is equal to $c^{-n} \cdot \Psi_c$.

Proposition 6.27 gives:

Corollary 6.36. *The set of homogeneous stable additive operations on $\mathrm{CK}^*_{\widehat{\mathbb{Z}}}$ is the closure in the topology τ_o of the set of (finite) $\widehat{\mathbb{Z}}$-linear combinations of stable multiplicative operations.*

Remark 6.37. Note that the respective statement for $\mathbb{Z}$-coefficients is not true, as there are only two stable multiplicative operations on CK^*, namely, Ψ^1_1 and Ψ^{-1}_1, while the group of stable additive operations there has infinite (uncountable) rank.

Now we consider stable multiplicative operations on K^*_{gr} over $\mathbb{Z}$.

Proposition 6.38. *Stable multiplicative operations $K^*_{gr} \to K^*_{gr}$ are exactly operations Ψ^c_1, for $c = \pm 1$. These are invertible and form a group isomorphic to $\mathbb{Z}^{\times} \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. The linear coefficient of γ_G for the operation ${}^l\Psi^c_b$ is $t^{1-l}b$ - see 5.2. This will be equal to 1 exactly when $l = 1$ and $b = 1$. □

As above, the operation Ψ^c_1 corresponds to the power series $A_c = (1 - x)^c$. On K^n_{gr} it coinsides with $c^{-n} \cdot \Psi^1_c$.

References

[1] Adams, J. F., and Clarke, F. W. Stable operations on complex K-theory. *Illinois J. Math. 21*, 4 (1977), 826–829.

[2] Adams, J. F., Harris, A. S., and Switzer, R. M. Hopf algebras of cooperations for real and complex K-theory. *Proc. London Math. Soc. (3) 23* (1971), 385–408.

[3] Cai, S. Algebraic connective K-theory and the niveau filtration. *J. Pure Appl. Algebra 212*, 7 (2008), 1695–1715.

[4] Clarke, F., Crossley, M. D., and Whitehouse, S. Bases for cooperations in K-theory. *K-Theory 23*, 3 (2001), 237–250.

[5] Dai, S., and Levine, M. Connective algebraic K-theory. *J. K-Theory 13*, 1 (2014), 9–56.

[6] Johnson, K. The action of the stable operations of complex K-theory on coefficient groups. *Illinois J. Math. 28*, 1 (1984), 57–63.

[7] Levine, M., and Morel, F. *Algebraic cobordism*. Springer Monographs in Mathematics. Springer, Berlin, 2007.

[8] Panin, I. Oriented cohomology theories of algebraic varieties. vol. 30. 2003, pp. 265–314. Special issue in honor of Hyman Bass on his seventieth birthday. Part III.

[9] Smirnov, A. Orientations and transfers in cohomology of algebraic varieties. *St. Petersburg Math. J. 18*, 2 (2007), 305–346.

[10] Strong, M.-J., and Whitehouse, S. Integer-valued polynomials and K-theory operations. *Proc. Amer. Math. Soc. 138*, 6 (2010), 2221–2233.

[11] Vishik, A. Stable and unstable operations in algebraic cobordism. *Ann. Sci. Éc. Norm. Supér. (4) 52*, 3 (2019), 561–630.

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