ON THE CHOW GROUP OF CYCLES OF CODIMENSION 2

Let X be an algebraic variety over F. We write $A^i(X, K_n)$ for the homology group of the complex

$$\coprod_{x\in X^{(i-1)}} K_{n-i+1}\big(F(x)\big) \xrightarrow{\partial} \coprod_{x\in X^{(i)}} K_{n-i}\big(F(x)\big) \xrightarrow{\partial} \coprod_{x\in X^{(i+1)}} K_{n-i-1}\big(F(x)\big),$$

where K_j are the Milnor K-groups and $X^{(i)}$ is the set of points in X of codimension i (see [7, §5]). In particular, $A^i(X, K_i) = \mathrm{CH}^i(X)$ is the Chow group of classes of codimension i algebraic cycles on X.

Let X and Y be smooth complete geometrically irreducible varieties over F.

Proposition 0.1. *Suppose that for every field extension K/F we have:*

1. *The natural map $\mathrm{CH}^1(X) \longrightarrow \mathrm{CH}^1(X_K)$ is an isomorphism of torsion free groups,*
2. *The product map $\mathrm{CH}^1(X_K) \otimes K^\times \longrightarrow A^1(X_K, K_2)$ is an isomorphism.*

Then the natural sequence

$$0 \longrightarrow \big(\mathrm{CH}^1(X) \otimes \mathrm{CH}^1(Y)\big) \oplus \mathrm{CH}^2(Y) \longrightarrow \mathrm{CH}^2(X \times Y) \longrightarrow \mathrm{CH}^2(X_{F(Y)})$$

is exact.

Proof. Consider the spectral sequence

$$E_1^{p,q} = \coprod_{y\in Y^{(p)}} A^q(X_{F(y)}, K_{2-p}) \Longrightarrow A^{p+q}(X \times Y, K_2)$$

for the projection $X \times Y \longrightarrow Y$ (see [7, Cor. 8.2]). The nonzero terms of the first page are the following:

$$\begin{array}{l}
\mathrm{CH}^2(X_{F(Y)}) \\[2em]
A^1(X_{F(Y)}, K_2) \longrightarrow \coprod_{y\in Y^{(1)}} \mathrm{CH}^1(X_{F(y)}) \\[2em]
A^0(X_{F(Y)}, K_2) \longrightarrow \coprod_{y\in Y^{(1)}} A^0(X_{F(y)}, K_1) \longrightarrow \coprod_{y\in Y^{(2)}} \mathrm{CH}^0(X_{F(y)}).
\end{array}$$

Then $E_1^{2,0} = \coprod_{y\in Y^{(2)}} \mathbb{Z}$ is the group of cycles on X of codimension 2 and $E_1^{1,0} = \coprod_{y\in Y^{(1)}} F(y)^\times$ as X is complete. It follows that $E_2^{2,0} = \mathrm{CH}^2(Y)$.

By assumption, the differential $E_1^{0,1} \longrightarrow E_1^{1,1}$ is identified with the map

$$\mathrm{CH}^1(X) \otimes \Big(F(Y)^\times \longrightarrow \coprod_{y\in Y^{(1)}} \mathbb{Z}\Big).$$

Since Y is complete and $\mathrm{CH}^1(X)$ is torsion free, we have $E_2^{0,1} = \mathrm{CH}^1(X) \otimes F^\times$ and $E_\infty^{1,1} = E_2^{1,1} = \mathrm{CH}^1(X) \otimes \mathrm{CH}^1(Y)$.

The edge map

$$A^1(X \times Y, K_2) \longrightarrow E_2^{0,1} = \mathrm{CH}^1(X) \otimes F^\times$$

is split by the product map

$$\mathrm{CH}^1(X) \otimes F^\times = A^1(X, K_1) \otimes A^0(Y, K_1) \longrightarrow A^1(X \times Y, K_2),$$

hence the edge map is surjective. Therefore, the differential $E_2^{0,1} \longrightarrow E_2^{2,0}$ is trivial and hence $E_\infty^{2,0} = E_2^{2,0} = \mathrm{CH}^2(Y)$. Thus, the natural homomorphism

$$\mathrm{CH}^2(Y) \longrightarrow \mathrm{Ker}\big(\mathrm{CH}^2(X \times Y) \longrightarrow \mathrm{CH}^2(X_{F(Y)})\big)$$

is injective and its cokernel is isomorphic to $\mathrm{CH}^1(X) \otimes \mathrm{CH}^1(Y)$. The statement follows. □

Example 0.2. Let X be a projective homogeneous variety of a semisimple algebraic group over F. There exist an étale F-algebra E and an Azumaya E-algebra A such that for $i = 0$ and 1, we have an exact sequence

$$0 \longrightarrow A^1(X, K_{i+1}) \longrightarrow K_i(E) \xrightarrow{\rho} H^{i+2}\big(F, \mathbb{Q}/\mathbb{Z}(i+1)\big),$$

where $\rho(x) = N_{E/F}\big((x) \cup [A]\big)$ (see [4] and [5]). If the algebras E and A are split, then ρ is trivial and for every field extension K/F,

$$\mathrm{CH}^1(X) \simeq K_0(E) \simeq K_0(E \otimes K) \simeq \mathrm{CH}^1(X_K),$$

$$A^1(X_K, K_2) \simeq K_1(E \otimes K) \simeq K_0(E) \otimes K^\times \simeq \mathrm{CH}^1(X_K) \otimes K^\times.$$

Therefore, the condition (1) and (2) in Proposition 0.1 hold. For example, if X is a smooth projective quadric of dimension at least 3, then $E = F$ and A is split.

Now consider the natural complex

$$\mathrm{CH}^2(X) \oplus \big(\mathrm{CH}^1(X) \otimes \mathrm{CH}^1(Y)\big) \longrightarrow \mathrm{CH}^2(X \times Y) \longrightarrow \mathrm{CH}^2(Y_{F(X)}). \tag{1}$$

Proposition 0.3. *Suppose that*

(1) *The Grothendieck group $K_0(Y)$ is torsion-free,*
(2) *The product map $K_0(X) \otimes K_0(Y) \longrightarrow K_0(X \times Y)$ is an isomorphism.*

Then the sequence (1) is exact.

Proof. It follows from the assumptions that the map $K_0(Y) \longrightarrow K_0(Y_{F(X)})$ is injective and the kernel of the natural homomorphism $K_0(X \times Y) \longrightarrow K_0(Y_{F(X)})$ coincides with

$$I_0(X) \otimes K_0(Y),$$

where $I_0(X)$ is the kernel of the rank homomorphism $K_0(X) \longrightarrow \mathbb{Z}$.

The kernel of the second homomorphism in the sequence (1) is generated by the classes of closed integral subschemes $Z \subset X \times Y$ that are not dominant over X. By Riemann-Roch (see [2]), we have $[Z] = -c_2\big([O_Z]\big)$ in $\mathrm{CH}^2(X \times Y)$, where $c_i : K_0(X \times Y) \longrightarrow \mathrm{CH}^i(X \times Y)$ is the i-th Chern class map. As

$$[O_Z] \in \mathrm{Ker}\big(K_0(X \times Y) \longrightarrow K_0(Y_{F(X)})\big) = I_0(X) \otimes K_0(Y),$$

it suffices to to show that $c_2\big(I_0(X) \otimes K_0(Y)\big)$ is contained in the image M of the first map in the sequence (1).

The formula $c_2(x + y) = c_2(x) + c_1(x)c_1(y) + c_2(y)$ shows that it suffices to prove that for all $a, a' \in I_0(X)$ and $b, b' \in K_0(Y)$, the elements $c_1(ab) \cdot c_1\big(a'b'\big)$ and $c_2(ab)$ are contained in M. This follows from the formulas (see [1, Remark 3.2.3 and Example 14.5.2]): $c_1(ab) = mc_1(a) + nc_1(b)$ and

$$c_2(ab) = \frac{m^2 - m}{2} c_1(a)^2 + mc_2(a) + (nm - 1)c_1(a)c_1(b) + \frac{n^2 - n}{2} c_1(b)^2 + nc_2(b),$$

where $n = \mathrm{rank}(a)$ and $m = \mathrm{rank}(b)$. □

Example 0.4. If Y is a projective homogeneous variety, then the condition (1) holds by [6]. If X is a projective homogeneous variety of a semisimple algebraic group G over F and the Tits algebras of G are split, then it follows from [6] that the condition (2) also holds for any Y. For example, if the even Clifford algebra of a nondegenerate quadratic form is split, then the corresponding projective quadric X satisfies (2) for any Y.

For any field extension K/F, let K^s denote the subfield of elements that are algebraic and separable over F.

Proposition 0.5. *Suppose that for every field extension K/F we have:*

(1) *The natural map $\mathrm{CH}^1(X) \longrightarrow \mathrm{CH}^1(X_K)$ is an isomorphism,*
(2) *The natural map $\mathrm{CH}^1(Y_{K^s}) \to \mathrm{CH}^1(Y_K)$ is an isomorphism.*

Then the sequence (1) is exact.

Proof. Consider the spectral sequence

$$E_1^{p,q}(F) = \coprod_{x \in X^{(p)}} A^q(Y_{F(x)}, K_{2-p}) \Longrightarrow A^{p+q}(X \times Y, K_2) \tag{2}$$

for the projection $X \times Y \longrightarrow X$. The nonzero terms of the first page are the following:

$$\mathrm{CH}^2(Y_{F(X)})$$

$$A^1(Y_{F(X)}, K_2) \longrightarrow \textstyle\coprod_{x \in X^{(1)}} \mathrm{CH}^1(Y_{F(x)})$$

$$A^0(Y_{F(X)}, K_2) \longrightarrow \textstyle\coprod_{x \in X^{(1)}} A^0(Y_{F(x)}, K_1) \longrightarrow \coprod_{x \in X^{(2)}} \mathrm{CH}^0(Y_{F(x)}).$$

As in the proof of Proposition 0.1, we have $E_2^{2,0}(F) = \mathrm{CH}^2(X)$. For a field extension K/F, write $C(K)$ for the factor group

$$\mathrm{Ker}\big(\mathrm{CH}^2(X_K \times Y_K) \longrightarrow \mathrm{CH}^2(Y_{K(X)})\big) / \operatorname{Im}\big(\mathrm{CH}^2(X_K) \longrightarrow \mathrm{CH}^2(X_K \times Y_K)\big).$$

The spectral sequence (2) for the varieties X_K and Y_K over K yields an isomorphism $C(K) \simeq E_2^{1,1}(K)$. We have a natural composition

$$\mathrm{CH}^1(X_K) \otimes \mathrm{CH}^1(Y_K) \longrightarrow E_1^{1,1}(K) \longrightarrow E_2^{1,1}(K) \simeq C(K).$$

We claim that the group $C(F)$ is generated by images of the compositions

$$\mathrm{CH}^1(X_K) \otimes \mathrm{CH}^1(Y_K) \longrightarrow C(K) \xrightarrow{N_{K/F}} C(F)$$

over all finite separable field extensions K/F (here $N_{K/F}$ is the norm map for the extension K/F).

The group $C(F)$ is generated by images of the maps

$$\varphi_x : \mathrm{CH}^1(Y_{F(x)}) \longrightarrow E_2^{1,1}(F) \simeq C(F)$$

over all points $x \in X^{(1)}$. Pick such a point x and let $K := F(x)^s$ be the subfield of elements that are separable over F. Then K/F is a finite separable field extension. Let $x' \in X_K^{(1)}$ be a point over x such that $K(x') \simeq F(x)$. Then φ_x coincides with the composition

$$\mathrm{CH}^1(Y_{K(x')}) \longrightarrow C(K) \xrightarrow{N_{K/F}} C(F).$$

By assumption, the map $\mathrm{CH}^1(Y_K) \longrightarrow \mathrm{CH}^1(Y_{K(x')})$ is an isomorphism, hence the image of φ_x coincides with the image of

$$[x'] \otimes \mathrm{CH}^1(Y_K) \longrightarrow C(K) \xrightarrow{N_{K/F}} C(F),$$

whence the claim.

As $\mathrm{CH}^1(X) \longrightarrow \mathrm{CH}^1(X_K)$ is an isomorphism for every field extension K/F, the projection formula shows that the map $\mathrm{CH}^1(X) \otimes \mathrm{CH}^1(Y) \longrightarrow C(F)$ is surjective. The statement follows. □

Example 0.6. Let Y be a projective homogeneous variety with the F-algebras E and A as in Example 0.2. If A is split, then $\mathrm{CH}^1(Y_K) = K_0(E \otimes K)$ for every field extension K/F. As K^s is separably closed in K, the natural map $K_0(E \otimes K^s) \longrightarrow K_0(E \otimes K)$ is an isomorphism, therefore, the condition (2) holds.

Write $\widetilde{\mathrm{CH}}^2(X \times Y)$ for the cokernel of the product map $\mathrm{CH}^1(X) \otimes \mathrm{CH}^1(Y) \longrightarrow \mathrm{CH}^2(X \times Y)$. We have the following commutative diagram:

$$\begin{tikzcd}
 & & \mathrm{CH}^2(X) \arrow[d] \arrow[dr] & \\
0 \arrow[r] & \mathrm{CH}^2(Y) \arrow[r] \arrow[dr] & \widetilde{\mathrm{CH}}^2(X \times Y) \arrow[r] \arrow[d] & \mathrm{CH}^2(X_{F(Y)}) \\
 & & \mathrm{CH}^2(Y_{F(X)}) &
\end{tikzcd}$$

Proposition 0.1 gives conditions for the exactness of the row in the diagram and Propositions 0.3 and 0.5 - for the exactness of the column in the diagram.

A diagram chase yields together with Propositions 0.1, 0.3 and 0.5 yields the following statements.

Theorem 0.7. *Let X and Y be smooth complete geometrically irreducible varieties such that for every field extension K/F:*

(1) *The natural map $\mathrm{CH}^1(X) \longrightarrow \mathrm{CH}^1(X_K)$ is an isomorphism of torsion free groups,*
(2) *The natural map $\mathrm{CH}^2(X) \longrightarrow \mathrm{CH}^2(X_K)$ is injective,*
(3) *The product map $\mathrm{CH}^1(X_K) \otimes K^\times \longrightarrow A^1(X_K, K_2)$ is an isomorphism,*
(4) *The Grothendieck group $K_0(Y)$ is torsion-free,*
(5) *The product map $K_0(X) \otimes K_0(Y) \longrightarrow K_0(X \times Y)$ is an isomorphism.*

Then the natural map $\mathrm{CH}^2(Y) \longrightarrow \mathrm{CH}^2(Y_{F(X)})$ is injective.

Remark 0.8. The conditions (1) – (3) hold for a smooth projective quadric X of dimension at least 7 by [3, Theorem 6.1] and Example 0.2. By Example 0.4, the conditions (4) and (5) hold if the even Clifford algebra of X is split and Y is a projective homogeneous variety.

Theorem 0.9. *Let X and Y be smooth complete geometrically irreducible varieties such that for every field extension K/F:*

(1) *The natural map $\mathrm{CH}^1(X) \longrightarrow \mathrm{CH}^1(X_K)$ is an isomorphism of torsion free groups,*
(2) *The natural map $\mathrm{CH}^2(X) \longrightarrow \mathrm{CH}^2(X_K)$ is injective,*
(3) *The product map $\mathrm{CH}^1(X_K) \otimes K^\times \longrightarrow A^1(X_K, K_2)$ is an isomorphism,*
(4) *The natural homomorphism $\mathrm{CH}^1(Y_{K^s}) \to \mathrm{CH}^1(Y_K)$ is an isomorphism.*

Then the natural map $\mathrm{CH}^2(Y) \longrightarrow \mathrm{CH}^2(Y_{F(X)})$ is injective.

Remark 0.10. The conditions (1) – (3) hold for a smooth projective quadric X of dimension at least 7 and a projective homogeneous variety Y with the split Azumaya algebra by Remark 0.8 and Example 0.6.

References

[1] W. Fulton, *Intersection theory*, Springer-Verlag, Berlin, 1984.
[2] A. Grothendieck, *La théorie des classes de Chern*, Bull. Soc. Math. France **86** (1958), 137–154.

[3] N. A. Karpenko, *Algebro-geometric invariants of quadratic forms*, Algebra i Analiz **2** (1990), no. 1, 141–162.
[4] A. S. Merkurjev, *The group $H^1(X, K_2)$ for projective homogeneous varieties*, Algebra i Analiz **7** (1995), no. 3, 136–164.
[5] A. S. Merkurjev and J.-P. Tignol, *The multipliers of similitudes and the Brauer group of homogeneous varieties*, J. Reine Angew. Math. **461** (1995), 13–47.
[6] I. A. Panin, *On the algebraic K-theory of twisted flag varieties*, K-Theory **8** (1994), no. 6, 541–585.
[7] M. Rost, *Chow groups with coefficients*, Doc. Math. **1** (1996), No. 16, 319–393 (electronic).