COHOMOLOGICAL INVARIANTS OF ALGEBRAIC TORI

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Abstract. Let G be an algebraic group over a field F. As defined by Serre, a cohomological invariant of G of degree n with values in $\mathbb{Q}/\mathbb{Z}(j)$ is a functorial in K collection of maps of sets $\mathrm{Tors}_G(K) \longrightarrow H^n\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$ for all field extensions K/F, where $\mathrm{Tors}_G(K)$ is the set of isomorphism classes of G-torsors over $\operatorname{Spec} K$. We study the group of degree 3 invariants of an algebraic torus with values in $\mathbb{Q}/\mathbb{Z}(2)$. In particular, we compute the group $H^3_{\mathrm{nr}}\big(F(S), \mathbb{Q}/\mathbb{Z}(2)\big)$ of unramified cohomology of an algebraic torus S.

1. Introduction

Let G be a linear algebraic group over a field F (of arbitrary characteristic). The notion of an *invariant* of G was defined in [15] as follows. Consider the category Fields_F of field extensions of F and the functor

$$\mathrm{Tors}_G : \mathit{Fields}_F \longrightarrow \mathit{Sets}$$

taking a field K to the set $\mathrm{Tors}_G(K)$ of isomorphism classes of (right) G-torsors over $\operatorname{Spec} K$. Let

$$H : \mathit{Fields}_F \longrightarrow \mathit{Abelian\ Groups}$$

be another functor. An H*-invariant* of G is then a morphism of functors

$$i : \mathrm{Tors}_G \longrightarrow H,$$

viewing H with values in Sets, i.e., a functorial in K collection of maps of sets $\mathrm{Tors}_G(K) \longrightarrow H(K)$ for all field extensions K/F. We denote the group of H-invariants of G by $\mathrm{Inv}(G, H)$.

An invariant $i \in \mathrm{Inv}(G, H)$ is called *normalized* if $i(I) = 0$ for the trivial G-torsor I. The normalized invariants form a subgroup $\mathrm{Inv}(G, H)_{\mathrm{norm}}$ of $\mathrm{Inv}(G, H)$ and there is a natural isomorphism

$$\mathrm{Inv}(G, H) \simeq H(F) \oplus \mathrm{Inv}(G, H)_{\mathrm{norm}},$$

so it is sufficient to study normalized invariants.

Typically, H is a cohomological functor given by Galois cohomology groups with values in a fixed Galois module. Of particular interest to us is the functor H which takes a field K/F to the Galois cohomology group $H^n\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$, where the coefficients $\mathbb{Q}/\mathbb{Z}(j)$ are defined as follows. For a prime integer p different from $\mathrm{char}(F)$, the p-component $\mathbb{Q}_p/\mathbb{Z}_p(j)$ of $\mathbb{Q}/\mathbb{Z}(j)$ is the colimit

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over n of the étale sheaves $\mu_{p^n}^{\otimes j}$, where μ_m is the sheaf of m^{th} roots of unity. In the case $p = \mathrm{char}(F) > 0$, $\mathbb{Q}_p/\mathbb{Z}_p(j)$ is defined via logarithmic de Rham-Witt differentials (see §3b).

We write $\mathrm{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big)$ for the group of *cohomological invariants of G of degree n with values in* $\mathbb{Q}/\mathbb{Z}(j)$.

The second cohomology group $H^2\big(K, \mathbb{Q}/\mathbb{Z}(1)\big)$ is canonically isomorphic to the Brauer group $\mathrm{Br}(K)$ of the field K. In §2c we prove (Theorem 2.4) that if G is a connected group (reductive if F is not perfect), then $\mathrm{Inv}(G, \mathrm{Br})_{\mathrm{norm}} \simeq \mathrm{Pic}(G)$. The group $\mathrm{Inv}^3\big(G, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}}$ for a semisimple simply connected group G has been studied by Rost (see [15]).

An essential object in the study of cohomological invariants is the notion of a *classifying torsor*: a G-torsor $E \longrightarrow X$ for a smooth variety X over F such that every G-torsor over an infinite field K/F is isomorphic to the pull-back of $E \longrightarrow X$ along a K-point of X. If V is a generically free linear representation of G with a nonempty open subset $U \subset V$ such that there is a G-torsor $\pi : U \longrightarrow X$, then π is classifying. Such representations exist (see Section 2b).

The generic fiber of π is the *generic torsor* over $\operatorname{Spec} F(X)$ attached to π. Evaluation at the generic torsor yields a homomorphism

$$\mathrm{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big), \tag{1.1}$$

and in §3 we show that the image of this map is contained in the subgroup $H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)$ of $H^n\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big)$, where $\mathcal{H}^n\big(\mathbb{Q}/\mathbb{Z}(j)\big)$ is the Zariski sheaf associated to the presheaf $W \mapsto H^n\big(W, \mathbb{Q}/\mathbb{Z}(j)\big)$ of the étale cohomology groups. In fact, the image is contained in the subgroup $H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)_{\mathrm{bal}}$ of *balanced* elements, i.e., elements that have the same images under the pull-back homomorphisms with respect to the two projections $(U \times U)/G \longrightarrow X$. Moreover, the balanced elements precisely describe the image and we prove (Theorem 3.4):

Theorem A. *Let G be a smooth linear algebraic group over a field F. We assume that G is connected if F is a finite field. Let $E \longrightarrow X$ be a classifying G-torsor with E a G-rational variety such that $E(F) \neq \emptyset$. Then (1.1) yields an isomorphism* $\mathrm{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big) \simeq H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)_{\mathrm{bal}}$.

At this point it is convenient to make use of a construction due to Totaro [33]: because the Chow groups are homotopy invariant, the groups $\mathrm{CH}^n(X)$ do not depend on the choice of the representation V and the open set $U \subset V$ provided the codimension of $V \setminus U$ in V is large enough. This leads to the notation $\mathrm{CH}^n(BG)$, the Chow groups of the so-called *classifying space* BG, although BG itself is not defined in this paper.

Unfortunately, the étale cohomology groups with values in $\mathbb{Q}_p/\mathbb{Z}_p(j)$, where $p = \mathrm{char}(F) > 0$, are not homotopy invariant. In particular, we cannot use the theory of cycle modules of Rost [31].

The main result of this paper is the exact sequence in Theorem 4.3 describing degree 3 cohomological invariants of an algebraic torus T. Writing $\widehat{T}_{\mathrm{sep}}$ for the character lattice of T over a separable closure of F and T° for the dual torus, we prove our main result:

Theorem B. *Let T be an algebraic torus over a field F. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{CH}^2(BT)_{\mathrm{tors}} \longrightarrow H^1(F, T^0) \xrightarrow{\alpha} \mathrm{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} \longrightarrow H^0\big(F, S^2(\widehat{T}_{\mathrm{sep}})\big)/\,\mathrm{Dec} \longrightarrow H^2(F, T^0).$$

The homomorphism α is given by $\alpha(a)(b) = a_K \cup b$ for every $a \in H^1(F, T^0)$ and $b \in H^1(K, T)$ and every field extension K/F, where the cup-product is defined in (4.5), and Dec *is the subgroup of decomposable elements in the symmetric square $S^2(\widehat{T}_{\mathrm{sep}})$ defined in Appendix A-II.*

In the proof of the theorem we compute the group of balanced elements in the motivic cohomology group $H^4\big(BT, \mathbb{Z}(2)\big)$ and relate it, using an exact sequence of B. Kahn and Theorem A, with the group of invariants $\mathrm{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}}$.

We also prove that the torsion group $\mathrm{CH}^2(BT)_{\mathrm{tors}}$ is finite of exponent 2 (Theorem 4.7) and the last homomorphism in the sequence is also of exponent 2 (see the discussion before Theorem 4.13).

Moreover, if p is an odd prime, the group $\mathrm{Inv}^3\big(T, \mathbb{Q}_p/\mathbb{Z}_p(2)\big)_{\mathrm{norm}}$, which is the p-primary component of $\mathrm{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}}$, splits canonically into the direct sum of *linear* invariants (those that induce group homomorphisms from Tors_T to H^3) and *quadratic* invariants, i.e., the invariants i such that the function $h(a,b) := i(a+b) - i(a) - i(b)$ is bilinear and $h(a,a) = 2i(a)$ for all a and b. Furthermore, the groups of linear and quadratic invariants with values in $\mathbb{Q}_p/\mathbb{Z}_p(2)$ are canonically isomorphic to $H^1(F, T^\circ)\{p\}$ and $\big(H^0\big(F, S^2(\widehat{T}_{\mathrm{sep}})\big)/\,\mathrm{Dec}\big)\{p\}$, respectively.

We also prove (Theorem 4.10) that the degree 3 invariants have control over the structure of all invariants. Precisely, the group $\mathrm{Inv}^3(T_K, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ is trivial for all K/F if and only if T is *special*, i.e., T has no nontrivial torsors over any field K/F, which in particular means T has no nonconstant H-invariants for every functor H.

Our motivation for considering invariants of tori comes from their connection with unramified cohomology (defined in §5). Specifically, this work began as an investigation of a problem posed by Colliot-Thélène in [4, p. 39]: for n prime to $\mathrm{char}(F)$ and $i \geq 0$, determine the unramified cohomology group $H^i_{\mathrm{nr}}\big(F(S), \mu_n^{\otimes(i-1)}\big)$, where $F(S)$ is the function field of a torus S over F. The connection is provided by Theorem 5.7 where we show that the unramified cohomology of a torus S is calculated by the invariants of an auxiliary torus:

Theorem C. *Let S be a torus over F and let $1 \longrightarrow T \longrightarrow P \longrightarrow S \longrightarrow 1$ be a flasque resolution of S, i.e., T is flasque and P is quasi-split. Then there is a natural isomorphism*

$$H^n_{\mathrm{nr}}\big(F(S), \mathbb{Q}/\mathbb{Z}(j)\big) \simeq \mathrm{Inv}^n\big(T, \mathbb{Q}/\mathbb{Z}(j)\big).$$

By Theorems B and C, we have an exact sequence

$$0 \longrightarrow \mathrm{CH}^2(BT)_{\mathrm{tors}} \longrightarrow H^1(F,T^0) \xrightarrow{\alpha} \\ \overline{H}^3_{\mathrm{nr}}\big(F(S),\mathbb{Q}/\mathbb{Z}(2)\big) \longrightarrow H^0\big(F,\mathsf{S}^2(\widehat{T}_{\mathrm{sep}})\big)/\operatorname{Dec} \longrightarrow H^2(F,T^0)$$

describing the reduced third cohomology group

$$\overline{H}^3_{\mathrm{nr}}\big(F(S),\mathbb{Q}/\mathbb{Z}(2))\big) := H^3_{\mathrm{nr}}\big(F(S),\mathbb{Q}/\mathbb{Z}(2)\big)/H^3\big(F,\mathbb{Q}/\mathbb{Z}(2)\big).$$

Moreover, for an odd prime p, we have a canonical direct sum decomposition of the p-primary components:

$$\overline{H}^3_{\mathrm{nr}}\big(F(S),\mathbb{Q}_p/\mathbb{Z}_p(2)\big) = H^1(F,T^0)\{p\} \oplus \big(H^0\big(F,\mathsf{S}^2(\widehat{T}_{\mathrm{sep}})\big)/\operatorname{Dec}\big)\{p\}.$$

Note that the torus S determines T up to multiplication by a quasisplit torus. If X is a smooth compactification of S, one can take the torus T with $\widehat{T}_{\mathrm{sep}} = \operatorname{Pic}(X_{\mathrm{sep}})$ (see [9, §2]).

In the present paper, F denotes a field of arbitrary characteristic, F_{sep} a separable closure of F, and Γ the absolute Galois group $\operatorname{Gal}(F_{\mathrm{sep}}/F)$ of F.

The word "scheme" over a field F means a separated scheme over F and, following [14], a "variety" over F is an integral scheme of finite type over F. If X is a scheme over F and L/F is a field extension then we write X_L for $X \times_F \operatorname{Spec} L$. When $L = F_{\mathrm{sep}}$ we write simply X_{sep}.

A "linear algebraic group over F" is an affine group scheme of finite type over F, not necessarily smooth.

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2. Invariants of algebraic groups

2a. **Definitions and basic properties.** Let G be a linear algebraic group over a field F. Consider the functor

$$\operatorname{Tors}_G : \mathsf{Fields}_F \longrightarrow \mathsf{Sets}$$

from the category of field extensions of F to the category of sets taking a field K to the set $\operatorname{Tors}_G(K)$ of isomorphism classes of (right) G-torsors over $\operatorname{Spec} K$. Note that if G is a smooth group, then there is a natural bijection

$$\operatorname{Tors}_G(K) \simeq H^1(K,G) := H^1\big(\operatorname{Gal}(K_{\mathrm{sep}}/K), G(K_{\mathrm{sep}})\big).$$

Let $H : \mathsf{Fields}_F \longrightarrow \mathsf{Abelian\ Groups}$ be a functor. We also view H as a functor with values in Sets. Following [15], we define an *H-invariant of G* as a morphism of functors $\operatorname{Tors}_G \longrightarrow H$ from the category Fields_F to Sets. All the H-invariants of G form the abelian group of invariants $\operatorname{Inv}(G,H)$.

An invariant $i \in \operatorname{Inv}(G,H)$ is called *constant* if there is an element $h \in H(F)$ such that $i(I) = h_K$ for every G-torsor $I \longrightarrow \operatorname{Spec} K$, where h_K is the image of h under natural map $H(F) \longrightarrow H(K)$. The constant invariants form a subgroup $\operatorname{Inv}(G,H)_{\mathrm{const}}$ of $\operatorname{Inv}(G,H)$ isomorphic to $H(F)$. An invariant

$i \in \operatorname{Inv}(G,H)$ is called *normalized*, if $i(I) = 0$ for the trivial G-torsor I. The normalized invariants form a subgroup $\operatorname{Inv}(G,H)_{\mathrm{norm}}$ of $\operatorname{Inv}(G,H)$ and we have the decomposition

$$\operatorname{Inv}(G,H) = \operatorname{Inv}(G,H)_{\mathrm{const}} \oplus \operatorname{Inv}(G,H)_{\mathrm{norm}} \simeq H(F) \oplus \operatorname{Inv}(G,H)_{\mathrm{norm}},$$

so it suffices to determine the normalized invariants.

2b. **Classifying torsors.** Let G be a linear algebraic group over a field F. A G-torsor $E \longrightarrow X$ over a smooth variety X over F is called *classifying* if for every field extension K/F, with K infinite, and for every G-torsor $I \longrightarrow \operatorname{Spec} K$, there is a point $x : \operatorname{Spec} K \longrightarrow X$ such that the torsor I is isomorphic to the fiber $E(x)$ of $E \longrightarrow X$ over x, i.e., $I \simeq E(x) := x^*(E) = \operatorname{Spec}(K) \times_X E$. The generic fiber $E_{\mathrm{gen}} \longrightarrow \operatorname{Spec} F(X)$ of a classifying torsor is called a *generic G-torsor* (see [15, Part 1, §5.3]).

If V is a generically free linear representation of G with a nonempty open subset $U \subset V$ such that there is a G-torsor $\pi : U \longrightarrow X$, then π is classifying (see [15, Part 1, §5.4]). We will write U/G for X and call π a *standard classifying G-torsor*. Standard classifying G-torsors exist: we can embed G into $U := \mathbf{GL}_{n,F}$ for some n as a closed subgroup. Then U is an open subset in the affine space $M_n(F)$ on which G acts linearly and the canonical morphism $U \to X := U/G$ is a G-torsor. Note that $U(F) \neq \emptyset$.

We say that a G-variety Y is *G-rational* if there is an affine space V with a linear G-action such that Y and V have G-isomorphic nonempty open G-invariant subvarieties. Note that if $U \longrightarrow U/G$ is a standard classifying G-torsor, then U is a G-rational variety.

Let $E \longrightarrow X$ be a classifying G-torsor and let $H : \mathsf{Fields}_F \longrightarrow \mathsf{Abelian\ Groups}$ be a functor. Define the map

$$\theta_G : \operatorname{Inv}(G,H) \longrightarrow H\big(F(X)\big) \qquad i \longmapsto i(E_{\mathrm{gen}}), \tag{2.1}$$

by sending an invariant to its value at the generic torsor E_{gen}.

Consider the following property of the functor H:

Property 2.1. *The map $H(K) \longrightarrow H\big(K((t))\big)$ is injective for any field extension K/F.*

The following theorem, due to M. Rost, was proved in [15, Part II, Th. 3.3]. For completeness, we give a slightly modified proof in Appendix A-I.

Theorem 2.2. *Let G be a smooth linear algebraic group over F. If a functor $H : \mathsf{Fields}_F \longrightarrow \mathsf{Abelian\ Groups}$ has Property 2.1, then the map θ_G is injective, i.e., every H-invariant of G is determined by its value at the generic G-torsor.*

Let G' be a (closed) subgroup of G over F. The map of sets $H^1(K,G') \to H^1(K,G)$ for every field extension K/F yields the *restriction map*

$$\operatorname{res} : \operatorname{Inv}(G,H) \to \operatorname{Inv}(G',H).$$

Choose standard torsors $\pi : U \to U/G$ and $\pi' : U \to U/G'$ (for example, with $U = \mathbf{GL}_{n,F}$ as above). The pull-back of π with respect to the natural morphism $\alpha : U/G' \to U/G$ is the push-forward of π' via the inclusion $G' \hookrightarrow G$. It follows that the diagram

$$\begin{tikzcd} \mathrm{Inv}(G,H) \arrow[r, "\mathrm{res}"] \arrow[d, "\theta_G"'] & \mathrm{Inv}(G',H) \arrow[d, "\theta_{G'}"'] \\ H\big(F(U/G)\big) \arrow[r, "\alpha^*"] & H\big(F(U/G')\big) \end{tikzcd}$$

is commutative.

2c. **The Brauer group invariants.** Let G be a smooth connected linear algebraic group over F. Every cohomological invariant of G of degree 1 is constant by [24, Prop. 31.15]. In this section we study (degree 2) Br-invariants for the Brauer group functor $K \longmapsto \mathrm{Br}(K)$. We assume that G is reductive if $\mathrm{char}(F) > 0$.

Lemma 2.3. *For any field extension K/F such that F is algebraically closed in K, the natural map $\mathrm{Pic}(G) \longrightarrow \mathrm{Pic}(G_K)$ is an isomorphism.*

Proof. We may assume that G is reductive by factoring out the unipotent radical in the case that F is perfect. There is an exact sequence (see [5, Th. 1.2])

$$1 \longrightarrow C \longrightarrow G' \longrightarrow G \longrightarrow 1$$

with C a torus and G' a reductive group with $\mathrm{Pic}(G'_L) = 0$ for any field extension L/F. Let T be the factor group of G' by the semisimple part. The result follows from the exact sequence [32, Prop. 6.10] (note that G is reductive if L is not perfect)

$$\widehat{T}(L) \longrightarrow \widehat{C}(L) \longrightarrow \mathrm{Pic}(G_L) \longrightarrow \mathrm{Pic}(G'_L) = 0$$

with $L = F$ and K since the groups $\widehat{T}(F)$ and $\widehat{C}(F)$ don't change when F is replaced by K. □

Since for any G_K-torsor $E \longrightarrow \mathrm{Spec}(K)$ over a field extension K/F one has [32, Prop. 6.10] the exact sequence

$$\mathrm{Pic}(E) \longrightarrow \mathrm{Pic}(G_K) \xrightarrow{\delta} \mathrm{Br}(K) \xrightarrow{\varepsilon} \mathrm{Br}(E), \tag{2.2}$$

we obtain the homomorphism

$$\nu : \mathrm{Pic}(G) \longrightarrow \mathrm{Inv}(G, \mathrm{Br})$$

which takes an element $\alpha \in \mathrm{Pic}(G)$ to the invariant that sends a G-torsor E over a field extension K/F to $\delta(\alpha_K)$. If E is a trivial torsor, i.e., $E(K) \neq \emptyset$, then ε is injective and hence $\delta = 0$. It follows that the invariant $\nu(\alpha)$ is normalized.

Theorem 2.4. *Let G be a smooth connected linear algebraic group over F. Assume that G is reductive if* $\mathrm{char}(F) > 0$. *Then the map $\nu : \mathrm{Pic}(G) \longrightarrow \mathrm{Inv}(G, \mathrm{Br})_{\mathrm{norm}}$ is an isomorphism.*

Proof. Choose a standard classifying G-torsor $U \longrightarrow U/G$. Write K for the function field $F(U/G)$ and let U_{gen} be the generic G-torsor over K. Consider the commutative diagram

$$\begin{tikzcd} & \mathrm{Pic}(G) \arrow[r, "\nu"] \arrow[d, "j"'] & \mathrm{Inv}(G, \mathrm{Br})_{\mathrm{norm}} \arrow[d, "\theta_G"'] & \\ \mathrm{Pic}(U_{\mathrm{gen}}) \arrow[r] & \mathrm{Pic}(G_K) \arrow[r, "\delta"] & \mathrm{Br}(K) \arrow[r, "i"] & \mathrm{Br}(K(U_{\mathrm{gen}})), \end{tikzcd}$$

where the bottom sequence is (2.2) for the G-torsor $U_{\mathrm{gen}} \longrightarrow \mathrm{Spec}(K)$ followed by the injection $\mathrm{Br}(U_{\mathrm{gen}}) \longrightarrow \mathrm{Br}\big(K(U_{\mathrm{gen}})\big)$ (see [29, Ch. IV, Cor. 2.6]), and the map θ_G is evaluation at the generic torsor U_{gen} given in (2.1) and is injective by Theorem 2.2. Since the generic torsor is split over $K(U_{\mathrm{gen}})$, $\mathrm{Im}(\theta_G) \subset \mathrm{Ker}(i) = \mathrm{Im}(\delta)$. By Lemma 2.3, j is an isomorphism, hence ν is surjective.

Note that U_{gen} is a localization of U, hence $\mathrm{Pic}(U_{\mathrm{gen}}) = 0$ as $\mathrm{Pic}(U) = 0$. It follows that ν is injective. □

An algebraic group G over a field F is called *special* if $H^1(K, G) = \{1\}$ for every field extension K/F, i.e., all G-torsors over any field extension of F are trivial.

Corollary 2.5. *If the group G is special, then* $\mathrm{Pic}(G) = 0$.

3. Invariants with values in $\mathbb{Q}/\mathbb{Z}(j)$

In this section we find a description for the group of cohomological invariants with values in $\mathbb{Q}/\mathbb{Z}(j)$ by identifying the image of the embedding θ_G in (2.1).

Let G be a linear algebraic group over a field F, let $H \subset G$ be a subgroup and let $E \longrightarrow X$ be a G-torsor. Suppose that G/H is affine. Consider a G-action on $E \times (G/H)$ by $(e, g'H)g = (eg, g^{-1}g'H)$. By [29, Th. I.2.23], the affine G-equivariant projection $E \times (G/H) \to E$ descends to an affine morphism $Y \to X$. The (trivial right) H-torsor $E \times G \to E \times (G/H)$ descends to an H-torsor $E \to Y$. We will write E/H for Y.

Example 3.1. Let G be a linear algebraic group over a field F and let $E \longrightarrow X$ be a G-torsor. Then for every $n > 0$, $E^n := E \times_F \cdots \times_F E$ (n times) is a G^n-torsor over X^n. Viewing G as the diagonal subgroup of G^n, we have the G-torsor $E^n \to E^n/G$.

3a. **Balanced elements.** Let G be a linear algebraic group over a field F. We assume that G is connected if F is finite. Let $E \longrightarrow X$ be a G-torsor such that $E(F) \neq \emptyset$. We write p_1 and p_2 for the two projections $E^2/G = (E \times_F E)/G \longrightarrow X$ (see Example 3.1).

Lemma 3.2. *Let K/F be a field extension and $x_1, x_2 \in X(K)$. Then the G-torsors $E(x_1)$ and $E(x_2)$ over K are isomorphic if and only if there is a point $y \in (E^2/G)(K)$ such that $p_1(y) = x_1$ and $p_2(y) = x_2$.*

Proof. "$\Rightarrow$": By construction, we have G-equivariant morphisms $f_i : E(x_i) \longrightarrow E$ for $i = 1, 2$. Choose an isomorphism $h : E(x_1) \xrightarrow{\sim} E(x_2)$ of G-torsors over K. The morphism $(f_1, f_2 h) : E(x_1) \longrightarrow E^2$ yields the required point $\operatorname{Spec} K = E(x_1)/G \longrightarrow E^2/G$.

"$\Leftarrow$": The pull-back of $E \longrightarrow X$ with respect to any projection $E^2/G \longrightarrow X$ coincides with the G-torsor $E^2 \longrightarrow E^2/G$, hence

$$E(x_1) = x_1^*(E) = y^* p_1^*(E) \simeq y^*(E^2) \simeq y^* p_2^*(E) = x_2^*(E) = E(x_2). \qquad \square$$

Let H be a (contravariant) functor from the category of schemes over F to the category of abelian groups. We have the two maps $p_i^* : H(X) \longrightarrow H(E^2/G)$, $i = 1, 2$. An element $h \in H(X)$ is called *balanced* if $p_1^*(h) = p_2^*(h)$. We write $H(X)_{\mathrm{bal}}$ for the subgroup of balanced elements in $H(X)$. In other words, $H(X)_{\mathrm{bal}} = h_0\big(H(E^\bullet/G)\big)$ in the notation of Appendix A-IV.

We can view H as a (covariant) functor $\mathit{Fields}_F \longrightarrow \mathit{Sets}$ taking a field K to $H(K) := H\big(\operatorname{Spec} K\big)$.

Lemma 3.3. *Let $h \in H(X)_{\mathrm{bal}}$ be a balanced element, K/F a field extension and I a G-torsor over* $\operatorname{Spec}(K)$. *Let $x \in X(K)$ be a point such that $E(x) \simeq I$. Then the element $x^*(h)$ in $H(K)$ does not depend on the choice of x.*

Proof. Let $x_1, x_2 \in X(K)$ be two points such that $E(x_1) \simeq E(x_2)$. By Lemma 3.2, there is a point $y \in (E^2/G)(K)$ such that $p_1(y) = x_1$ and $p_2(y) = x_2$. Therefore

$$x_1^*(h) = y^*\big(p_1^*(h)\big) = y^*\big(p_2^*(h)\big) = x_2^*(h). \qquad \square$$

It follows from Lemma 3.3 that if the torsor $E \longrightarrow X$ is classifying with $E(F) \neq \emptyset$, then every element $h \in H(X)_{\mathrm{bal}}$ determines an H-invariant i_h of G as follows. Let I be a G-torsor over a field extension K/F. We claim that there is a point $x \in X(K)$ such that $E(x) \simeq I$. If K is infinite, this follows from the definition of the classifying G-torsor. If K is finite then all G-torsors over K are trivial by [25], as G is connected. Since $E(K) \neq \emptyset$, we can take for x the image in $X(K)$ of any point in $E(K)$. Defining $i_h(E) = x^*(h) \in H(K)$, we have a group homomorphism

$$H(X)_{\mathrm{bal}} \longrightarrow \operatorname{Inv}(G, H), \quad h \mapsto i_h.$$

3b. **Cohomology with values in $\mathbb{Q}/\mathbb{Z}(j)$.** For every integer $j \geq 0$, the coefficients $\mathbb{Q}/\mathbb{Z}(j)$ are defined as the direct sum over all prime integers p of the objects $\mathbb{Q}_p/\mathbb{Z}_p(j)$ in the derived category of sheaves of abelian groups on the big étale site of $\operatorname{Spec} F$, where

$$\mathbb{Q}_p/\mathbb{Z}_p(j) = \operatorname*{colim}_n \mu_{p^n}^{\otimes j}$$

if $p \neq \operatorname{char} F$, with μ_{p^n} the sheaf of $(p^n)^{\text{th}}$ roots of unity, and

$$\mathbb{Q}_p/\mathbb{Z}_p(j) = \operatorname*{colim}_n W_n\Omega^j_{log}[-j]$$

if $p = \operatorname{char} F > 0$, with $W_n\Omega^j_{log}$ the sheaf of *logarithmic de Rham-Witt differentials* (see [20, I.5.7], [23]).

We write $H^m\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)$ for the étale cohomology of a scheme X with values in $\mathbb{Q}/\mathbb{Z}(j)$. Then

$$H^m\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)\{p\} = \operatorname*{colim}_n H^m\big(X, \mu_{p^n}^{\otimes j}\big)$$

if $p \neq \operatorname{char} F$ and

$$H^m\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)\{p\} = \operatorname*{colim}_n H^{m-j}\big(X, W_n\Omega^j_{log}\big)$$

if $p = \operatorname{char} F > 0$. In the latter case, the group $W_n\Omega^j_{log}(F)$ is canonically isomorphic to $K^M_j(F)/p^nK^M_j(F)$, where $K^M_j(F)$ is Milnor's K-group of F (see [1, Cor. 2.8]), hence by [21] and [15, Part II, Appendix A], $H^s\big(F, W_n\Omega^j_{log}\big)$ is isomorphic to

$$H^s\big(F, K^M_j(F_{\text{sep}})/p^nK^M_j(F_{\text{sep}})\big) = \begin{cases} K^M_j(F)/p^nK^M_j(F), & \text{if } s = 0; \\ H^2\big(F, K^M_j(F_{\text{sep}})\big)_{p^n}, & \text{if } s = 1; \\ 0, & \text{otherwise.} \end{cases}$$

It follows that in the case $p = \operatorname{char} F > 0$, we have

$$H^m\big(F, \mathbb{Q}/\mathbb{Z}(j)\big)\{p\} = \begin{cases} K^M_j(F) \otimes (\mathbb{Q}_p/\mathbb{Z}_p), & \text{if } m = j; \\ H^2\big(F, K^M_j(F_{\text{sep}})\big)\{p\}, & \text{if } m = j+1; \\ 0, & \text{otherwise.} \end{cases}$$

The *motivic complexes* $\mathbb{Z}(j)$, for $j = 0, 1, 2$, of étale sheaves on a smooth scheme X were defined in [26] and [27] by S. Lichtenbaum. We write $H^*\big(X, \mathbb{Z}(j)\big)$ for the étale (hyper)cohomology groups of X with values in $\mathbb{Z}(j)$.

The complex $\mathbb{Z}(0)$ is equal to the constant sheaf $\mathbb{Z}$ and $\mathbb{Z}(1) = \mathbb{G}_{m,X}[-1]$, thus $H^n\big(X, \mathbb{Z}(1)\big) = H^{n-1}(X, \mathbb{G}_{m,X})$. In particular, $H^3\big(X, \mathbb{Z}(1)\big) = \operatorname{Br}(X)$, the cohomological Brauer group of X. The complex $\mathbb{Z}(2)$ is concentrated in degrees 1 and 2 and there is a product map $\mathbb{Z}(1) \otimes^L \mathbb{Z}(1) \longrightarrow \mathbb{Z}(2)$ (see [26, Proposition 2.5]).

The exact triangle in the derived category of étale sheaves

$$\mathbb{Z}(j) \longrightarrow \mathbb{Q} \otimes \mathbb{Z}(j) \longrightarrow \mathbb{Q}/\mathbb{Z}(j) \longrightarrow \mathbb{Z}(j)[1]$$

yields the connecting homomorphism

$$H^i\big(X, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^{i+1}\big(X, \mathbb{Z}(j)\big),$$

which is an isomorphism if $X = \operatorname{Spec}(F)$ for a field F and $i > j$ [22, Lemme 1.1].

Write $\mathcal{H}^n\big(\mathbb{Q}/\mathbb{Z}(j)\big)$ for the Zariski sheaf on a smooth scheme X associated to the presheaf $U \mapsto H^n\big(U, \mathbb{Q}/\mathbb{Z}(j)\big)$ of étale cohomology groups.

Let G be a linear algebraic group over F. We assume that G is connected if F is a finite field and write $\operatorname{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big)$ for the group of *degree n invariants* of G for the functor $K \mapsto H^n\big(K, \mathbb{Q}/\mathbb{Z}(j)\big)$. Note that Property 2.1 holds for this functor by [15, Part 2, Prop. A.9].

Choose a classifying G-torsor $E \longrightarrow X$ with E a G-rational variety such that $E(F) \neq \emptyset$. Applying the construction given in §3a to the functor $U \mapsto H^0_{\mathrm{Zar}}\big(U, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)$, we get a homomorphism

$$\varphi : H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)_{\mathrm{bal}} \longrightarrow \operatorname{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big).$$

Theorem 3.4. *Let G be a smooth linear algebraic group over a field F. We assume that G is connected if F is a finite field. Let $E \longrightarrow X$ be a classifying G-torsor with E a G-rational variety such that $E(F) \neq \emptyset$. Then the homomorphism φ is an isomorphism.*

Proof. Let $E_{\mathrm{gen}} \longrightarrow F(X)$ be the generic fiber of the classifying G-torsor $E \longrightarrow X$. Note that since the pull-back of $E \longrightarrow X$ with respect to any of the two projections $E^2/G \longrightarrow X$ coincides with the G-torsor $E^2 \longrightarrow E^2/G$, the pull-backs of the generic G-torsor $E_{\mathrm{gen}} \longrightarrow \operatorname{Spec} F(X)$ with respect to the two morphisms $\operatorname{Spec} F(E^2/G) \longrightarrow \operatorname{Spec} F(X)$ induced by the projections are isomorphic. It follows that for every invariant $i \in \operatorname{Inv}\big(G, H^*(\mathbb{Q}/\mathbb{Z}(j))\big)$ we have

$$p_1^*\big(i(E_{\mathrm{gen}})\big) = i\big(p_1^*(E_{\mathrm{gen}})\big) = i\big(p_2^*(E_{\mathrm{gen}})\big) = p_2^*\big(i(E_{\mathrm{gen}})\big)$$

in $H^*\big(F(E^2/G), \mathbb{Q}/\mathbb{Z}(j)\big)$, i.e., $i(E_{\mathrm{gen}}) \in H^*\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big)_{\mathrm{bal}}$. By Proposition A.9, $\partial_x(h) = 0$ for every point $x \in X$ of codimension 1, hence

$$\theta_G(i) = i(E_{\mathrm{gen}}) \in H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)_{\mathrm{bal}}$$

by Proposition A.10. By Theorem 2.2, θ_G is injective and by construction, the composition $\theta_G \circ \varphi$ is the identity. It follows that φ is an isomorphism. □

Write $\overline{H}^0_{\mathrm{Zar}}\big(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)$ for the factor group of $H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)$ by the natural image of $H^n\big(F, \mathbb{Q}/\mathbb{Z}(j)\big)$.

Corollary 3.5. *The isomorphism φ yields an isomorphism*

$$\overline{H}^0_{\mathrm{Zar}}\big(X, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))\big)_{\mathrm{bal}} \xrightarrow{\sim} \operatorname{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big)_{\mathrm{norm}}.$$

4. Degree 3 invariants of algebraic tori

In this section we prove the main theorem that describes degree 3 invariants of an algebraic torus with values in $\mathbb{Q}/\mathbb{Z}(2)$.

4a. **Algebraic tori.** Let F be a field and $\Gamma = \operatorname{Gal}(F_{\mathrm{sep}}/F)$ the absolute Galois group of F. An *algebraic torus* of dimension n over F is an algebraic group T such that T_{sep} is isomorphic to the product of n copies of the multiplicative group $\mathbb{G}_m$ (see [9] or [35]). For an algebraic torus T over a field F, we write $\widehat{T}_{\mathrm{sep}}$ for the Γ-module of characters $\operatorname{Hom}(T_{\mathrm{sep}}, \mathbb{G}_m)$. The group $\widehat{T}_{\mathrm{sep}}$ is a Γ-*lattice*, i.e., a free abelian group of finite rank with a continuous Γ-action. The contravariant functor $T \mapsto \widehat{T}_{\mathrm{sep}}$ is an anti-equivalence between the category of algebraic tori and the category of Γ-lattices: the torus T and the group $T(F)$ can be reconstructed from the lattice $\widehat{T}_{\mathrm{sep}}$ by the formulas

$$T = \operatorname{Spec}\bigl(F_{\mathrm{sep}}[\widehat{T}_{\mathrm{sep}}]^{\Gamma}\bigr),$$
$$T(F) = \operatorname{Hom}_{\Gamma}(\widehat{T}_{\mathrm{sep}}, F_{\mathrm{sep}}^{\times}) = \bigl(\widehat{T}_{\mathrm{sep}}^{\circ} \otimes F_{\mathrm{sep}}^{\times}\bigr)^{\Gamma},$$

where $\widehat{T}_{\mathrm{sep}}^{\circ} = \operatorname{Hom}(\widehat{T}_{\mathrm{sep}}, \mathbb{Z})$.

We write $\widehat{T}$ for the character group $\operatorname{Hom}_F(T, \mathbb{G}_m) = (\widehat{T}_{\mathrm{sep}})^{\Gamma}$ and T° for the *dual torus* having character lattice $\widehat{T}_{\mathrm{sep}}^{\circ}$.

A torus T is called *quasi-split* if T is isomorphic to the group of invertible elements of an étale F-algebra, or equivalently, the Γ-lattice $\widehat{T}_{\mathrm{sep}}$ is permutation, i.e., $\widehat{T}_{\mathrm{sep}}$ has a Γ-invariant $\mathbb{Z}$-basis. An *invertible* torus is a direct factor of a quasi-split torus.

A torus T is called *flasque* (respectively, *coflasque*) if $H^1(L, \widehat{T}_{\mathrm{sep}}^{\circ}) = 0$ (respectively, $H^1(L, \widehat{T}_{\mathrm{sep}}) = 0$) for every finite field extension L/F. A *flasque resolution* of a torus S is an exact sequence of tori $1 \longrightarrow T \longrightarrow P \longrightarrow S \longrightarrow 1$ with T flasque and P quasi-split. By [9, §4], or [35, §4.7], the torus T in the flasque resolution is invertible if and only if S is a direct factor of a rational torus.

4b. **Products.** Let T be a torus over F and let $\widehat{T}(i)$ denote the complex $\widehat{T}_{\mathrm{sep}} \otimes \mathbb{Z}(i)$ of étale sheaves over F for $i = 0, 1, 2$. Thus, $\widehat{T}(0) = \widehat{T}_{\mathrm{sep}}$ and $\widehat{T}(1) = \bigl(\widehat{T}_{\mathrm{sep}} \otimes F_{\mathrm{sep}}^{\times}\bigr)[-1] = T^{\circ}(F_{\mathrm{sep}})[-1]$.

Let S and T be algebraic tori over F and let i and j be nonnegative integers with $i + j \leq 2$. For any smooth variety X over F, we have the product map

$$(4.1)\quad (\widehat{S}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})^{\Gamma} \otimes H^p\bigl(X, \widehat{S}^{\circ}(i)\bigr) \otimes H^q\bigl(X, \widehat{T}^{\circ}(j)\bigr) \longrightarrow H^{p+q}\bigl(X, \mathbb{Z}(i+j)\bigr)$$

taking $a \otimes b \otimes c$ to $a \cup b \cup c$, via the canonical pairings between $\widehat{S}_{\mathrm{sep}}$ and $\widehat{S}_{\mathrm{sep}}^{\circ}$, $\widehat{T}_{\mathrm{sep}}$ and $\widehat{T}_{\mathrm{sep}}^{\circ}$, and the product map $\mathbb{Z}(i) \otimes^L \mathbb{Z}(j) \longrightarrow \mathbb{Z}(i+j)$.

Recall that there is an isomorphism $H^n\bigl(F, \mathbb{Z}(k)\bigr) \simeq H^{n-1}\bigl(F, \mathbb{Q}/\mathbb{Z}(k)\bigr)$ for $n > k$. In particular, we have the cup-product map

$$(4.2)\qquad (\widehat{S}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})^{\Gamma} \otimes H^p(F, S) \otimes H^q(F, T) \longrightarrow H^3\bigl(F, \mathbb{Q}/\mathbb{Z}(2)\bigr)$$

if $p + q = 2$.

If $S = T^{\circ}$ is the dual torus, then $(\widehat{S}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})^{\Gamma} = \operatorname{End}_{\Gamma}(\widehat{T}_{\mathrm{sep}})$ contains the canonical element 1_T. We then have the product map

$$(4.3)\qquad H^p\bigl(X, \widehat{T}(i)\bigr) \otimes H^q\bigl(X, \widehat{T}^{\circ}(j)\bigr) \longrightarrow H^{p+q}\bigl(X, \mathbb{Z}(i+j)\bigr)$$

and in particular, the product maps

$$(4.4)\qquad H^1(F, \widehat{T}_{\mathrm{sep}}) \otimes H^1(F, T) \longrightarrow H^2\bigl(F, \mathbb{Q}/\mathbb{Z}(1)\bigr) = \operatorname{Br}(F),$$

$$(4.5)\qquad H^1(F, T^{\circ}) \otimes H^1(F, T) \longrightarrow H^3\bigl(F, \mathbb{Q}/\mathbb{Z}(2)\bigr),$$

$$H^2(F,T^\circ)\otimes H^0(F,T)\longrightarrow H^3\big(F,\mathbb{Q}/\mathbb{Z}(2)\big),\tag{4.6}$$

taking $a\otimes b$ to $1_T\cup a\cup b$ and applying (4.2).

As T is a commutative group, the set $H^1(K,T)$ is an abelian group. An invariant $i\in\operatorname{Inv}(T,H)$ for a functor H is called *linear* if $i_K: H^1(K,T)\longrightarrow H(K)$ is a group homomorphism for every K/F. In the next section we will see that a normalized degree 3 invariant of a torus need not be linear.

4c. **Main theorem.** Let T be a torus over F and choose a standard classifying T-torsor $U\longrightarrow U/T$ such that the codimension of $V\setminus U$ in V is at least 3. Such a torsor exists by [12, Lemma 9].

By [32, Prop. 6.10], there is an exact sequence

$$F_{\mathrm{sep}}[U]^\times/F_{\mathrm{sep}}^\times\longrightarrow\widehat{T}_{\mathrm{sep}}\longrightarrow\operatorname{Pic}\big((U/T)_{\mathrm{sep}}\big)\longrightarrow\operatorname{Pic}(U_{\mathrm{sep}}).$$

The codimension assumption implies that the side terms are trivial, hence the map $\widehat{T}_{\mathrm{sep}}\longrightarrow\operatorname{Pic}\big((U/T)_{\mathrm{sep}}\big)$ is an isomorphism. It follows that the classifying T-torsor $U\longrightarrow U/T$ is universal in the sense of [11].

Write $K_*(F)$ for the (Quillen) K-groups of F and $\mathcal{K}_*$ for the Zariski sheaf associated to the presheaf $U\mapsto K_*(U)$. Then the groups $H^n_{\mathrm{Zar}}\big(U/T,\mathcal{K}_2\big)$ are independent of the choice of the classifying torsor (cf. [12]). So we write $H^n_{\mathrm{Zar}}\big(BT,\mathcal{K}_2\big)$ for this group (see Appendix A-IV). As T_{sep} is a split torus, by the Künneth formula (see Example A.5),

$$H^n_{\mathrm{Zar}}\big(BT_{\mathrm{sep}},\mathcal{K}_2\big)=\begin{cases}K_2(F_{\mathrm{sep}}), & \text{if } n=0;\\ \operatorname{Pic}\big((U/T)_{\mathrm{sep}}\big)\otimes F_{\mathrm{sep}}^\times=\widehat{T}_{\mathrm{sep}}\otimes F_{\mathrm{sep}}^\times=T^\circ(F_{\mathrm{sep}}), & \text{if } n=1;\\ \mathrm{CH}^2\big((U/T)_{\mathrm{sep}}\big)=S^2(\widehat{T}_{\mathrm{sep}}), & \text{if } n=2.\end{cases}$$

Applying the calculation of the $\mathcal{K}$-cohomology groups to the standard classifying T-torsor $U^i\longrightarrow U^i/T$ for every $i>0$ instead of $U\longrightarrow U/T$, by Proposition B.3, we have the exact sequence

(4.7)

$$0\longrightarrow H^1(F,T^\circ)\xrightarrow{\alpha}\overline{H}^4\big(U^i/T,\mathbb{Z}(2)\big)\longrightarrow\overline{H}^4\big((U^i/T)_{\mathrm{sep}},\mathbb{Z}(2)\big)^\Gamma\longrightarrow H^2(F,T^\circ),$$

where $\overline{H}^4\big(U^i/T,\mathbb{Z}(2)\big)$ is the factor group of $H^4\big(U^i/T,\mathbb{Z}(2)\big)$ by $H^4\big(F,\mathbb{Z}(2)\big)$, the map α is given by $\alpha(a)=q^*(a)\cup[U^i]$ with $q:U^i/T\longrightarrow\operatorname{Spec}F$ the structure morphism, $[U^i]$ the class of the T-torsor $U^i\longrightarrow U^i/T$ in $H^1(U^i/T,T)$, and the cup-product is taken for the pairing (B.6).

Taking the sequences (4.7) for all i (see Section A-IV), we get the exact sequence of cosimplicial groups

$$0\longrightarrow H^1(F,T^\circ)\xrightarrow{\alpha}\overline{H}^4\big(U^\bullet/T,\mathbb{Z}(2)\big)\longrightarrow\overline{H}^4\big((U^\bullet/T)_{\mathrm{sep}},\mathbb{Z}(2)\big)^\Gamma\longrightarrow H^2\big(F,T^\circ\big).$$

The first and the last cosimplicial groups in the sequence are constant, hence by Lemma A.2, the sequence

$$0\longrightarrow H^1(F,T^\circ)\xrightarrow{\alpha}\overline{H}^4\big(U/T,\mathbb{Z}(2)\big)_{\mathrm{bal}}\longrightarrow \overline{H}^4\big((U/T)_{\mathrm{sep}},\mathbb{Z}(2)\big)^\Gamma_{\mathrm{bal}}\longrightarrow H^2(F,T^\circ)\tag{4.8}$$

is exact as

$$h_0\big(\overline{H}^4\big(U^\bullet/T,\mathbb{Z}(2)\big)\big) = H^4\big(U/T,\mathbb{Z}(2)\big)_{\mathrm{bal}}.$$

The following theorem was proved by B. Kahn in [23, Th. 1.1]:

Theorem 4.1. *Let X be a smooth variety over F. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{CH}^2(X) \longrightarrow H^4\big(X,\mathbb{Z}(2)\big) \longrightarrow H^0_{\mathrm{Zar}}\big(X,\mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))\big) \longrightarrow 0.$$

By Theorem 4.1, there is an exact sequence of cosimplicial groups

$$0 \longrightarrow \mathrm{CH}^2(U^\bullet/T) \longrightarrow \overline{H}^4\big(U^\bullet/T,\mathbb{Z}(2)\big) \longrightarrow \overline{H}^0_{\mathrm{Zar}}\big(U^\bullet/T,\mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))\big) \longrightarrow 0.$$

As the functor CH^2 is homotopy invariant, by Lemma A.4, the first group in the sequence is constant. In view of Lemma A.2, and following the notation for the $\mathcal{K}$-cohomology, the sequence

(4.9)

$$0 \longrightarrow \mathrm{CH}^2(BT) \longrightarrow \overline{H}^4\big(U/T,\mathbb{Z}(2)\big)_{\mathrm{bal}} \longrightarrow \overline{H}^0_{\mathrm{Zar}}\big(U/T,\mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))\big)_{\mathrm{bal}} \longrightarrow 0$$

is exact. By Corollary 3.5, the last group in the sequence is canonically isomorphic to $\mathrm{Inv}\big(T,H^3(\mathbb{Q}/\mathbb{Z}(2))\big)_{\mathrm{norm}}$.

As the torus T_{sep} is split, all the invariants of T_{sep} are trivial hence the sequence (4.9) over F_{sep} yields an isomorphism

$$\overline{H}^4\big((U/T)_{\mathrm{sep}},\mathbb{Z}(2)\big)_{\mathrm{bal}} \simeq \mathrm{CH}^2(BT_{\mathrm{sep}}) \simeq S^2(\widehat{T}_{\mathrm{sep}}). \tag{4.10}$$

Combining (4.8), (4.9) and (4.10), we get the following diagram with an exact row and column:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & H^1(F,T^0) \arrow[d, "\alpha"'] \arrow[dr] & & \\
0 \arrow[r] & \mathrm{CH}^2(BT) \arrow[r] \arrow[dr] & \overline{H}^4\big(U/T,\mathbb{Z}(2)\big)_{\mathrm{bal}} \arrow[r] \arrow[d] & \mathrm{Inv}^3\big(T,\mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} \arrow[r] & 0. \\
 & & S^2(\widehat{T}_{\mathrm{sep}})^\Gamma \arrow[d] & & \\
 & & H^2(F,T^0) & &
\end{tikzcd}$$

Write $\mathrm{Dec} = \mathrm{Dec}(\widehat{T}_{\mathrm{sep}})$ for the subgroup of *decomposable elements* in $S^2(\widehat{T}_{\mathrm{sep}})^\Gamma$ (see Appendix A-II).

Lemma 4.2. *The image of the homomorphism* $\mathrm{CH}^2(BT) \longrightarrow \mathrm{CH}^2(BT_{\mathrm{sep}})^\Gamma = S^2(\widehat{T}_{\mathrm{sep}})^\Gamma$ *in the diagram coincides with* Dec.

Proof. Consider the Grothendieck ring $K_0(BT)$ of the category of T-equivariant vector bundles over $\mathrm{Spec}(F)$, or equivalently, of the category of finite dimensional linear representations of T. If T is split, every linear representation of T is a direct sum of one-dimensional representations. Therefore, there is an isomorphism between the group ring $\mathbb{Z}[\widehat{T}]$ of all formal finite sums $\sum_{x\in\widehat{T}} a_x e^x$ and $K_0(BT)$, taking e^x with $x \in \widehat{T}$ to the class of the 1-dimensional representation given by x. In general, for every torus T, we have $K_0(BT_{\mathrm{sep}}) = \mathbb{Z}[\widehat{T}_{\mathrm{sep}}]$ and $K_0(BT) = \mathbb{Z}[\widehat{T}_{\mathrm{sep}}]^\Gamma = K_0(BT_{\mathrm{sep}})^\Gamma$ (see [28, page 136]). The group $\mathbb{Z}[\widehat{T}_{\mathrm{sep}}]^\Gamma$ is generated by the sums $\sum_{i=1}^n e^{\gamma_i x}$, where $\gamma_1, \gamma_2, \dots, \gamma_n$ are representatives of the left cosets of an arbitrary open subgroup Γ' in Γ and $x \in (\widehat{T}_{\mathrm{sep}})^{\Gamma'}$.

The equivariant Chern classes were defined in [12, §2.4]. The first Chern class $c_1 : K_0(BT_{\mathrm{sep}}) \longrightarrow \mathrm{CH}^1(BT_{\mathrm{sep}}) = \widehat{T}_{\mathrm{sep}}$ takes e^x to x. In the diagram

$$\begin{tikzcd}
\mathbb{Z}[\widehat{T}_{\mathrm{sep}}]^\Gamma \arrow[r, equal] \arrow[d] & K_0(BT) \arrow[r, "c_2"] \arrow[d] & \mathrm{CH}^2(BT) \arrow[d] & \\
\mathbb{Z}[\widehat{T}_{\mathrm{sep}}] \arrow[r, equal] & K_0(BT_{\mathrm{sep}}) \arrow[r, "c_2"] & \mathrm{CH}^2(BT_{\mathrm{sep}}) \arrow[r, equal] & S^2(\widehat{T}_{\mathrm{sep}})
\end{tikzcd}$$

the second Chern class maps c_2 are surjective by [13, Lemma C.3]. It follows from the formula $c_2(a+b) = c_2(a) + c_1(a)c_1(b) + c_2(b)$ that the composition

$$\mathbb{Z}[\widehat{T}_{\mathrm{sep}}]^\Gamma = K_0(BT) \longrightarrow K_0(BT_{\mathrm{sep}}) \xrightarrow{c_2} \mathrm{CH}^2(BT_{\mathrm{sep}}) = S^2(\widehat{T}_{\mathrm{sep}}) \longrightarrow S^2(\widehat{T}_{\mathrm{sep}})/(\widehat{T})^2$$

is a homomorphism and its image is generated by the elements (see Appendix A-II)

$$c_2\Big(\sum_{i=1}^n e^{\gamma_i x}\Big) = \sum_{i<j} (\gamma_i x)(\gamma_j x) = \mathrm{Qtr}(x). \qquad \square$$

By the restriction-corestriction argument, the kernel of the homomorphism $\mathrm{CH}^2(BT) \longrightarrow \mathrm{CH}^2(BT_{\mathrm{sep}})^\Gamma = S^2(\widehat{T}_{\mathrm{sep}})^\Gamma$ coincides with the torsion subgroup $\mathrm{CH}^2(BT)_{\mathrm{tors}}$ in $\mathrm{CH}^2(BT)$.

The following theorem describes degree 3 invariants of an algebraic torus with values in $\mathbb{Q}/\mathbb{Z}(2)$:

Theorem 4.3. *Let T be an algebraic torus a field F. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{CH}^2(BT)_{\mathrm{tors}} \longrightarrow H^1(F, T^0) \xrightarrow{\alpha}$$
$$\mathrm{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} \longrightarrow S^2(\widehat{T}_{\mathrm{sep}})^\Gamma / \mathrm{Dec} \longrightarrow H^2(F, T^0).$$

The homomorphism α is given by $\alpha(a)(b) = a_K \cup b$ for every $a \in H^1(F, T^0)$ and $b \in H^1(K, T)$ and every field extension K/F, where the cup-product is defined in (4.5).

Proof. The exactness of the sequence follows from the diagram before Lemma 4.2. It remains to describe the map α. Take an $a \in H^1(F, T^0)$ and consider the invariant i defined by $i(b) = a_K \cup b$, where the cup-product is given by

(4.5). We need to prove that $i = \alpha(a)$. Choose a standard classifying T-torsor $U \longrightarrow U/T$ and set $K = F(U/T)$. Let U_{gen} be the generic fiber of the classifying torsor. By Theorem 2.2, it suffices to show that $i(U_{\mathrm{gen}}) = \alpha(a)(U_{\mathrm{gen}})$ over K. This follows from the description of the map α in the exact sequence (4.7). □

Remark 4.4. In a similar (and much simpler) fashion one can describes degree 2 invariants of an algebraic torus T with values in $\mathbb{Q}/\mathbb{Z}(1)$, i.e., invariants with values in the Brauer group by computing the étale motivic cohomology group $H^3\big(U/T, \mathbb{Z}(1)\big) = H^2\big(U/T, \mathbb{G}_m\big) = \mathrm{Br}(U/T)$ for a standard classifying T-torsor $U \longrightarrow U/T$. One establishes canonical isomorphisms

$$H^1(F, \widehat{T}_{\mathrm{sep}}) \simeq \overline{H}^3\big(U/T, \mathbb{Z}(1)\big)_{\mathrm{bal}} \simeq \mathrm{Inv}^2\big(T, \mathbb{Q}/\mathbb{Z}(1)\big)_{\mathrm{norm}} = \mathrm{Inv}\big(T, \mathrm{Br}\big)_{\mathrm{norm}}.$$

The composition takes an element $a \in H^1(F, \widehat{T}_{\mathrm{sep}})$ to the invariant $b \mapsto a_K \cup b$ for $b \in H^1(K, T)$ and a field extension K/F. This description shows that every normalized Br-invariant of T is linear.

4d. **Torsion in** $\mathrm{CH}^2(BT)$**.** We investigate the group $\mathrm{CH}^2(BT)_{\mathrm{tors}}$, the first term of the exact sequence in Theorem 4.3.

Let S be an algebraic torus over F. Using the Gersten resolution, [30, Prop. 5.8] we identify the group $H^0(S_{\mathrm{sep}}, \mathcal{K}_2)$ with a subgroup in $K_2\big(F_{\mathrm{sep}}(S)\big)$. Set $\overline{H}^0(S_{\mathrm{sep}}, \mathcal{K}_2) := H^0(S_{\mathrm{sep}}, \mathcal{K}_2)/K_2(F_{\mathrm{sep}})$. By [15, Part 2, §5.7], we have an exact sequence

$$0 \longrightarrow \widehat{S}_{\mathrm{sep}} \otimes F_{\mathrm{sep}}^{\times} \longrightarrow \overline{H}^0(S_{\mathrm{sep}}, \mathcal{K}_2) \xrightarrow{\lambda} \Lambda^2 \widehat{S}_{\mathrm{sep}} \longrightarrow 0 \tag{4.11}$$

of Γ-modules, where $\lambda\big(\{e^x, e^y\}\big) = x \wedge y$ for $x, y \in \widehat{S}_{\mathrm{sep}}$.

Consider the Γ-homomorphism

$$\begin{aligned} \gamma : \Lambda^2 \widehat{S}_{\mathrm{sep}} &\longrightarrow \overline{H}^0(S_{\mathrm{sep}}, \mathcal{K}_2) \\ x \wedge y &\longmapsto \{e^x, e^y\} - \{e^y, e^x\}. \end{aligned}$$

We have $\lambda \circ \gamma = 2 \cdot \mathrm{Id}$, hence the connecting homomorphism

$$\partial : H^i(F, \Lambda^2 \widehat{S}_{\mathrm{sep}}) \longrightarrow H^{i+1}(F, \widehat{S}_{\mathrm{sep}} \otimes F_{\mathrm{sep}}^{\times}) \tag{4.12}$$

satisfies $2\partial = 0$.

Lemma 4.5. *If S is an invertible torus, then the sequence of Γ-modules (4.11) is split.*

Proof. Suppose first that S is quasi-split. Let $\{x_1, x_2, \ldots, x_m\}$ be a permutation basis for $\widehat{S}_{\mathrm{sep}}$. Then the elements $x_i \wedge x_j$ for $i < j$ form a $\mathbb{Z}$-basis for $\Lambda^2 \widehat{S}_{\mathrm{sep}}$. The map $\Lambda^2 \widehat{S}_{\mathrm{sep}} \longrightarrow \overline{H}^0(S_{\mathrm{sep}}, \mathcal{K}_2)$, taking $x_i \wedge x_j$ to $\{e^{x_i}, e^{x_j}\}$ is a splitting for γ.

In general, find a torus S' such that $S \times S'$ is quasi-split. The desired splitting is the composition

$$\Lambda^2 \widehat{S}_{\mathrm{sep}} \longrightarrow \Lambda^2 \widehat{(S_{\mathrm{sep}} \times S'_{\mathrm{sep}})} \xrightarrow{\alpha} \overline{H}^0(S_{\mathrm{sep}} \times S'_{\mathrm{sep}}, \mathcal{K}_2) \xrightarrow{\beta} \overline{H}^0(S_{\mathrm{sep}}, \mathcal{K}_2),$$

where α is a splitting for the torus $S \times S'$ and β is the pull-back map for the canonical inclusion $S \hookrightarrow S \times S'$. $\square$

Let

$$1 \longrightarrow T \longrightarrow P \longrightarrow Q \longrightarrow 1$$

be a coflasque resolution of T, i.e., P is a quasi-split torus and Q is a coflasque torus (see [9]). The torus P is an open set in the affine space of a separable F-algebra on which T acts linearly. Hence $P \longrightarrow Q$ is a standard classifying T-torsor. By Theorem 2.2, the natural map

$$\theta_T : \operatorname{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big) \longrightarrow H^3\big(F(Q), \mathbb{Q}/\mathbb{Z}(2)\big)$$

is injective.

Consider the spectral sequence (B.10) for the variety $X = Q$. We have $H^1(Q_{\mathrm{sep}}, \mathcal{K}_2) = 0$ by [15, Part 2, Cor. 5.6]. In view of Proposition B.4, we have an injective homomorphism

$$\beta : H^2\big(F, \overline{H}^0(Q_{\mathrm{sep}}, \mathcal{K}_2)\big) \longrightarrow \overline{H}^4\big(Q, \mathbb{Z}(2)\big) \tag{4.13}$$

such that the composition of β with the homomorphism

$$H^2(F, Q^\circ) \longrightarrow H^2\big(F, \overline{H}^0(Q_{\mathrm{sep}}, \mathcal{K}_2)\big)$$

is given by the cup-product with the class of the identity in $H^0(Q, Q)$.

Lemma 4.6. *For a coflasque torus Q, the group $\mathrm{CH}^2(Q)$ is trivial.*

Proof. By [28, Th. 9.1], for every torus Q, the Grothendieck group $K_0(Q)$ is generated by the classes of the sheaves $i_*(P)$, where P is an invertible sheaf on Q_L, L/F a finite separable field extension and $i : Q_L \longrightarrow Q$ is the natural morphism. By definition of a coflasque torus,

$$\operatorname{Pic}(Q_L) = H^1(L, \widehat{Q}_{\mathrm{sep}}) = 0.$$

It follows that every invertible sheaf on Q_L is trivial, hence $K_0(Q) = \mathbb{Z} \cdot 1$. Since the group $\mathrm{CH}^2(Q)$ is generated by the second Chern classes of vector bundles on Q [13, Lemma C.3], we have $\mathrm{CH}^2(Q) = 0$. $\square$

It follows from Proposition A.10, Theorem 4.1, and Lemma 4.6 that the homomorphism

$$\kappa : \overline{H}^4\big(Q, \mathbb{Z}(2)\big) \longrightarrow \overline{H}^4\big(F(Q), \mathbb{Z}(2)\big) = \overline{H}^3\big(F(Q), \mathbb{Q}/\mathbb{Z}(2)\big) \tag{4.14}$$

is injective.

Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Q^\circ(F_{\mathrm{sep}}) & \longrightarrow & \overline{H}^0(Q_{\mathrm{sep}}, \mathcal{K}_2) & \longrightarrow & \Lambda^2 \widehat{Q}_{\mathrm{sep}} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} & & \\
0 & \longrightarrow & Q^\circ(F_{\mathrm{sep}}) & \longrightarrow & P^\circ(F_{\mathrm{sep}}) & \longrightarrow & T^\circ(F_{\mathrm{sep}}) & \longrightarrow & 0
\end{array}$$

where s is the composition of the natural map $\overline{H}^0(Q_{\mathrm{sep}}, \mathcal{K}_2) \longrightarrow \overline{H}^0(P_{\mathrm{sep}}, \mathcal{K}_2)$ and a splitting of $P^\circ(F_{\mathrm{sep}}) \longrightarrow \overline{H}^0(P_{\mathrm{sep}}, \mathcal{K}_2)$ (see Lemma 4.5).

We have the following diagram

$$\begin{tikzcd} & & H^1(F,T^\circ) \arrow[r, "\alpha"] \arrow[d, hook, "\partial_1"] & \mathrm{Inv}^3\big(T,\mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} \arrow[d, hook, "\theta_T"] \\ H^1\big(F,\overline{H}^0(Q_{\mathrm{sep}},\mathcal{K}_2)\big) \arrow[r, "\varphi"] & H^1(F,\Lambda^2\widehat{Q}_{\mathrm{sep}}) \arrow[r, "\partial"] \arrow[ur, "t^*"] & H^2(F,Q^\circ) \arrow[r, "\sigma"] & \overline{H}^3\big(F(Q),\mathbb{Q}/\mathbb{Z}(2)\big) \end{tikzcd}$$

with the bottom sequence a complex, where σ is the composition of the maps in (4.13) and (4.14):

$$H^2(F,Q^0) \xrightarrow{\psi} H^2\big(F,\overline{H}^0(Q_{\mathrm{sep}},\mathcal{K}_2)\big) \xrightarrow{\beta} \overline{H}^4\big(Q,\mathbb{Z}(2)\big) \xrightarrow{\kappa} \\ \overline{H}^4\big(F(Q),\mathbb{Z}(2)\big) = \overline{H}^3\big(F(Q),\mathbb{Q}/\mathbb{Z}(2)\big),$$

with φ and ψ given by Galois cohomology applied to the exact sequence (4.11) for the torus Q. Note that the connecting map ∂_1 is injective as $H^1(F,P^\circ)=0$ since P° is a quasi-split torus. As $2\partial=0$ in (4.12), we have $2t^*=0$.

The commutativity of the triangle follows from the definition of t^*. We claim that the square in the diagram is anti-commutative. Note that $\partial_2(\xi)=[P_{\mathrm{gen}}]$, where $\partial_2 : H^0(F,Q) \longrightarrow H^1(F,T)$ is the connecting homomorphism, P_{gen} is the generic fiber of the morphism $P \longrightarrow Q$, and $\xi \in H^0(K,Q)$ is the generic point of Q with $K=F(Q)$. It follows from the description of the maps α and β in (4.7) and (4.13), respectively, and Lemma A.1 that

$$\sigma\big(\partial_1(a)\big) = \partial_1(a)_K \cup \xi = (-a_K)\cup\partial_2(\xi) = (-a_K)\cup[P_{\mathrm{gen}}] = -\theta_T\big(\alpha(a)\big).$$

for every $a \in H^1(F,T^\circ)$.

The maps β and κ are injective, hence the bottom sequence in the diagram is exact. Thus, we have an exact sequence

$$H^1\big(F,\overline{H}^0(Q_{\mathrm{sep}},\mathcal{K}_2)\big) \longrightarrow H^1\big(F,\Lambda^2\widehat{Q}_{\mathrm{sep}}\big) \longrightarrow \mathrm{Ker}(\alpha) \longrightarrow 0$$

and $2\cdot\mathrm{Ker}(\alpha) = 2\cdot\mathrm{Im}(t^*) = 0$. Furthermore, $\mathrm{Ker}(\alpha) \simeq \mathrm{CH}^2(BT)_{\mathrm{tors}}$ by Theorem 4.3 and the group $H^1(F,\Lambda^2\widehat{Q}_{\mathrm{sep}})$ is finite as $\Lambda^2\widehat{Q}_{\mathrm{sep}}$ is a lattice.

We have proved:

Theorem 4.7. *Let $1 \longrightarrow T \longrightarrow P \longrightarrow Q \longrightarrow 1$ be a coflasque resolution of a torus T. Then there is an exact sequence*

$$H^1\big(F,\overline{H}^0(Q_{\mathrm{sep}},\mathcal{K}_2)\big) \longrightarrow H^1\big(F,\Lambda^2\widehat{Q}_{\mathrm{sep}}\big) \longrightarrow \mathrm{CH}^2(BT)_{\mathrm{tors}} \longrightarrow 0.$$

Moreover, $\mathrm{CH}^2(BT)_{\mathrm{tors}}$ is a finite group satisfying $2\cdot\mathrm{CH}^2(BT)_{\mathrm{tors}}=0$.

Corollary 4.8. *If T° is a birational direct factor of a rational torus, or if T is split over a cyclic field extension, then $\mathrm{CH}^2(BT)_{\mathrm{tors}}=0$, i.e., the map α in Theorem 4.3 is injective.*

Proof. The exact sequence $1 \longrightarrow Q^\circ \longrightarrow P^\circ \longrightarrow T^\circ \longrightarrow 1$ is a flasque resolution of T°. If T° is a birational direct factor of a rational torus, or if T is split over a cyclic field extension, the torus Q° ,and hence Q, is invertible (see §4a and [35, §4, Th. 3]). By Lemma 4.5, the sequence (4.11) for the torus Q is split, hence the first map in Theorem 4.7 is surjective. □

Question 4.9. Is $\mathrm{CH}^2(BT)_{\mathrm{tors}}$ trivial for every torus T?

4e. **Special tori.** Let T be an algebraic torus over a field F. The *tautological invariant* of the torus $T^\circ \times T$ is the normalized invariant taking a pair $(a,b) \in H^1(K,T^\circ) \times H^1(K,T)$ to the cup-product $a \cup b \in H^3\big(K, \mathbb{Q}/\mathbb{Z}(2)\big)$ defined in (4.5).

The following theorem shows that if a torus T has only trivial degree 3 normalized invariants with values in $\mathbb{Q}/\mathbb{Z}(2)$ universally, i.e., over all field extensions of F, then T has no non-constant invariants at all by the simple reason: every T-torsor over a field is trivial. Note that it follows from Theorem 2.4 that T has no degree 2 normalized invariants with values in $\mathbb{Q}/\mathbb{Z}(1)$ universally if and only if T is coflasque.

Theorem 4.10. *Let T be an algebraic torus over a field F. Then the following are equivalent:*

1. $\mathrm{Inv}^3\big(T_K, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} = 0$ *for every field extension K of F.*
2. *The tautological invariant of the torus $T^\circ \times T$ is trivial.*
3. *The torus T is invertible.*
4. *The torus T is special.*

Proof. (1) $\Rightarrow$ (2): Let K/F be a field extension and $a \in H^1(K,T^\circ)$. By assumption, the degree 3 normalized invariant $i = \alpha(a)$ with values in $\mathbb{Q}/\mathbb{Z}(2)$, defined by $i(b) = a \cup b$ for every $b \in H^1(K,T)$, is trivial. In other words, the tautological invariant of the torus $T^\circ \times T$ is trivial.

(2) $\Rightarrow$ (3): The image of the tautological invariant in the group $S^2(\widehat{T}^\circ_{\mathrm{sep}} \oplus \widehat{T}_{\mathrm{sep}})^\Gamma / \operatorname{Dec}$ is represented by the identity $1_{\widehat{T}}$ in the direct factor $(\widehat{T}^\circ_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})^\Gamma = \operatorname{End}_\Gamma(\widehat{T}_{\mathrm{sep}})$ of $S^2(\widehat{T}^\circ_{\mathrm{sep}} \oplus \widehat{T}_{\mathrm{sep}})^\Gamma$ (see Appendix A-II). The projection of Dec on the direct summand $(\widehat{T}^\circ_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})^\Gamma$ is generated by the traces $\operatorname{Tr}(a \otimes b)$ for all open subgroups $\Gamma' \subset \Gamma$ and all $a \in (\widehat{T}^\circ_{\mathrm{sep}})^{\Gamma'}$ and $b \in (\widehat{T}_{\mathrm{sep}})^{\Gamma'}$. Hence $1_{\widehat{T}} = \sum_i \operatorname{Tr}(a_i \otimes b_i)$ for some open subgroups $\Gamma_i \subset \Gamma$, $a_i \in (\widehat{T}^\circ)^{\Gamma_i}$ and $b_i \in (\widehat{T})^{\Gamma_i}$. If $P_i = \mathbb{Z}[\Gamma/\Gamma_i]$, then a_i can be viewed as a Γ-homomorphism $\widehat{T} \longrightarrow P_i$ and b_i can be viewed as a Γ-homomorphism $P_i \longrightarrow \widehat{T}$ such that the composition

$$\widehat{T} \xrightarrow{(b_i)} P \xrightarrow{(a_i)} \widehat{T},$$

where $P = \coprod P_i$, is the identity. It follows that $\widehat{T}$ is a direct summand of a permutation Γ-module P and hence T is invertible.

(3) $\Rightarrow$ (4): Obvious as every invertible torus is special.

(4) $\Rightarrow$ (1): Obvious. □

Remark 4.11. The equivalence (3) $\Leftrightarrow$ (4) was essentially proved in [10, Proposition 7.4].

4f. **Linear and quadratic invariants.** Let T be a torus over F. By Theorem 4.3, we have a natural homomorphism to the group of linear invariants:

$$\alpha : H^1(F, T^\circ) \longrightarrow \mathrm{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{lin}}.$$

Let S and T be algebraic tori over F. For every field extension K/F, the cup-product (4.2) yields a homomorphism

$$\varepsilon : (\widehat{T}_{\mathrm{sep}}^{\otimes 2})^\Gamma \longrightarrow \mathrm{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)$$

defined by $\varepsilon(a)(b) = a_K \cup b \cup b$ for $a \in (\widehat{T}_{\mathrm{sep}}^{\otimes 2})^\Gamma$ and $b \in H^1(K, T)$.

We say that an invariant $i \in \mathrm{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)$ is *quadratic* if the function $h(a, b) := i(a + b) - i(a) - i(b)$ is bilinear and $h(a, a) = 2i(a)$ for all a and b. For example, the tautological invariant of the torus $T^\circ \times T$ in §4e is quadratic. We write $\mathrm{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{quad}}$ for the subgroup of all quadratic invariants in $\mathrm{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)$. The image of ε is contained in $\mathrm{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{quad}}$.

Lemma 4.12. *The composition of* ε *with the map* $\mathrm{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big) \longrightarrow S^2(\widehat{T}_{\mathrm{sep}})^\Gamma / \mathrm{Dec}$ *in Theorem 4.3 is induced by the natural homomorphism* $\widehat{T}_{\mathrm{sep}}^{\otimes 2} \longrightarrow S^2(\widehat{T}_{\mathrm{sep}})$.

Proof. Let $U \longrightarrow U/T =: X$ be a standard classifying T-torsor as in §4c. Consider the commutative diagram

$$\begin{tikzcd}
(\widehat{T}_{\mathrm{sep}}^{\otimes 2})^\Gamma \otimes H^1(X, T)^{\otimes 2} \arrow[r, "prod"] \arrow[d] & \overline{H}^4\big(X, \mathbb{Z}(2)\big)_{\mathrm{bal}} \arrow[r] \arrow[d] & \mathrm{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} \\
\widehat{T}_{\mathrm{sep}}^{\otimes 2} \otimes H^1(X_{\mathrm{sep}}, T)^{\otimes 2} \arrow[r, "prod"] \arrow[d, "\eta", "\wr"'] & \overline{H}^4\big(X_{\mathrm{sep}}, \mathbb{Z}(2)\big)_{\mathrm{bal}} \arrow[d, "\wr"] & \\
\widehat{T}_{\mathrm{sep}}^{\otimes 2} \otimes (\widehat{T}^\circ_{\mathrm{sep}})^{\otimes 2} \otimes \widehat{T}_{\mathrm{sep}}^{\otimes 2} \arrow[r, "\kappa"] & S^2(\widehat{T}_{\mathrm{sep}})^\Gamma / \mathrm{Dec} &
\end{tikzcd}$$

where the product maps are given by (4.1), η identifies $H^1(X_{\mathrm{sep}}, T) = \widehat{T}^\circ_{\mathrm{sep}} \otimes \mathrm{Pic}(X_{\mathrm{sep}})$ with $\widehat{T}^\circ_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}}$ and κ is given by the pairing between the first and second factors. Write $[U]$ for the class of the classifying torsor in $H^1(X, T)$. The image of $[U]$ in $H^1(X_{\mathrm{sep}}, T_{\mathrm{sep}}) = \widehat{T}^\circ_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}} = \mathrm{End}(\widehat{T}_{\mathrm{sep}})$ is the identity $1_{\widehat{T}_{\mathrm{sep}}}$. Hence for every $a \in (\widehat{T}_{\mathrm{sep}}^{\otimes 2})^\Gamma$, the image of $a \otimes [U] \otimes [U]$ under the diagonal map in the diagram coincides with the canonical image of a in $S^2(\widehat{T}_{\mathrm{sep}})^\Gamma / \mathrm{Dec}$. $\square$

The composition of the map $S^2(\widehat{T}_{\mathrm{sep}})^\Gamma \longrightarrow (\widehat{T}_{\mathrm{sep}}^{\otimes 2})^\Gamma$ given by $a \cdot b \longmapsto a \otimes b + b \otimes a$ with the natural map $(\widehat{T}_{\mathrm{sep}}^{\otimes 2})^\Gamma \longrightarrow S^2(\widehat{T}_{\mathrm{sep}})^\Gamma$ is multiplication by 2. Then by Lemma 4.12, $2 \cdot S^2(\widehat{T}_{\mathrm{sep}})^\Gamma / \mathrm{Dec}$ is contained in the image of the map $\mathrm{Inv}^3\big(T, \mathbb{Q}/\mathbb{Z}(2)\big) \longrightarrow S^2(\widehat{T}_{\mathrm{sep}})^\Gamma / \mathrm{Dec}$.

Theorem 4.3 then yields:

Theorem 4.13. *Let T be an algebraic torus over F. Then 2 times the homomorphism $S^2(\widehat{T}_{\mathrm{sep}})^\Gamma/\operatorname{Dec} \longrightarrow H^2(F,T^0)$ from Theorem 4.3 is trivial. If p is an odd prime,*

$$\operatorname{Inv}^3\big(T,\mathbb{Q}_p/\mathbb{Z}_p(2)\big)_{\mathrm{norm}} = \operatorname{Inv}^3\big(T,\mathbb{Q}_p/\mathbb{Z}_p(2)\big)_{\mathrm{lin}} \oplus \operatorname{Inv}^3\big(T,\mathbb{Q}_p/\mathbb{Z}_p(2)\big)_{\mathrm{quad}}$$

and there are natural isomorphisms $\operatorname{Inv}^3\big(T,\mathbb{Q}_p/\mathbb{Z}_p(2)\big)_{\mathrm{lin}} \simeq H^1(F,T^\circ)\{p\}$ *and* $\operatorname{Inv}^3\big(T,\mathbb{Q}_p/\mathbb{Z}_p(2)\big)_{\mathrm{quad}} \simeq \big(S^2(\widehat{T}_{\mathrm{sep}})^\Gamma/\operatorname{Dec}\big)\{p\}$.

Example 4.14. Let $X = \{x_1, x_2, \dots, x_n\}$ be a set of n elements with the natural action of the symmetric group S_n. A continuous surjective group homomorphism $\Gamma \longrightarrow S_n$ yields a separable field extension L/F of degree n. Consider the torus $T = R_{L/F}(\mathbb{G}_{m,L})/\mathbb{G}_m$, where $R_{L/F}$ is the Weil restriction (see [35, Ch. 1, §3.12]). Note that the generic maximal torus of the group $\mathbf{PGL}_n$ is of this form (see §5b). The character lattice $\widehat{T}_{\mathrm{sep}}$ is the kernel of the augmentation homomorphism $\mathbb{Z}[X] \longrightarrow \mathbb{Z}$.

The dual torus T° is the norm one torus $R^{(1)}_{L/F}(\mathbb{G}_{m,L})$. For every field extension K/F, we have:

$$H^1(K,T) = \operatorname{Br}(KL/K), \qquad H^1(K,T^\circ) = K^\times/N(KL)^\times,$$

where $KL := K \otimes L$, N is the norm map for the extension KL/K and $\operatorname{Br}(KL/K) = \operatorname{Ker}\big(\operatorname{Br}(K) \longrightarrow \operatorname{Br}(KL)\big)$. The pairing

$$K^\times/N(KL)^\times \otimes \operatorname{Br}(KL/K) \longrightarrow H^3\big(F,\mathbb{Q}/\mathbb{Z}(2)\big)$$

defines linear degree 3 invariants of both T and T°.

We claim that $S^2(\widehat{T}_{\mathrm{sep}})^\Gamma/\operatorname{Dec} = 0$ and $S^2(\widehat{T}^\circ_{\mathrm{sep}})^\Gamma/\operatorname{Dec} = 0$, i.e., T and T° have no nontrivial quadratic degree 3 invariants. We have $\widehat{T}^\circ_{\mathrm{sep}} = \mathbb{Z}[X]/\mathbb{Z}N_X$, where $N_X = \sum x_i$. The group $S^2(\widehat{T}^\circ_{\mathrm{sep}})^\Gamma$ is generated by $S := \sum_{i<j} x_i \cdot x_j$. As $S \in \operatorname{Dec}$, we have $S^2(\widehat{T}^\circ_{\mathrm{sep}})^\Gamma/\operatorname{Dec} = 0$.

Let $D = \sum x_i^2$ and $E := \operatorname{Qtr}(x_1 - x_2) = 2S - (n-1)D$, where the quadratic map Qtr is defined in Appendix A-II. The group $S^2(\widehat{T}_{\mathrm{sep}})^\Gamma$ is generated by E if n is even and by $E/2$ if n is odd. A computation shows that $nE/2 = \operatorname{Qtr}(nx_1 - N_X)$. It follows that the generator of $S^2(\widehat{T}_{\mathrm{sep}})^\Gamma$ belongs to Dec, hence $S^2(\widehat{T}_{\mathrm{sep}})^\Gamma/\operatorname{Dec}$ is trivial.

Note that as the torus T is rational, it follows from Theorem 4.3 and Corollary 4.8 that $\operatorname{Inv}^3\big(T^\circ,\mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} \simeq \operatorname{Br}(L/F)$.

5. Unramified invariants

Let K/F be a field extension and v a discrete valuation of K over F with valuation ring O_v. We say that an element $a \in H^n\big(K,\mathbb{Q}/\mathbb{Z}(j)\big)$ is *unramified with respect to v* if a belongs to the image of the map $H^n\big(O_v,\mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n\big(K,\mathbb{Q}/\mathbb{Z}(j)\big)$ (see [8]). We write $H^n_{\mathrm{nr}}\big(K,\mathbb{Q}/\mathbb{Z}(j)\big)$ for the subgroup of the elements in $H^n\big(K,\mathbb{Q}/\mathbb{Z}(j)\big)$ that are unramified with respect to all discrete valuations of K over F. We have a natural homomorphism

$$H^n\big(F,\mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n_{\mathrm{nr}}\big(K,\mathbb{Q}/\mathbb{Z}(j)\big). \tag{5.1}$$

A dominant morphism of varieties $Y \longrightarrow X$ yields a homomorphism

$$H^n_{\mathrm{nr}}\big(F(X),\mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n_{\mathrm{nr}}\big(F(Y),\mathbb{Q}/\mathbb{Z}(j)\big). \tag{5.2}$$

Proposition 5.1. *Let K/F be a purely transcendental field extension. Then the homomorphism (5.1) is an isomorphism.*

Proof. The statement is well known for the p-components if $p \neq \operatorname{char} F$ (see, for example, [8, Prop. 1.2]). It suffices to consider the case $K = F(t)$ and prove the surjectivity of (5.1). The coniveau spectral sequence for the projective line $\mathbb{P}^1$ (see Appendix (A.1)) yields an exact sequence

$$H^n\big(\mathbb{P}^1,\mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n\big(K,\mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow \coprod_{x\in\mathbb{P}^1} H^{n+1}_x\big(\mathbb{P}^1,\mathbb{Q}/\mathbb{Z}(j)\big)$$

and, therefore, a surjective homomorphism $H^n\big(\mathbb{P}^1,\mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n_{\mathrm{nr}}\big(K,\mathbb{Q}/\mathbb{Z}(j)\big)$. By the projective bundle theorem (classical if $p \neq \operatorname{char}(F)$ and [16, Th. 2.1.11] if $p = \operatorname{char}(F) > 0$), we have

$$H^n\big(\mathbb{P}^1,\mathbb{Q}/\mathbb{Z}(j)\big) = H^n\big(F,\mathbb{Q}/\mathbb{Z}(j)\big) \oplus H^{n-2}\big(F,\mathbb{Q}/\mathbb{Z}(j-1)\big)t,$$

where t is a generator of $H^2\big(\mathbb{P}^1,\mathbb{Z}(1)\big) = \operatorname{Pic}(\mathbb{P}^1) = \mathbb{Z}$. As t vanishes over the generic point of $\mathbb{P}^1$, the result follows. □

Let G be a linear algebraic group over F. Choose a standard classifying G-torsor $U \longrightarrow U/G$. An invariant $i \in \operatorname{Inv}^n\big(G,\mathbb{Q}/\mathbb{Z}(j)\big)$ is called *unramified* if the image of i under $\theta_G : \operatorname{Inv}^n\big(G,\mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n\big(F(U/G),\mathbb{Q}/\mathbb{Z}(j)\big)$ is unramified. This is independent of the choice of standard classifying torsor. Indeed, if $U' \longrightarrow U'/G$ is another standard classifying G-torsor, then $(U \times V')/G \longrightarrow U/G$ and $(V \times U')/G \longrightarrow U'/G$ are vector bundles. Hence the field $F\big((U\times U')/G\big)$ is a purely transcendental extension of $F(U/G)$ and $F(U'/G)$ and by Proposition 5.1,

$$H^n_{\mathrm{nr}}\big(F(U/G),\mathbb{Q}/\mathbb{Z}(j)\big) \simeq H^n_{\mathrm{nr}}\big(F((U\times U')/G),\mathbb{Q}/\mathbb{Z}(j)\big) \simeq H^n_{\mathrm{nr}}\big(F(U'/G),\mathbb{Q}/\mathbb{Z}(j)\big).$$

We write $H^n_{\mathrm{nr}}\big(F(BG),\mathbb{Q}/\mathbb{Z}(j)\big)$ for this common value and $\operatorname{Inv}^n_{\mathrm{nr}}\big(G,\mathbb{Q}/\mathbb{Z}(j)\big)$ for the subgroup of unramified invariants. Similarly, we write $\operatorname{Br}_{\mathrm{nr}}\big(F(BG)\big)$ for the *unramified Brauer group* $H^2_{\mathrm{nr}}\big(F(BG),\mathbb{Q}/\mathbb{Z}(1)\big)$.

Proposition 5.2. *If G' be a subgroup of G and $i \in \operatorname{Inv}^n_{\mathrm{nr}}\big(G,\mathbb{Q}/\mathbb{Z}(j)\big)$, then $\operatorname{res}(i) \in \operatorname{Inv}^n_{\mathrm{nr}}\big(G',\mathbb{Q}/\mathbb{Z}(j)\big)$.*

Proof. It is shown in Section 2b that there is a surjective morphism $X' \to X$ of classifying varieties of G' and G respectively, such that

$$\theta_G(i)_{F(X')} = \theta_{G'}(\operatorname{res}(i)).$$

Applying the homomorphism (5.2) we see that $\operatorname{res}(i)$ is unramified. □

Proposition 5.3. *Let G be a smooth linear algebraic group over a field F. The map $\mathrm{Inv}^n_{\mathrm{nr}}\big(G, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n_{\mathrm{nr}}\big(F(BG), \mathbb{Q}/\mathbb{Z}(j)\big)$ induced by θ_G is an isomorphism.*

Proof. By Theorem 3.4, it suffices to show that $H^n_{\mathrm{nr}}\big(F(U/G), \mathbb{Q}/\mathbb{Z}(j)\big) \subset H^n\big(F(U/G), \mathbb{Q}/\mathbb{Z}(j)\big)_{\mathrm{bal}}$. We follow Totaro's approach (see [15, p. 99]). Consider the open subscheme W of $(U^2/G) \times \mathbb{A}^1$ of all triples (u, u', t) such that $(2-t)u + (t-1)u' \in U$. We have the projection $q : W \longrightarrow U^2/G$, the morphisms $f : W \longrightarrow U/G$ defined by $f(u, u', t) = (2-t)u + (t-1)u'$, and $h_i : U^2/G \longrightarrow W$ defined by $h_i(u, u') = (u, u', i)$ for $i = 1$ and 2. The composition $f \circ h_i$ is the projection $p_i : U^2/G \longrightarrow U/G$ and $q \circ h_i$ is the identity of U^2/G.

Let w_i be the generic point of the pre-image of i with respect to the projection $W \longrightarrow \mathbb{A}^1$ and write O_i for the local ring of W at w_i. The morphisms q, f, and h_i yield F-algebra homomorphisms $F(U^2/G) \longrightarrow O_i$, $F(U/G) \longrightarrow O_i$ and $O_i \longrightarrow F(U/G)$. Note that by Proposition A.11, we have $H^n_{\mathrm{nr}}\big(F(W), \mathbb{Q}/\mathbb{Z}(j)\big) \subset H^n\big(O_i, \mathbb{Q}/\mathbb{Z}(j)\big)$. In the commutative diagram

$$\begin{tikzcd}
H^n_{\mathrm{nr}}\big(F(U/G), \mathbb{Q}/\mathbb{Z}(j)\big) \arrow[r, "f^*"] \arrow[d, hook] & H^n_{\mathrm{nr}}\big(F(W), \mathbb{Q}/\mathbb{Z}(j)\big) \arrow[d, hook] & H^n_{\mathrm{nr}}\big(F(U^2/G), \mathbb{Q}/\mathbb{Z}(j)\big) \arrow[l, "q^*"', "\sim"] \arrow[d, hook] \\
H^n\big(F(U/G), \mathbb{Q}/\mathbb{Z}(j)\big) \arrow[r, "f^*"] \arrow[dr, "p_i^*"'] & H^n\big(O_i, \mathbb{Q}/\mathbb{Z}(j)\big) \arrow[d, "h_i^*"] & H^n\big(F(U^2/G), \mathbb{Q}/\mathbb{Z}(j)\big) \arrow[l, "q^*"'] \arrow[dl, equal] \\
 & H^n\big(F(U^2/G), \mathbb{Q}/\mathbb{Z}(j)\big) &
\end{tikzcd}$$

the top right map q^* is an isomorphism by Proposition 5.1 since the field extension $F(W)/F(U^2/G)$ is purely transcendental. It follows that the restriction of p_i^* on $H^n_{\mathrm{nr}}\big(F(U/G), \mathbb{Q}/\mathbb{Z}(j)\big)$ coincides with $(q^*)^{-1} \circ f^*$ and hence is independent of i. ☐

5a. **Unramified invariants of tori.**

Proposition 5.4. *If T is a flasque torus, then every invariant in $\mathrm{Inv}^n\big(T, \mathbb{Q}/\mathbb{Z}(j)\big)$ is unramified.*

Proof. Consider an exact sequence of tori $1 \longrightarrow T \longrightarrow P \longrightarrow Q \longrightarrow 1$ with P quasi-split. Choose a smooth projective compactification X of Q (see [6]). As T is flasque, by [9, Prop. 9], there is a T-torsor $E \longrightarrow X$ extending the T-torsor $P \longrightarrow Q$. The torsor E is classifying and T-rational. Choose an invariant in $\mathrm{Inv}^n\big(G, \mathbb{Q}/\mathbb{Z}(j)\big)$ and consider its image a in $H^n\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big)_{\mathrm{bal}}$ (see Theorem 3.4). We show that a is unramified with respect to every discrete valuation v on $F(X)$ over F (cf., [4, Prop. 2.1.8]). By Proposition A.9, a is unramified with respect to the discrete valuation associated to every point $x \in X$ of codimension 1, i.e., $\partial_x(a) = 0$.

As X is projective, the valuation ring O_v of the valuation v dominates a point $x \in X$. It follows from Proposition A.11 that a belongs to the image of

$H^n\big(O_{X,x},\mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n\big(F(X),\mathbb{Q}/\mathbb{Z}(j)\big)$. As the local ring $O_{X,x}$ is a subring of O_v, a belongs to the image of $H^n\big(O_v,\mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^n\big(F(X),\mathbb{Q}/\mathbb{Z}(j)\big)$ and hence a is unramified with respect to v. □

Let T be a torus over F. By [10, Lemma 0.6], there is an exact sequence of tori $1 \longrightarrow T \longrightarrow T' \longrightarrow P \longrightarrow 1$ with T' flasque and P quasi-split. The following theorem computes the unramified invariants of T in terms of the invariants of T'.

Theorem 5.5. *There is a natural isomorphism*

$$\mathrm{Inv}^n_{\mathrm{nr}}\big(T,\mathbb{Q}/\mathbb{Z}(j)\big) \simeq \mathrm{Inv}^n\big(T',\mathbb{Q}/\mathbb{Z}(j)\big).$$

Proof. Choose an exact sequence $1 \longrightarrow T' \longrightarrow P' \longrightarrow S \longrightarrow 1$ with P' a quasi-split torus. Let S' be the cokernel of the composition $T \longrightarrow T' \longrightarrow P'$. We have an exact sequence $1 \longrightarrow P \longrightarrow S' \longrightarrow S \longrightarrow 1$. As P is quasi-split, the latter exact sequence splits at the generic point of S and therefore, $F(S')$ is a purely transcendental field extension of $F(S)$. It follows from Propositions 5.1, 5.3, and 5.4 that

$$\mathrm{Inv}^n_{\mathrm{nr}}\big(T,\mathbb{Q}/\mathbb{Z}(j)\big) \simeq H^n_{\mathrm{nr}}\big(F(S'),\mathbb{Q}/\mathbb{Z}(j)\big) \simeq H^n_{\mathrm{nr}}\big(F(S),\mathbb{Q}/\mathbb{Z}(j)\big) \simeq \mathrm{Inv}^n_{\mathrm{nr}}\big(T',\mathbb{Q}/\mathbb{Z}(j)\big) = \mathrm{Inv}^n\big(T',\mathbb{Q}/\mathbb{Z}(j)\big). \quad \square$$

The following corollary is essentially equivalent to [10, Prop. 9.5] in the case when F is of zero characteristic.

Corollary 5.6. *With notation as above, the isomorphism* $\mathrm{Inv}(T,\mathrm{Br}) \xrightarrow{\sim} \mathrm{Pic}(T) = H^1(F,\widehat{T})$ *identifies* $\mathrm{Inv}_{\mathrm{nr}}(T,\mathrm{Br})$ *with the subgroup* $H^1(F,\widehat{T}')$ *of* $H^1(F,\widehat{T})$ *of all elements that are trivial when restricted to all cyclic subgroups of the decomposition group of* T.

Proof. The description of $H^1(F,\widehat{T}')$ as a subgroup of $H^1(F,\widehat{T})$ is given in [10, Prop. 9.5], and this part of the proof is characteristic free. □

In view of Propositions 5.1 and 5.3 we can calculate the group of unramified cohomology for the function field of an arbitrary torus in terms of the invariants of a flasque torus:

Theorem 5.7. *Let S be a torus over F and let $1 \longrightarrow T \longrightarrow P \longrightarrow S \longrightarrow 1$ be a flasque resolution of S, i.e., T is flasque and P is quasi-split. Then there is a natural isomorphism*

$$H^n_{\mathrm{nr}}\big(F(S),\mathbb{Q}/\mathbb{Z}(j)\big) \simeq \mathrm{Inv}^n\big(T,\mathbb{Q}/\mathbb{Z}(j)\big).$$

Note that the torus S determines T up to multiplication by a quasisplit torus. If X is a smooth compactification of S, then one can take a torus T with $\widehat{T}_{\mathrm{sep}} \simeq \mathrm{Pic}(X_{\mathrm{sep}})$ (see [9, Proposition 6] or [35, §4.6]).

Corollary 5.8. *A torus S has no nonconstant unramified degree 3 cohomology with values in $\mathbb{Q}/\mathbb{Z}(2)$ universally, i.e., $H^3_{\mathrm{nr}}\big(K(S),\mathbb{Q}/\mathbb{Z}(2)\big) = H^3\big(K,\mathbb{Q}/\mathbb{Z}(2)\big)$ for any field extension K/F, if and only if S is a direct factor of a rational torus.*

Proof. If S is a direct factor of a rational torus, then S has no nonconstant unramified cohomology by Proposition 5.1.

Conversely, let $1 \longrightarrow T \longrightarrow P \longrightarrow S \longrightarrow 1$ be a flasque resolution of S. By Theorem 5.7, $\mathrm{Inv}^3\big(T_K, \mathbb{Q}/\mathbb{Z}(2)\big)_{\mathrm{norm}} = 0$ for every K/F. It follows from Theorem 4.10 that T is invertible and hence S is a factor of a rational torus (see §4a). □

Theorems 4.3, 5.7 and [9, §2] yield the following proposition.

Proposition 5.9. *Let S be a torus over F and let $1 \longrightarrow T \longrightarrow P \longrightarrow S \longrightarrow 1$ be a flasque resolution of S. Then we have an exact sequence*

$$0 \longrightarrow \mathrm{CH}^2(BT)_{\mathrm{tors}} \longrightarrow H^1(F, T^0) \longrightarrow \overline{H}^3_{\mathrm{nr}}\big(F(S), \mathbb{Q}/\mathbb{Z}(2)\big) \longrightarrow H^0\big(F, \mathsf{S}^2(\widehat{T}_{\mathrm{sep}})\big)/\operatorname{Dec} \longrightarrow H^2(F, T^0).$$

For an odd prime p, there is a canonical direct sum decomposition

$$\overline{H}^3_{\mathrm{nr}}\big(F(S), \mathbb{Q}_p/\mathbb{Z}_p(2)\big) = H^1(F, T^0)\{p\} \oplus \big(H^0\big(F, \mathsf{S}^2(\widehat{T}_{\mathrm{sep}})\big)/\operatorname{Dec}\big)\{p\}.$$

If X is a smooth compactification of S, one can take the torus T with $\widehat{T}_{\mathrm{sep}} = \mathrm{Pic}(X_{\mathrm{sep}})$.

5b. **The Brauer invariant for semisimple groups.** The following theorem was proved by Bogomolov [2, Lemma 5.7] in characteristic zero:

Theorem 5.10. *Let G be a (connected) semisimple group over a field F. Then $\mathrm{Inv}_{\mathrm{nr}}(G, \mathrm{Br}) = \mathrm{Inv}(G, \mathrm{Br})_{\mathrm{const}} = \mathrm{Br}(F)$ and $\mathrm{Br}_{\mathrm{nr}}\big(F(BG)\big) = \mathrm{Br}(F)$.*

Proof. Let $G' \longrightarrow G$ be a simply connected cover of G and C the kernel of $G' \longrightarrow G$. By Theorem 2.4 and [32, Lemme 6.9(iii)], we have

$$\mathrm{Inv}(G, \mathrm{Br})_{\mathrm{norm}} = \mathrm{Pic}(G) = \widehat{C}(F).$$

As the map $\widehat{C}(F) \longrightarrow \widehat{C}(F_{\mathrm{sep}})$ is injective, we can replace F by F_{sep} and assume that the group G is split.

Consider the variety $\mathcal{T}$ of maximal tori in G and the closed subscheme $\mathcal{X} \subset G \times \mathcal{T}$ of all pairs (g, T) with $g \in T$. The generic fiber of the projection $\mathcal{X} \longrightarrow \mathcal{T}$ is the *generic torus* T_{gen} of G. Then T_{gen} is a maximal torus of G_K, where $K := F(\mathcal{T})$.

Every maximal torus in G is the factor torus of a maximal torus in G' by C. It follows that the variety $\mathcal{T}'$ of maximal tori in G' is naturally isomorphic to $\mathcal{T}$. Moreover, as the generic torus T'_{gen} of G' is a maximal torus of G'_K, we have $T_{\mathrm{gen}} \simeq T'_{\mathrm{gen}}/C_K$ and therefore, an exact sequence of character groups

$$0 \longrightarrow \widehat{T}_{\mathrm{gen}} \longrightarrow \widehat{T}'_{\mathrm{gen}} \longrightarrow \widehat{C}_K \longrightarrow 0.$$

By Theorem 2.4, the compositon of the natural homomorphism $\mathrm{Inv}(G, \mathrm{Br})_{\mathrm{norm}} \longrightarrow \mathrm{Inv}(G_K, \mathrm{Br})_{\mathrm{norm}}$ with the restriction $\mathrm{Inv}(G_K, \mathrm{Br})_{\mathrm{norm}} \longrightarrow \mathrm{Inv}(T_{\mathrm{gen}}, \mathrm{Br})_{\mathrm{norm}}$ can be identified with the natural composition $\mathrm{Pic}(G) \to \mathrm{Pic}(G_K) \to \mathrm{Pic}(T_{\mathrm{gen}})$ and hence with the connecting homomorphism $\widehat{C}(F) = \widehat{C}(K) \longrightarrow H^1(K, \widehat{T}_{\mathrm{gen}})$.

Note that as $F = F_{\mathrm{sep}}$, the decomposition group of T_{gen} coincides with the Weyl group W of G by [34, Th. 1], hence $H^1(K, \widehat{T}_{\mathrm{gen}}) \simeq H^1(W, \widehat{T}_{\mathrm{gen}})$.

Let w be a Coxeter element in W.[1] It is the product of reflections with respect to all simple roots (in some order). By [19, Lemma, p. 76], 1 is not an eigenvalue of w on the space of weights $\widehat{T'}_{\mathrm{gen}} \otimes \mathbb{R}$. Let W_0 be the cyclic subgroup in W generated by w. It follows that the first term in the exact sequence

$$(\widehat{T'}_{\mathrm{gen}})^{W_0} \longrightarrow \widehat{C}(K) \longrightarrow H^1(W_0, \widehat{T}_{\mathrm{gen}})$$

is trivial, i.e., the second map is injective. Hence every nonzero character χ in $\widehat{C}(K)$ restricts to a nonzero element in $H^1(W_0, \widehat{T}_{\mathrm{gen}})$. It follows that the image of χ in $H^1(W, \widehat{T}_{\mathrm{gen}})$ is ramified by Corollary 5.6, hence so is χ by Proposition 5.2. $\square$

Appendix A. Generalities

A-I. **Proof of Theorem 2.2.** Suppose that $i(E_{\mathrm{gen}}) = 0$ for an H-invariant i of G. Let K/F be a field extension and $I \longrightarrow \operatorname{Spec} K$ a G-torsor. We need to show that $i(I) = 0$ in $H(K)$.

Suppose first that K is infinite. Find a point $x \in X(K)$ such that I is isomorphic to the pull-back of the classifying torsor with respect to x. Let x' be a rational point of X_K above x and write O for the local ring $O_{X_K,x'}$. The K-algebra O is a regular local ring with residue field K. Therefore, the completion $\widehat{O}$ is isomorphic to $K[[t_1, t_2, \dots, t_n]]$ over K. Let L be the quotient field of $\widehat{O}$, a field extension of $K(X)$. We have the following diagram of morphisms with a commutative square and three triangles:

$$\begin{tikzcd}
 & \operatorname{Spec} K \arrow[d, shift left] \arrow[dr, "x"] & \\
\operatorname{Spec} L \arrow[ur] \arrow[r] \arrow[d] & \operatorname{Spec} \widehat{O} \arrow[u, shift left] \arrow[r] \arrow[d] & X \\
\operatorname{Spec} K(X) \arrow[r] & \operatorname{Spec} O \arrow[ur] &
\end{tikzcd}$$

The pull-back of the classifying torsor $E \longrightarrow X$ via $\operatorname{Spec} K(X) \longrightarrow X$ is $(E_{\mathrm{gen}})_{K(X)}$. The G-torsor I is the pull-back of $E \longrightarrow X$ with respect to x. Let $\widehat{E}$ be the pull-back of $E \longrightarrow X$ via $\operatorname{Spec} \widehat{O} \longrightarrow X$. Therefore, I is the pull-back of $\widehat{E}$. Since G is smooth, by a theorem of Grothendieck [18, XXIV, Prop. 8.1], $\widehat{E}$ is the pull-back of I with respect to $\operatorname{Spec} \widehat{O} \longrightarrow \operatorname{Spec}(K)$. It follows that $I_L \simeq (E_{\mathrm{gen}})_L$ as torsors over L. Hence the images of $i(I)$ and $i(E_{\mathrm{gen}})$ in $H(L)$ are equal and therefore, $i(I)_L = 0$. By Property 2.1, we have $i(I) = 0$.

[1]We owe the idea to use the Coxeter element and the reference below to S. Garibaldi.

If K is finite, we replace F by $F((t))$ and K by $K((t))$. By the first part of the proof, $i(I)$ belongs to the kernel of $H(K) \longrightarrow H\big(K((t))\big)$ and hence is trivial by Property 2.1 again.

A-II. **Decomposable elements.** Let Γ be a profinite group and A a Γ-lattice. Write A^Γ for the subgroup of Γ-invariant elements in A. Let $\Gamma' \subset \Gamma$ be an open subgroup and choose representatives $\gamma_1, \gamma_2, \ldots, \gamma_n$ for the left cosets of Γ' in Γ. We have the *trace map* $\mathrm{Tr} : A^{\Gamma'} \longrightarrow A^\Gamma$ defined by $\mathrm{Tr}(a) = \sum_{i=1}^n \gamma_i a$.

Let $S^2(A)$ be the symmetric square of A. Consider the *quadratic trace map* $\mathrm{Qtr} : A^{\Gamma'} \longrightarrow S^2(A)^\Gamma$ defined by $\mathrm{Qtr}(a) = \sum_{i<j} (\gamma_i a)(\gamma_j a)$. Write $\mathrm{Dec}(A)$ for the subgroup of *decomposable elements* in $S^2(A)^\Gamma$ generated by the square $(A^\Gamma)^2$ of A^Γ and the elements $\mathrm{Qtr}(a)$ for all open subgroups $\Gamma' \subset \Gamma$ and all $a \in A^{\Gamma'}$.

Let B be another Γ-lattice. We write $\mathrm{Dec}(A, B)$ for the subgroup of $(A \otimes B)^\Gamma$ generated by elements of the form $\mathrm{Tr}(a \otimes b)$ for all open subgroups $\Gamma' \subset \Gamma$ and all $a \in A^{\Gamma'}$, $b \in B^{\Gamma'}$.

There is a natural isomorphism

$$S^2(A \oplus B) \simeq S^2(A) \oplus (A \otimes B) \oplus S^2(B).$$

Moreover, the equality

$$\mathrm{Qtr}(a + b) = \mathrm{Qtr}(a) + \big(\mathrm{Tr}(a) \otimes \mathrm{Tr}(b) - \mathrm{Tr}(a \otimes b)\big) + \mathrm{Qtr}(b)$$

yields the decomposition

$$\mathrm{Dec}(A \oplus B) \simeq \mathrm{Dec}(A) \oplus \mathrm{Dec}(A, B) \oplus \mathrm{Dec}(B).$$

A-III. **Cup-products.** Let $1 \longrightarrow T \longrightarrow P \longrightarrow Q \longrightarrow 1$ be an exact sequence of tori. We consider the connecting maps

$$\partial_1 : H^p\big(F, \widehat{T}(i)\big) \longrightarrow H^{p+1}\big(F, \widehat{Q}(i)\big)$$

for the exact sequence $0 \longrightarrow \widehat{Q}_{\mathrm{sep}} \longrightarrow \widehat{P}_{\mathrm{sep}} \longrightarrow \widehat{T}_{\mathrm{sep}} \longrightarrow 0$ of character Γ-lattices and

$$\partial_2 : H^q\big(F, \widehat{Q}^\circ(j)\big) \longrightarrow H^{q+1}\big(F, \widehat{T}^\circ(j)\big)$$

for the dual sequence of lattices (see notation in §4b).

Lemma A.1. *Let $a \in H^p\big(F, \widehat{T}(i)\big)$ and $b \in H^q\big(F, \widehat{Q}^\circ(j)\big)$ with $i + j \leq 2$. Then $\partial_1(a) \cup b = (-1)^{p+1} a \cup \partial_2(b)$ in $H^{p+q+1}\big(F, \mathbb{Z}(i + j)\big)$, where the cup-product is defined in (4.3).*

Proof. By [3, Ch. V, Prop. 4.1], the elements $\partial_1(1_T)$ and $\partial_2(1_Q)$ in

$$H^1\big(F, \widehat{T}^\circ_{\mathrm{sep}} \otimes \widehat{Q}_{\mathrm{sep}}\big) = \mathrm{Ext}^1_\Gamma\big(\widehat{T}_{\mathrm{sep}}, \widehat{Q}_{\mathrm{sep}}\big)$$

differ by a sign. Write τ for the isomorphism induced by permutation of the factors. By the standard properties of the cup-product, we have

$$\begin{aligned}\partial_1(a) \cup b &= 1_T \cup \partial_1(a) \cup b \\ &= \partial_1(1_T) \cup a \cup b \\ &= (-1)^{pq} \tau\big(\partial_1(1_T) \cup b \cup a\big) \\ &= (-1)^{pq+1} \tau\big(\partial_2(1_Q) \cup b \cup a\big) \\ &= (-1)^{pq+1} \tau\big(1_Q \cup \partial_2(b) \cup a\big) \\ &= (-1)^{p+1} 1_Q \cup a \cup \partial_2(b) \\ &= (-1)^{p+1} a \cup \partial_2(b). \end{aligned}$$ □

A-IV. **Cosimplicial abelian groups.** Let $A^\bullet$ be a cosimplicial abelian group

$$A^0 \underset{d^1}{\overset{d^0}{\rightrightarrows}} A^1 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; A^2 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \cdots$$

and write $h_*(A^\bullet)$ for the homology groups of the associated complex of abelian groups. In particular,

$$h_0(A^\bullet) = \operatorname{Ker}\big[(d^0 - d^1) : A^0 \longrightarrow A^1\big].$$

We say that the cosimplicial abelian group $A^\bullet$ is *constant* if for every i, all the coface maps $d_j : A^{i-1} \longrightarrow A^i$, $j = 0, 1, \dots i$, are isomorphisms. In this case all the d_j are equal as $d_j = s_j^{-1} = d_{j+1}$, where the s_j are the codegeneracy maps. For a constant cosimplicial abelian group $A^\bullet$, we have $h_0(A^\bullet) = A^0$ and $h_i(A^\bullet) = 0$ for all $i > 0$. We will need the following straightforward statement.

Lemma A.2. *Let $0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow D^\bullet$ be an exact sequence of cosimplicial abelian groups with $A^\bullet$ a constant cosimplicial group. Then the sequence of groups $0 \longrightarrow A^0 \longrightarrow h_0(B^\bullet) \longrightarrow h_0(C^\bullet) \longrightarrow h_0(D^\bullet)$ is exact.*

Let H be a contravariant functor from the category of schemes over F to the category of abelian groups. We say that H is *homotopy invariant* if for every vector bundle $E \longrightarrow X$ over F, the induced map $H(X) \longrightarrow H(E)$ is an isomorphism.

For an integer $d > 0$ consider the following property of the functor H:

Property A.3. *For every closed subscheme Z of a scheme X with $\operatorname{codim}_X(Z) \geq d$, the natural homomorphism $H(X) \longrightarrow H(X \setminus Z)$ is an isomorphism.*

Let G be a linear algebraic group over a field F and choose a standard classifying G-torsor $U \longrightarrow U/G$. Let U^i denote the product of i copies of U. We have the G-torsors $U^i \longrightarrow U^i/G$.

Consider the cosimplicial abelian group $A^\bullet = H(U^\bullet/G)$ with $A^i = H(U^{i+1}/G)$ and coface maps $A^{i-1} \longrightarrow A^i$ induced by the projections $U^{i+1}/G \longrightarrow U^i/G$.

Lemma A.4. *Let H be a homotopy invariant functor satisfying Property A.3 for some d. Let $U \longrightarrow U/G$ be a standard classifying G-torsor and U' an open subset of a G-representation V'.*

1. *If* $\operatorname{codim}_{V'}(V' \setminus U') \geq d$, *then the natural homomorphism* $H(U/G) \longrightarrow H\big((U \times U')/G\big)$ *is an isomorphism.*
2. *If* $\operatorname{codim}_V(V \setminus U) \geq d$, *then the cosimplicial group* $H(U^\bullet/G)$ *is constant.*

Proof. 1. The scheme $(U \times U')/G$ is an open subset of the vector bundle $(U \times V')/G$ over U/G with complement of codimension at least d. The map in question is the composition $H(U/G) \longrightarrow H\big((U \times V')/G\big) \longrightarrow H\big((U \times U')/G\big)$ and both maps in the composition are isomorphisms since H is homotopy invariant and satisfies Property A.3.

2. By the first part of the lemma applied to the G-torsor $U^i \longrightarrow U^i/G$ and $U' = U$, the map $H(U^i/G) \longrightarrow H(U^{i+1}/G)$ induced by a projection $U^{i+1}/G \longrightarrow U^i/G$ is an isomorphism. □

By Lemma A.4, if H is a homotopy invariant functor satisfying Property A.3 for some d, then the group $H(U/G)$ does not depend on the choice of the representation V and the open set $U \subset V$ provided $\operatorname{codim}_V(V \setminus U) \geq d$. Following [33], we denote this group by $H(BG)$.

Example A.5. The split torus $T = (\mathbb{G}_m)^n$ over F acts freely on the product U of n copies of $\mathbb{A}^{r+1} \setminus \{0\}$ with $U/T \simeq (\mathbb{P}^r)^n$, i.e., BT is "approximated" by the varieties $(\mathbb{P}^r)^n$ if "$r \gg 0$." We then have $\mathrm{CH}^*(BT) = S^*(\widehat{T})$, where S^* represents the symmetric algebra and $\widehat{T}$ is the character group of T (see [12, p. 607]). In particular, $\operatorname{Pic}(BT) = \mathrm{CH}^1(BT) = \widehat{T}$. More generally, by the Künneth formula [13, Prop. 3.7],

$$H^*_{\mathrm{Zar}}(BT, \mathcal{K}_*) \simeq \mathrm{CH}^*(BT) \otimes K_*(F) \simeq S^*(\widehat{T}) \otimes K_*(F),$$

where $K_n(F)$ is the Quillen K-group of F and $\mathcal{K}_n$ is the Zariski sheaf associated to the presheaf $U \longmapsto K_n(U)$.

A-V. **Étale cohomology.** For a scheme X and a closed subscheme $Z \subset X$ we write $H^*_Z\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)$ for the étale cohomology group of X with support in Z and values in $\mathbb{Q}/\mathbb{Z}(j)$ [29, Ch. III, §1]. Write $X^{(i)}$ for the set of points in X of codimension i. For a point $x \in X^{(1)}$ set

$$H^*_x\big(X, \mathbb{Q}/\mathbb{Z}(j)\big) = \operatorname*{colim}_{x \in U} H^*_{\overline{\{x\}} \cap U}\big(U, \mathbb{Q}/\mathbb{Z}(j)\big),$$

where the colimit is taken over all open subsets $U \subset X$ containing x. If X is a variety, write

$$\partial_x : H^*\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^{*+1}_x\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)$$

for the residue homomorphisms arising from the *coniveau spectral sequence* [7, 1.2]

$$E_1^{p,q} = \coprod_{x \in X^{(p)}} H^{p+q}_x\big(X, \mathbb{Q}/\mathbb{Z}(j)\big) \Rightarrow H^{p+q}\big(X, \mathbb{Q}/\mathbb{Z}(j)\big). \tag{A.1}$$

Let $f : Y \longrightarrow X$ be a dominant morphism of varieties over F, $y \in Y^{(1)}$, and $x = f(y)$. If $x \in X^{(1)}$, there is a natural homomorphism

$$f_y^* : H_x^*\big(X, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H_y^*\big(Y, \mathbb{Q}/\mathbb{Z}(j)\big).$$

The following lemma is straightforward.

Lemma A.6. *Let $f : Y \longrightarrow X$ be a dominant morphism of varieties over F, $y \in Y^{(1)}$ and $x = f(y)$.*

(1) *If x is the generic point of X, then the composition*

$$H^*\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big) \xrightarrow{f^*} H^*\big(F(Y), \mathbb{Q}/\mathbb{Z}(j)\big) \xrightarrow{\partial_y} H_y^{*+1}\big(Y, \mathbb{Q}/\mathbb{Z}(j)\big)$$

is trivial.

(2) *If $x \in X^{(1)}$, the diagram*

$$\begin{array}{ccc} H^*\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big) & \xrightarrow{\partial_x} & H_x^{*+1}\big(X, \mathbb{Q}/\mathbb{Z}(j)\big) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f_y^*} \\ H^*\big(F(Y), \mathbb{Q}/\mathbb{Z}(j)\big) & \xrightarrow{\partial_y} & H_y^{*+1}\big(Y, \mathbb{Q}/\mathbb{Z}(j)\big). \end{array}$$

is commutative.

Lemma A.7. *Let X be a geometrically irreducible variety, $Z \subset X$ a closed subvariety of codimension* 1*, and x the generic point of Z. Let P be a variety over F such that $P(K)$ is dense in P for every field extension K/F with K infinite, and let y be the generic point of $Z \times P$ in $Y := X \times P$. Then the homomorphism $f_y^* : H_x^*\big(X, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H_y^*\big(Y, \mathbb{Q}/\mathbb{Z}(j)\big)$ induced by the projection $f : Y \longrightarrow X$ is injective.*

Proof. Assume first that the field F is infinite. An element $\alpha \in H_x^*\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)$ is represented by an element $h \in H_{Z\cap U}^*\big(U, \mathbb{Q}/\mathbb{Z}(j)\big)$ for a nonempty open set $U \subset X$ containing x. If α belongs to the kernel of

$$f_y^* : H_x^*\big(X, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H_y^*\big(Y, \mathbb{Q}/\mathbb{Z}(j)\big),$$

then there is an open subset $W \subset U \times P$ containing y such that h belongs to the kernel of the composition

$$g : H_{Z\cap U}^*\big(U, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H_{(Z\cap U)\times P}^*\big(U\times P, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H_{(Z\times P)\cap W}^*\big(W, \mathbb{Q}/\mathbb{Z}(j)\big).$$

As F is infinite, by the assumption on P, there is a rational point $t \in P$ in the image of the dominant composition $(Z\times P)\cap W \hookrightarrow Z\times P \longrightarrow P$. We have $(U\times t)\cap W = U'\times t$ for an open subset $U' \subset U$ such that $x \in U'$. Composing g with the homomorphism $H_{(Z\times P)\cap W}^*\big(W, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H_{Z\cap U'}^*\big(U', \mathbb{Q}/\mathbb{Z}(j)\big)$ induced by the morphism $(U', Z\cap U') \longrightarrow (W, (Z\times P)\cap W)$, $u \longmapsto (u, t)$, we see that h belongs to the kernel of the restriction homomorphism $H_{Z\cap U}^*\big(U, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H_{Z\cap U'}^*\big(U', \mathbb{Q}/\mathbb{Z}(j)\big)$, hence the image of α in $H_x^*\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)$ is trivial.

Suppose now that F is a finite field. Choose a prime integer p and an infinite algebraic pro-p-extension L/F. By the first part of the proof, the statement holds for the variety X_L over L. By the restriction-corestriction argument,

$\operatorname{Ker}(f_y^*)$ is a p-primary torsion group. Since this holds for every prime p, we have $\operatorname{Ker}(f_y^*)=0$. $\square$

Corollary A.8. *Let G be a linear algebraic group over F, let $E \longrightarrow X$ be a G-torsor over a geometrically irreducible variety X with E a G-rational variety and consider the first projection $p : E^2/G \longrightarrow X$. Let $x \in X$ and $y \in E^2/G$ be points of codimension 1 such that $p(y) = x$. Then the homomorphism $p_y^* : H_x^*\big(X, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H_y^*\big(E^2/G, \mathbb{Q}/\mathbb{Z}(j)\big)$ is injective.*

Proof. Choose a linear G-space V and a nonempty G-variety U that is G-isomorphic to open subschemes of E and V. We can replace the variety E^2/G by $(E\times U)/G$, an open subscheme in the vector bundle $(E\times V)/G$ over X. Shrinking X around x we may assume that the vector bundle is trivial, i.e., $(E\times U)/G$ is isomorphic to an open subscheme in $X\times V$. The statement then follows from Lemma A.7. $\square$

Proposition A.9. *In the conditions of Corollary A.8, let $h \in H^*\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big)_{\mathrm{bal}}$. Then $\partial_x(h) = 0$ for every point $x \in X$ of codimension 1.*

Proof. Let $y \in E^2/G$ be the point of codimension 1 such that $p_1(y) = x$. As $p_2(y)$ is the generic point of X, by Lemma A.6(1), $\partial_y(h') = 0$, where $h' = p_1^*(h) = p_2^*(h)$ in $H^*\big(F(E^2/G), \mathbb{Q}/\mathbb{Z}(j)\big)$. It follows from Lemma A.6(2) that $\partial_x(h)$ is in the kernel of $(p_1)_y^* : H_x^*(X, \mathbb{Q}/\mathbb{Z}(j)) \longrightarrow H_y^*(E^2/G, \mathbb{Q}/\mathbb{Z}(j))$ and hence is trivial by Corollary A.8. $\square$

The sheaf $\mathcal{H}^*\big(\mathbb{Q}/\mathbb{Z}(j)\big)$ defined in §3 has a flasque resolution related to the Cousin complex by [7, §2] (for the p-components with $p \neq \operatorname{char} F$) and [17, Th. 1.4] (for the p-component with $p = \operatorname{char} F > 0$):

$$0 \longrightarrow \mathcal{H}^n\big(\mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow \coprod_{x\in X^{(0)}} i_{x*} H_x^n\big(X, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow \coprod_{x\in X^{(1)}} i_{x*} H_x^{n+1}\big(X, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow \cdots,$$

where $i_x : \operatorname{Spec} F(x) \longrightarrow X$ are the canonical morphisms. In particular, we have:

Proposition A.10. *Let X be a smooth variety over F. The sequence*

$$0 \longrightarrow H^0_{\mathrm{Zar}}\big(X, \mathcal{H}^*(\mathbb{Q}/\mathbb{Z}(j))\big) \longrightarrow H^*\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big) \xrightarrow{\partial} \coprod_{x\in X^{(1)}} H_x^{*+1}\big(X, \mathbb{Q}/\mathbb{Z}(j)\big),$$

where $\partial = \coprod \partial_x$, is exact.

Proposition A.11. *Let X be a smooth variety over F and $x \in X$. The sequence*

$$0 \longrightarrow H^*\big(O_{X,x}, \mathbb{Q}/\mathbb{Z}(j)\big) \longrightarrow H^*\big(F(X), \mathbb{Q}/\mathbb{Z}(j)\big) \xrightarrow{\partial} \coprod_{\substack{x'\in X^{(1)}\\ x'\in\overline{\{x\}}}} H_{x'}^{*+1}\big(X, \mathbb{Q}/\mathbb{Z}(j)\big)$$

is exact.

Appendix B. Spectral sequences

B-I. **Hochschild-Serre spectral sequence.** Let

$$\mathcal{A} \xrightarrow{W} \mathcal{B} \xrightarrow{V} \mathcal{C}$$

be additive left exact functors between abelian categories with enough injective objects. If W takes injective objects to V-acyclic ones, there is a spectral sequence

$$E_2^{p,q} = R^pV\big(R^qW(A)\big) \Rightarrow R^{p+q}(VW)(A)$$

for every complex A in $\mathcal{A}$ bounded from below.

We have exact triangles in the derived category of $\mathcal{B}$:

$$\tau_{\leq n}RW(A) \longrightarrow RW(A) \longrightarrow \tau_{\geq n+1}RW(A) \longrightarrow \tau_{\leq n}RW(A)[1], \tag{B.1}$$

$$\tau_{\leq n-1}RW(A) \longrightarrow \tau_{\leq n}RW(A) \longrightarrow R^nW(A)[-n] \longrightarrow \tau_{\leq n-1}RW(A)[1]. \tag{B.2}$$

The filtration on $R^n(VW)(A)$ is defined by

$$F^jR^n(VW)(A) = \mathrm{Im}\big(R^nV(\tau_{\leq(n-j)}RW(A)) \longrightarrow R^nV(RW(A)) = R^n(VW)(A)\big).$$

As $\tau_{\geq n+1}RW(A)$ is acyclic in degrees $\leq n$, the morphism

$$R^nV\big(\tau_{\leq n}RW(A)\big) \longrightarrow R^nV(RW(A)) = R^n(VW)(A)$$

is an isomorphism, in particular, $F^0R^n(VW)(A) = R^n(VW)(A)$.

The *edge* homomorphism is defined as the composition

$$R^n(VW)(A) \xrightarrow{\sim} R^nV\big(\tau_{\leq n}RW(A)\big) \longrightarrow R^nV\big(R^nW(A)[-n]\big) = V\big(R^nW(A)\big).$$

Moreover, the kernel $F^1R^n(VW)(A)$ of the edge homomorphism is isomorphic to $R^nV\big(\tau_{\leq n-1}RW(A)\big)$. We define the morphism d_n as the composition

$$d_n : F^1R^n(VW)(A) \longrightarrow R^nV\big(R^{n-1}W(A)[-n+1]\big) = R^1V\big(R^{n-1}W(A)\big) = E_2^{1,n-1}.$$

B-II. **First spectral sequence.** Let X be a smooth variety over a field F. We have the functors

$$\mathrm{Sheaves}_{\acute{e}t}(X) \xrightarrow{q_*} \mathrm{Sheaves}_{\acute{e}t}(F) \xrightarrow{V} Ab,$$

where q_* is the push-forward map for the structure morphism $q : X \longrightarrow \mathrm{Spec}(F)$ and $V(M) = H^0(F, M)$.

Consider the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p\big(F, H^q(X_{\mathrm{sep}}, \mathbb{Z}(2)\big) \Rightarrow H^{p+q}\big(X, \mathbb{Z}(2)\big). \tag{B.3}$$

Set $\Delta(i) := Rq_*\big(\mathbb{Z}(i)\big)$ for $i = 1$ or 2. Then $\Delta(i)$ is the complex of étale sheaves on F concentrated in degrees ≥ 1. The j^{th} term $F^jH^n(X, \mathbb{Z}(i))$ of the filtration on $H^n\big(X, \mathbb{Z}(i)\big)$ coincides with the image of the canonical homomorphism

$$H^n\big(F, \tau_{\leq(n-j)}\Delta(i)\big) \longrightarrow H^n\big(F, \Delta(i)\big) = H^n\big(X, \mathbb{Z}(i)\big).$$

Let M be a Γ-lattice viewed as an étale sheaf over F. Note that there are canonical isomorphisms

$$H^*\big(F, M^\circ \otimes \Delta(i)\big) = \mathrm{Ext}^*_F\big(M, \Delta(i)\big) = \mathrm{Ext}^*_X\big(q^*M, \mathbb{Z}(i)\big), \tag{B.4}$$

where $M^\circ := \mathrm{Hom}(M,\mathbb{Z})$ is the dual lattice.

Consider also the following product map

$$\mathbb{Z}(1)\otimes^L\Delta(1) \longrightarrow Rq_*\big(q^*\mathbb{Z}(1)\otimes^L\mathbb{Z}(1)\big) \longrightarrow Rq_*\big(\mathbb{Z}(1)\otimes^L\mathbb{Z}(1)\big) \longrightarrow Rq_*\big(\mathbb{Z}(2)\big).$$

The complex $\mathbb{Z}(1)\otimes^L\tau_{\le 2}\Delta(1)$ is trivial in degrees >3, hence we have a commutative diagram

$$\begin{tikzcd}
\mathbb{Z}(1)\otimes^L\tau_{\le 2}\Delta(1) \arrow[r, "prod"] \arrow[d] & \tau_{\le 3}Rq_*\big(\mathbb{Z}(2)\big) \arrow[d] \arrow[r, phantom, "="] & \tau_{\le 3}\Delta(2) \\
\mathbb{Z}(1)\otimes^L\Delta(1) \arrow[r, "prod"] & Rq_*\big(\mathbb{Z}(2)\big) \arrow[r, phantom, "="] & \Delta(2).
\end{tikzcd}$$

There are canonical morphisms from (B.2):

$$h_2 : \tau_{\le 2}\Delta(1)[2] \longrightarrow H^2\big(X_{\mathrm{sep}},\mathbb{Z}(1)\big),$$

$$h_3 : \tau_{\le 3}\Delta(2)[3] \longrightarrow H^3\big(X_{\mathrm{sep}},\mathbb{Z}(2)\big).$$

Consider an element

$$\delta\in H^1(F, M\otimes F_{\mathrm{sep}}^\times) = \mathrm{Ext}^1_F(M^\circ,\mathbb{G}_{m,F}) = \mathrm{Ext}^2_F\big(M^\circ,\mathbb{Z}(1)\big),$$

and view δ as a morphism

$$\delta : M^\circ \longrightarrow \mathbb{Z}(1)[2]$$

in $D^+\big(\mathrm{Sheaves}_{\acute{e}t}(F)\big)$.

The following diagram

$$\begin{tikzcd}
M^\circ\otimes\Delta(1)[2] \arrow[r, "\delta\otimes 1"] & \mathbb{Z}(1)\otimes^L\Delta(1)[4] \arrow[r, "prod"] & \Delta(2)[4] \\
M^\circ\otimes\tau_{\le 2}\Delta(1)[2] \arrow[r, "\delta\otimes 1"] \arrow[u, "(1\otimes i_2)[2]"] \arrow[d, "1\otimes h_2"'] & \mathbb{Z}(1)\otimes^L\tau_{\le 2}\Delta(1)[4] \arrow[r, "prod"] \arrow[u, "(1\otimes i_2)[4]"] \arrow[d, "1\otimes h_2"'] & \tau_{\le 3}\Delta(2)[4] \arrow[u, "(i_3)[4]"] \arrow[d, "h_3"'] \\
M^\circ\otimes H^2\big(X_{\mathrm{sep}},\mathbb{Z}(1)\big) \arrow[r, "\delta\otimes 1"] & \mathbb{Z}(1)\otimes^L H^2\big(X_{\mathrm{sep}},\mathbb{Z}(1)\big)[2] \arrow[r, "prod"] & H^3\big(X_{\mathrm{sep}},\mathbb{Z}(2)\big)[1],
\end{tikzcd}$$

where $i_2 : \tau_{\le 2}\Delta(1)\longrightarrow\Delta(1)$ and $i_3 : \tau_{\le 3}\Delta(2)\longrightarrow\Delta(2)$ are natural morphisms, is commutative.

By (B.4), we have

$$H^0\big(F, M^\circ\otimes\Delta(1)[2]\big) = \mathrm{Ext}^2_F\big(M,\Delta(1)\big) = \mathrm{Ext}^2_X\big(q^*M,\mathbb{Z}(1)\big).$$

Furthermore, the diagram above yields a commutative square

$$\begin{tikzcd}
\mathrm{Ext}^2_X\big(q^*M,\mathbb{Z}(1)\big) \arrow[r, "q^*(\delta)\,\cup\,-"] \arrow[d, "d_2"'] & F^1H^4\big(X,\mathbb{Z}(2)\big) \arrow[d, "d_4"'] \\
\mathrm{Hom}_\Gamma\big(M, H^2(X_{\mathrm{sep}},\mathbb{Z}(1)\big) \arrow[r, "j"] & H^1\big(F, H^3(X_{\mathrm{sep}},\mathbb{Z}(2))\big),
\end{tikzcd}$$

where d_2 is the edge map coming from the spectral sequence

$$\mathrm{Ext}^p_F\big(M, H^q(X_{\mathrm{sep}}, \mathbb{Z}(1))\big) \Rightarrow \mathrm{Ext}^{p+q}_X\big(q^*M, \mathbb{Z}(1)\big) \tag{B.5}$$

and j coincides with the composition

$$\begin{aligned}\mathrm{Hom}_\Gamma\big(M, H^2(X_{\mathrm{sep}}, \mathbb{Z}(1)\big) = H^0\big(F, M^\circ \otimes H^2(X_{\mathrm{sep}}, \mathbb{Z}(1))\big) &\xrightarrow{\delta \cup -} \\ H^1\big(F, F^\times_{\mathrm{sep}} \otimes H^2(X_{\mathrm{sep}}, \mathbb{Z}(1))\big) &\xrightarrow{\rho} H^1\big(F, H^3\big(X_{\mathrm{sep}}, \mathbb{Z}(2)\big)\big),\end{aligned}$$

with ρ given by the product map.

Now suppose the group $H^2\big(X_{\mathrm{sep}}, \mathbb{Z}(1)\big)$, which is canonically isomorphic to $\mathrm{Pic}(X_{\mathrm{sep}})$, is a lattice. Let $M = \mathrm{Pic}(X_{\mathrm{sep}})$ and consider the torus T over F with $\widehat{T}_{\mathrm{sep}} = M$. It follows that

$$\delta \in H^1(F, T^\circ) = H^1(F, \widehat{T}_{\mathrm{sep}} \otimes F^\times_{\mathrm{sep}}) = H^2\big(F, \widehat{T}_{\mathrm{sep}} \otimes \mathbb{Z}(1)\big),$$

where T° is the dual torus. Note that $\delta \cup 1_M = \delta$, where

$$1_M \in H^0\big(F, M^\circ \otimes H^2(X_{\mathrm{sep}}, \mathbb{Z}(1))\big) = \mathrm{End}_\Gamma(M)$$

is the identity.

The top map in the last diagram is given by the pairing

$$\begin{aligned} H^1(X, T^0) \otimes H^1(X, T) &\longrightarrow F^1H^4\big(X, \mathbb{Z}(2)\big), \\ a \otimes b &\longmapsto a \cup b \end{aligned} \tag{B.6}$$

defined as the cup-product in (4.3),

$$H^2\big(X, \widehat{T}(1)\big) \otimes H^2\big(X, \widehat{T}^\circ(1)\big) \longrightarrow F^1H^4\big(X, \mathbb{Z}(2)\big),$$

if we identify $\mathrm{Ext}^2_X(q^*M, \mathbb{Z}(1))$ with $H^2\big(X, \widehat{T}^\circ(1)\big) = H^1(X, T)$.

In this case, the homomorphism

$$\rho : H^1(F, T^\circ) \longrightarrow H^1\big(F, H^3\big(X_{\mathrm{sep}}, \mathbb{Z}(2)\big)\big) \tag{B.7}$$

is given by the product homomorphism

$$T^\circ(F_{\mathrm{sep}}) = F^\times_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}} = F^\times_{\mathrm{sep}} \otimes \mathrm{Pic}(X_{\mathrm{sep}}) \longrightarrow H^3(X_{\mathrm{sep}}, \mathbb{Z}(2)).$$

A T-torsor $E \longrightarrow X$ is called *universal* if the class of E in $H^1(X, T) = \mathrm{Ext}^2_X\big(q^*M, \mathbb{Z}(1)\big)$ satisfies $d_2\big([E]\big) = 1_M$ (see [11]).

Commutativity of the previous diagram gives:

Proposition B.1. *Let X be a smooth variety over F such that $\mathrm{Pic}(X_{\mathrm{sep}})$ is a lattice. Let T be the torus over F satisfying $\widehat{T}_{\mathrm{sep}} = \mathrm{Pic}(X_{\mathrm{sep}})$ and let E be a universal T-torsor over X with the class $[E] \in H^1(X, T)$. Then for every $\delta \in H^1(F, T^\circ)$, we have*

$$d_4\big(q^*(\delta) \cup [E]\big) = \rho(\delta),$$

where $d_4 : F^1H^4\big(X, \mathbb{Z}(2)\big) \longrightarrow H^1\big(F, H^3(X_{\mathrm{sep}}, \mathbb{Z}(2))\big)$ is the map induced by the Hochschild-Serre spectral sequence (B.3) and the cup-product is taken for the pairing (B.6).

B-III. **Second spectral sequence.** We assume that $H^3\big(X_{\mathrm{sep}}, \mathbb{Z}(2)\big) = 0$, hence in particular $E_2^{0,3} = 0$ in the spectral sequence (B.3) and so $E_\infty^{2,2} \subset E_2^{2,2}$. Therefore, we have a canonical map

$$e_4 : F^2H^4\big(X, \mathbb{Z}(2)\big) \longrightarrow E_\infty^{2,2} \hookrightarrow E_2^{2,2} = H^2\big(F, H^2(X_{\mathrm{sep}}, \mathbb{Z}(2)\big).$$

Let N be a Γ-lattice. Consider an element

$$\gamma \in H^2(F, N \otimes F_{\mathrm{sep}}^\times) = \mathrm{Ext}^2_F(N^\circ, \mathbb{G}_{m,F}) = \mathrm{Ext}^3_F\big(N^\circ, \mathbb{Z}(1)\big),$$

and view γ as a morphism

$$\gamma : N^\circ \longrightarrow \mathbb{Z}(1)[3]$$

in $D^+\big(\mathrm{Sheaves}_{\acute{e}t}(F)\big)$.

As above, the commutative diagram

$$\begin{tikzcd}
N^\circ \otimes \Delta(1)[1] \arrow[r, "\gamma\otimes 1"] & \mathbb{Z}(1) \otimes^L \Delta(1)[4] \arrow[r, "prod"] & \Delta(2)[4] \\
N^\circ \otimes \tau_{\le 1}\Delta(1)[1] \arrow[r, "\gamma\otimes 1"] \arrow[u, "(1\otimes i_1)[1]"] \arrow[d, "1\otimes h_1"'] & \mathbb{Z}(1) \otimes^L \tau_{\le 1}\Delta(1)[4] \arrow[r, "prod"] \arrow[u, "(1\otimes i_1)[4]"] \arrow[d, "1\otimes h_1"'] & \tau_{\le 2}\Delta(2)[4] \arrow[u, "(i_2)[4]"] \arrow[d, "h_2"'] \\
N^\circ \otimes H^1\big(X_{\mathrm{sep}}, \mathbb{Z}(1)\big) \arrow[r, "\gamma\otimes 1"] & \mathbb{Z}(1) \otimes^L H^1\big(X_{\mathrm{sep}}, \mathbb{Z}(1)\big)[3] \arrow[r, "prod"] & H^2\big(X_{\mathrm{sep}}, \mathbb{Z}(2)\big)[2],
\end{tikzcd}$$

where i_1, i_2, h_1 and h_2 are defined in a similar fashion as in §B-II, yields a commutative square

$$\begin{tikzcd}
\mathrm{Ext}^1_X(q^*N, \mathbb{Z}(1)) \arrow[r, "q^*(\gamma)\,\cup\,-"] \arrow[d, "d_1"'] & F^2H^4\big(X, \mathbb{Z}(2)\big) \arrow[d, "e_4"] \\
\mathrm{Hom}_\Gamma\big(N, H^1(X_{\mathrm{sep}}, \mathbb{Z}(1)\big) \arrow[r, "k"] & H^2\big(F, H^2(X_{\mathrm{sep}}, \mathbb{Z}(2))\big),
\end{tikzcd}$$

where d_1 is the edge map coming from the spectral sequence

$$\mathrm{Ext}^p_F\big(N, H^q(X_{\mathrm{sep}}, \mathbb{Z}(1))\big) \Rightarrow \mathrm{Ext}^{p+q}_X\big(q^*N, \mathbb{Z}(1)\big)$$

and k coincides with the composition

$$\mathrm{Hom}_\Gamma\big(N, H^1(X_{\mathrm{sep}}, \mathbb{Z}(1)\big) = H^0\big(F, N^\circ \otimes H^1(X_{\mathrm{sep}}, \mathbb{Z}(1))\big) \xrightarrow{\gamma\,\cup\,-}$$
$$H^2\big(F, F_{\mathrm{sep}}^\times \otimes H^1(X_{\mathrm{sep}}, \mathbb{Z}(1))\big) \longrightarrow H^2\big(F, H^2\big(X_{\mathrm{sep}}, \mathbb{Z}(2)\big)\big)$$

with the last homomorphism given by the product map.

Suppose N is a Γ-lattice in $F_{\mathrm{sep}}[X]^\times$ such that the composition $N \hookrightarrow F_{\mathrm{sep}}[X]^\times \longrightarrow F_{\mathrm{sep}}[X]^\times / F_{\mathrm{sep}}^\times$ is an isomorphism. Consider the torus Q with $\widehat{Q}_{\mathrm{sep}} = N$, so that $\gamma \in H^2(F, Q^\circ)$.

Note that $\gamma \cup i_N = \gamma$, where

$$i_N \in H^0\big(F, N^\circ \otimes H^1(X_{\mathrm{sep}}, \mathbb{Z}(1))\big) = \mathrm{Hom}_\Gamma(N, F_{\mathrm{sep}}[X]^\times)$$

is the embedding.

The top map in the previous diagram is given by the pairing

$$\begin{aligned} H^2(X,Q^0)\otimes H^0(X,Q) &\longrightarrow F^2H^4\big(X,\mathbb{Z}(2)\big) \\ a\otimes b &\longmapsto a\cup b, \end{aligned} \tag{B.8}$$

defined as the cup-product in (4.3),

$$H^3\big(X,\widehat{Q}(1)\big)\otimes H^1\big(X,\widehat{Q}^\circ(1))\big)\longrightarrow H^4\big(X,\mathbb{Z}(2)\big),$$

if we identify $\mathrm{Ext}^1_X\big(q^*N,\mathbb{Z}(1)\big)$ with $H^1\big(X,\widehat{Q}^\circ(1)\big)=H^0(X,Q)$.

The inclusion of $\widehat{Q}_{\mathrm{sep}}$ into $F_{\mathrm{sep}}[X]^\times$ yields a morphism $\varepsilon: X\longrightarrow Q$ that can be viewed as an element of $H^0(X,Q)$. Consider the map

$$\mu: H^2(F,Q^\circ)\longrightarrow H^2\big(F,H^2\big(X_{\mathrm{sep}},\mathbb{Z}(2)\big)\big) \tag{B.9}$$

given by composition with the product homomorphism

$$Q^\circ(F_{\mathrm{sep}})=F^\times_{\mathrm{sep}}\otimes\widehat{Q}_{\mathrm{sep}}\longrightarrow F^\times_{\mathrm{sep}}\otimes H^1\big(X_{\mathrm{sep}},\mathbb{Z}(1)\big)\longrightarrow H^2\big(X_{\mathrm{sep}},\mathbb{Z}(2)\big).$$

We have proved:

Proposition B.2. *Let X be a smooth variety over F such that $H^3\big(X_{\mathrm{sep}},\mathbb{Z}(2)\big)=0$. Let N be a Γ-lattice in $F_{\mathrm{sep}}[X]^\times$ such that the composition $N\hookrightarrow F_{\mathrm{sep}}[X]^\times\longrightarrow F_{\mathrm{sep}}[X]^\times/F^\times_{\mathrm{sep}}$ is an isomorphism. Let Q be the torus over F satisfying $\widehat{Q}_{\mathrm{sep}}=N$. Then for every $\gamma\in H^2(F,Q^\circ)$, we have*

$$e_4\big(q^*(\gamma)\cup\varepsilon\big)=\mu(\gamma),$$

where $e_4: F^2H^4\big(X,\mathbb{Z}(2)\big)\longrightarrow H^2\big(F,H^2(X_{\mathrm{sep}},\mathbb{Z}(2))\big)$ is the map induced by the Hochschild-Serre spectral sequence (B.3) and the cup-product is taken for the pairing (B.8).

B-IV. **Relative étale cohomology.** Let X be a smooth variety over F. Following B. Kahn [23, §3], we define the relative étale cohomology groups as follows. Recall that $\Delta(i)=Rq_*\big(\mathbb{Z}(i)\big)$ for $i=1$ and 2, where $q: X\longrightarrow \mathrm{Spec}(F)$ is the structure morphism, and let $\Delta'(i)$ be the cone of the natural morphism $\mathbb{Z}(i)\longrightarrow\Delta(i)$ in $D_+\big(\mathrm{Sheaves}_{\acute{e}t}(F)\big)$. Define

$$H^*\big(X/F,\mathbb{Z}(2)\big):=H^*\big(F,\Delta'(2)\big).$$

(Note that our indexing is different from the one in [23, §3].)

There is an infinite exact sequence

$$\ldots\longrightarrow H^i\big(F,\mathbb{Z}(2)\big)\longrightarrow H^i\big(X,\mathbb{Z}(2)\big)\longrightarrow H^i\big(X/F,\mathbb{Z}(2)\big)\longrightarrow H^{i+1}\big(F,\mathbb{Z}(2)\big)\longrightarrow\cdots$$

If X has a rational point, we have

$$H^i\big(X/F,\mathbb{Z}(2)\big)=\overline{H}^i\big(X,\mathbb{Z}(2)\big):=H^i\big(X,\mathbb{Z}(2)\big)/H^i\big(F,\mathbb{Z}(2)\big).$$

There is a Hochschild-Serre type spectral sequence [23, §3]

$$E_2^{p,q}=H^p\big(F,H^q(X_{\mathrm{sep}}/F_{\mathrm{sep}},\mathbb{Z}(2))\big)\Rightarrow H^{p+q}\big(X/F,\mathbb{Z}(2)\big), \tag{B.10}$$

and we have by [23, Lemma 3.1] that

$$H^q\big(X_{\mathrm{sep}}/F_{\mathrm{sep}}, \mathbb{Z}(2)\big) = \begin{cases} 0, & \text{if } q \leq 0; \\ \text{uniquely divisible group}, & \text{if } q = 1; \\ \overline{H}^0_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2), & \text{if } q = 2; \\ H^1_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2), & \text{if } q = 3. \end{cases}$$

It follows that $E_2^{p,q} = 0$ if $q \leq 1$ and $p > 0$. Comparing the spectral sequences (B.3) and (B.10), by Proposition B.1 we have:

Proposition B.3. *Let X be a smooth variety over F such that $X(F) \neq \emptyset$. If $H^0_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2) = K_2(F_{\mathrm{sep}})$, then the spectral sequence (B.10) yields an exact sequence*

$$0 \longrightarrow H^1\big(F, H^1_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2)\big) \xrightarrow{\alpha} \overline{H}^4\big(X, \mathbb{Z}(2)\big) \longrightarrow \overline{H}^4\big(X_{\mathrm{sep}}, \mathbb{Z}(2)\big)^{\Gamma} \longrightarrow H^2\big(F, H^1_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2)\big).$$

If, moreover, the group $\mathrm{Pic}(X_{\mathrm{sep}})$ *is a lattice and T is the torus over F such that $\widehat{T}_{\mathrm{sep}} = \mathrm{Pic}(X_{\mathrm{sep}})$, then $\alpha\big(\rho(\delta)\big) = q^*(\delta) \cup [E]$ for every $\delta \in H^1(F, T^\circ)$, where ρ is defined in (B.7) and E is a universal T-torsor over X.*

Comparing the spectral sequences (B.3) and (B.10), by Proposition B.2 we have:

Proposition B.4. *Let X be a smooth variety over F such that $X(F) \neq \emptyset$. If $H^1_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2) = 0$, then the spectral sequence (B.10) yields an exact sequence*

$$0 \longrightarrow H^2\big(F, \overline{H}^0_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2)\big) \xrightarrow{\beta} \overline{H}^4\big(X, \mathbb{Z}(2)\big) \longrightarrow \overline{H}^4\big(X_{\mathrm{sep}}, \mathbb{Z}(2)\big)^{\Gamma} \longrightarrow H^3\big(F, \overline{H}^0_{\mathrm{Zar}}(X_{\mathrm{sep}}, \mathcal{K}_2)\big).$$

If N is a Γ-lattice in $F_{\mathrm{sep}}[X]^\times$ such that the composition $N \hookrightarrow F_{\mathrm{sep}}[X]^\times \longrightarrow F_{\mathrm{sep}}[X]^\times / F_{\mathrm{sep}}^\times$ is an isomorphism and Q is the torus over F satisfying $\widehat{Q}_{\mathrm{sep}} = N$, then $\beta\big(\mu(\gamma)\big) = q^*(\gamma) \cup \varepsilon$ for every $\gamma \in H^2(F, Q^\circ)$, where μ is defined in (B.9) and $\varepsilon \in H^0(X, Q)$ is given by the inclusion of $\widehat{Q}_{\mathrm{sep}}$ into $F_{\mathrm{sep}}[X]^\times$.

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