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THE GROUP OF K_1-ZERO-CYCLES ON SEVERI-BRAUER VARIETIES

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For any algebraic variety X of dimension d over a field F, one can define the following complex [7]:

$$\bigcup_{x\in X^0} K_nF(x) \to \bigcup_{x\in X^1} K_{n-1}F(x) \to \ldots \to \bigcup_{x\in X^d} K_{n-d}F(x)$$

where X^i is the set of points of codimension i in X. Cohomology groups of this complex we'll denote by $H^i(X,K_n)$ and call K-cohomology groups. In particular, the group $H^i(X,K_i)$ coincides with the Chow group of the cycles of codimension i [7]. The group of K_n-zero-cycles $H^d(X,K_{n+d})$ we'll denote by $H_0(X,K_n)$.

Let X be a Severi-Brauer variety associated with a central simple F-algebra D [2]. In the case when the index of D is a prime number, some K-cohomology groups were computed in [3]. The group of zero-cycles $H_0(X,K_0)$ was computed in [5] for any Severi-Brauer variety X. The present paper is devoted to the computation of the group $H_0(X,K_1)$ also for any Severi-Brauer variety X.

For any $n \geq 0$ we construct a homomorphism

$$p_n: H_0(X,K_n) \to K_nD.$$

The result of Panin mentioned above shows that p_0 is an isomorphism. It is not difficult to show that for $n \geq 3$ in general p_n is neither injective nor surjective. The main result of the present paper is the proof of bijectivity of p_1. It seems reasonable that p_2 is also always an isomorphism. (At least this is true for one-dimensional Severi-Brauer varieties).

The paper is organized as follows. In the first section the technique of specialization is developed. In Section 2 we define the homomorphism p_n (the definition of p_0 and p_1 is possible without using the higher algebraic K-theory). The rest of the paper is devoted to the construction of the inverse map to p_1 which at first is defined with help of the technique of specialization of some "dense" subset (Section 3) and then is extended to the whole group K_1D.

Some words about notation. If X is a variety over a field F, D is any F-algebra then for any commutative F-algebra B we write:

$$X_B = X x_{\mathrm{Spec} F} \mathrm{Spec} B, D_B = D \otimes_F B.$$

1. Specialization

In this section we develop the technique which will be used in consequence. Let X be an algebraic variety over a field F, R be an F-algebra which is a discrete valuation ring with residue field k and fraction field K, $\pi \in R$ be any prime element, and D be a central simple F-algebra. We construct the homomorphisms of specialization in the following three situations:
1. The category of coherent X_K-modules $M(X_K)$ is equivalent to the factor category $M(X_R)/B$, where B is the full subcategory in $M(X_A)$ consisting of the sheaves with support in $X_k \subset X_R$ [1]. Hence we can define the following connecting homomorphisms [7]:

$$\partial : K'_{*+1}(X_K) \to K'_*(B) = K'_*(X_k).$$

The composition

$$s_\pi : K'_*(X_K) \to K'_{*+1}(X_K) \xrightarrow{\partial} = K'_*(X_k).$$

where the first homomorphism is the multiplication by the inverse image of the prime element π in the map $K_1(K) \to K_1(X_k)$ is called the specialization homomorphism.

2. The category of finitely generated D_K-modules D_K-mod is equivalent to the factor category D_R- mod$/C$, where C is the full subcategory in D_R-mod, consisting of all torsion D_R-modules. Hence we can define the following connecting homomorphism [7]:

$$\partial: K_{*+1} + (D_k) \to K_*(C) = K_*(D_k).$$

The composition

$$s_\pi: K_*(D_K) \to K_{*+1}(D_K) \to = K_*(D_k),$$

where the first homomorphism is the multiplication by the prime element π is also called the specialization homomorphism.

3. The exact sequence of complexes

$$0 \to \bigcup_{x \in X_k{}^{*+1}} K_*F(x) \to \bigcup_{x \in X_{R^*}} K_*F(x) \to \bigcup_{x \in X_{K^*}} K_*F(x) \to 0$$

induces the following connecting homomorphism

$$\partial: H^i(X_K, K_{*+1}) \to H^i(X_k, K_*).$$

The composition

$$s_\pi: H^i(X_K, K_*) \to H^i(X_K, K_{*+1}) \xrightarrow{\partial} H^i(X_k, K_*),$$

where the first homomorphism is the multiplication by $\pi \in H^0(X_K, K_1)$ is also called the specialization homomorphism.

Let $u \in H^i(X, K_*)$; for any field extension L/F by $u_L \in H^i(X_L, K_*)$ we denote the image of u under the homomorphism $H^i(X, K_*) \to H^i(X_L, K_*)$.

Lemma 1. For any prime element π of the ring R the equality $s_\pi{:}(u_K) = u_k$ holds.

Proof. By the product formula $s_\pi(u_K) = \partial(u_K \cdot \pi) = u_k \cdot \partial(\pi) = u_k$ since $\partial(\pi) = 1 \in H^0(X, K_0)$.

Example. Let C be an irreducible curve over the field F, $c \in C$ be a nonsingular point, $R = 0_{C.c}$ be the local ring of the point c. In this case $k = F(c)$, $K = F(C)$ and we have the following

Corollary. For any nonsingular rational point $c \in C$, prime element $\pi \in 0_{C.c}$ and $u \in H^i(X, K_*)$ the equality $s_\pi\left(u_{F(c)}\right) = u$ holds, i.e. the result of the specialization in this case does not depend on the choice of c and π.

The category $M(X)$ has the following filtration: $M(X)_0 \subset M(X)_1 \subset \ldots \subset M(X)_d = M(X)$ where $d = \dim X$ and $M(X)_i$ is the full subcategory in $M(X)$ consisting of all sheaves G such that $\dim \operatorname{supp} G \leq i$. Since $K_*\left(M(X)_i / M(X)_{i-1}\right) = \bigcup_{x \in X_i} K_* F(x)$ [7] the inclusion $M(X)_0 \subset M(X)$ induces the homomorphism

$$t: H_0(X, K_*) \to K'_*(X)..$$

Lemma 2. For any discrete valuation ring R with the fraction field K and the residue field k for any prime element $\pi \in R$ the following diagram

$$\begin{array}{ccc} H_0(X_K, K_*) & \xrightarrow{t} & K'_*(X_K) \\ \downarrow s_\pi & & \downarrow s_\pi \\ H_0(X_k, K_*) & \xrightarrow{t} & K'(X_k) \end{array}$$

is commutative.

Proof. It is clearly sufficient to prove the commutativity of the following diagram:

$$\begin{array}{ccc} H_0(X_{K'}, K_{*+1}) & \xrightarrow{t} & K'_{*+1}(X_K) \\ \downarrow \partial' & & \downarrow \partial'' \\ H_0(X_k, K_*) & \xrightarrow{t} & K'_*(X_k). \end{array}$$

Since $M(X_R)_0 = M(X_k)_0$ the functor $M(X_R)_1 \to M(X_R) \to M(X_K)$ induces the functor $M(X_R)_1/M(X_R)_0 \to M(X_K)_0 \to M(X_K)$. Therefore we have the following commutative diagram

$$\begin{array}{ccccc} M(X_R)_0 \to & M(X_R)_1 \to & M(X_R)_1/M(X_R)_0 \\ \| & | & \downarrow \\ M(X_k)_0 & | & M(X_K)_0 \\ \downarrow & \downarrow & \downarrow \\ M(X_k) \to & M(X_R) \to & M(X_K), \end{array}$$

which induces the commutative diagram

$$\begin{array}{ccc} K_{*+1}(M(X_R)_1/M(X_R)_0) & \xrightarrow{\partial} & K_*(M(X_R)_0) \\ \Downarrow & & \| \\ K_{*+1}(M(X_K)_0) & & K_*(M(X_k)_0) \\ \downarrow & & \downarrow \\ K_{*+1}(X_K) & \xrightarrow{\partial''} & K_*(X_k). \end{array}$$

By definition of ∂' the following diagram

$$\begin{array}{ccc}
K_{*+1}\left(M(X_R)_1/M(X_R)_0\right) & \xrightarrow{\partial} & K_*\left(M(X_R)_0\right) \\
\downarrow & & \| \\
K_{*+1}(M(X_K)_0) & & K_*(M(X_k)_0) \\
\downarrow & & \downarrow \\
H_0(X_{K'}, K_{*+1}) & \xrightarrow{\partial'} & H_0(X_k, K_*)
\end{array}$$

is commutative. Comparing the two last diagrams we get the result we need.

2. The definition of $p_n: H_0(X(D), K_n) \to K_nD$

Let $X = X(D)$ be a Severi-Brauer variety over a field F, associated to the central simple F-algebra D of dimension n^2, J be the canonical locally free O_X-module of rank n, and $D = \mathrm{End}_X(J)$ [7,9].

For any commutative F-algebra B consider the full subcategory $M'(X_B)$ in $M(X_B)$ consisting of X_B-modules G such that $R^i f_*(J \otimes_X G) = 0$ for any $i > 0$, where $f: X_B \to \mathrm{Spec} B$ is the structural morphism. By this theorem of Quillen [7] the inclusion $M'(X_B)$ in $M(X_B)$ induces an isomorphism $K_*(M'(X_B)) \to K_*(X_B)$.

It is clear that for any $G \in M(X_B)$ B-module $f_*(J \otimes_X G)$ has a structure of the left D_B-module. The exact functor

$$j_B: M'(X_B) \to D_B - \mathrm{mod}, G: \to f_*(J \otimes_X G)$$

induces the homomorphism $K_*(M'(X_B)) \to K_*(D_B)$. We define p_B as a composition

$$p_B : H_0(X_B, K_*) \xrightarrow{t} K_*(X_B) = K_*(M'(X_B)) \to K_*(D_B)$$

Let R be a discrete valuation ring with the fraction field K and the residue field k, and $\pi \in R$ be any prime element. The following statement shows that the homomorphism p_K and p_k are compatible with the specialization.

Proposition 1. The diagram

$$\begin{array}{ccc} H_0(X_K, K_*) & \xrightarrow{p_K} & K_*(D_K) \\ \downarrow s_\pi & & \downarrow s_\pi \\ H_0(X_k, K_*) & \xrightarrow{p_k} & K_*(D_k) \end{array}$$

is commutative.

Proof. By Lemma 2 it is sufficient to prove that the following diagram is commutative

$$\begin{array}{ccc} K_{*+1}(X_k K) & \to & K_{*+1}(D_K) \\ \downarrow\partial & & \downarrow\partial \\ K_*(X_k) & \to & K_*(D_k). \end{array}$$

But this follows from the commutative diagram of functors

$$\begin{array}{ccc} M'(X_k) \to & M'(X_R) \to & M'(X_K) \\ \downarrow j_k & \downarrow j_R & \downarrow j_K \\ D_k - \text{mod} \to & D_R - \text{mod} \to & D_k - \text{mod}. \end{array}$$

Let now $B = F, x \in X$ be any closed point. We want to compute the following composition

$$r_x : K_* F(x) \to H_0(X, K_*) \xrightarrow{p} K_*(D),$$

where $p = p_F$. Consider the diagram of functors

$$\begin{array}{ccc} F(x)-\text{mod} & \longrightarrow & D_F(x)-\text{mod} \\ \downarrow i_* & & \downarrow \\ M'(X) & \xrightarrow{j} & D-\text{mod}, \end{array}$$

where $i{:}\mathrm{Spec}F(x) \to X$ is the closed immersion, right functor is induced by the inclusion $D \subset D_{F(x)}$ and the top arrow sends $F(x)$ – module M to $D_{F(x)}$ – module $J(x) \otimes_{F(x)} M$. Since $\dim_{F(x)} J(x) = n, J(x)$ is a simple $D_{F(x)}$-module and therefore the top arrow is the equivalence of categories. The commutativity of the diagram following from the natural isomorphism $J \otimes_X (i_* M) = i_*\left(J(x) \otimes_{F(x)} M\right)$ for any $F(x)$ – module M shows that r_x is induced by the functor

$$F(x)-\text{mod} \to \text{D}-\text{mod}; M{:}\to J(x) \otimes_{F(x)} M$$

and therefore can be decomposed as follows:

$$r_x{:}K_*F(x) \to K_*\left(D_{F(x)}\right) \to K_*(D),$$

where the first map is an isomorphism induced by the equivalence of categories and the second map is the homomorphism of transfer.

Let now D be a skew field and x be a point of degree n. We embed $F(x)$ in D as a maximal subfield. Since $F(x)$–modules $J(x)$ and D are isomorphic, $J(x) \otimes_{F(x)} M = D \otimes_{F(x)} M$ for any $F(x)$–module M and therefore the homomorphism $r_x{:}K_*F(x) \to K_*(D)$ is induced by the inclusion of $F(x)$ in D.

Lemma 3. If D is split then $p{:}H_0(X, K_*) \to K_*(D)$ is an isomorphism.

Proof. In this case $X = P_F^{n-1}$ is the projective space. Let $x \in X$ be any rational point. In the commutative diagram

$$\begin{array}{ccc} K_*F(x) & \xrightarrow{\sim} & K_*\left(D_{F(x)}\right) \\ \downarrow & \searrow^{r_x} & \downarrow \\ H_0(X,K_*) & \xrightarrow{p} & K_*D \end{array}$$

the vertical maps are isomorphisms since X is a projective space [8], $F(x) = F$ and therefore p is an isomorphism too.

Now we formulate the main result of the present paper.

Theorem. Let X be a Severi-Brauer variety corresponding to the central simple algebra D. Then the homomorphism $p_1: H_0(X,K_1) \to K_1(D)$ is an isomorphism.

The rest of the paper is devoted to the proof of this theorem.

3. The map $q: S(D) \to H_0(X(D),K_1)$

The idea is to construct the inverse map to $p = p_1$. In this section we build the "first approach" of this inverse map.

Let R be a commutative ring, B be an Azumaya algebra over R of rank n^2, and $X = X(B)$ be a Severi-Brauer scheme associated to B. For any commutative R-algebra S the set $X(S)$ of S–points of X coincides with the set of direct summands of the rank n of S–module $B \otimes_R S$ which are right ideals [9].

Let $A \subset B$ be a commutative R-subalgebra in B. Considering B as an A–module with respect to the right multiplication define the following homomorphism

$$f: B \otimes_R A \to \mathrm{End}_A(B); f(x \otimes a)(b) = xba.$$

Suppose that

1. A is the direct summand of the A–module B.
2. f is an isomorphism.

Then A–module $B = \mathrm{Hom}_A(A,B)$ is the direct summand of the projective A–module $\mathrm{End}_A(B) = B \otimes_R A$ and therefore is projective. Since f is an isomorphism, rank $_A B = n$. Hence $\mathrm{Hom}_A(B,A)$ is the right ideal of rank n and the direct summand in $\mathrm{End}_A(B) = B \otimes_R A$ and therefore defines the element in the set of points $X(A)$, i.e. the morphism $\mathrm{Spec} A \to X$.

Note that this construction is functional: for any R–algebra S the subalgebra $A \otimes_R S$ in $B \otimes_R S$ satisfies the conditions 1 and 2 and the corresponding morphism $\mathrm{Spec}(A \otimes_R S) \to X_S$ is the base change in the morphism $\mathrm{Spec} A \to X$.

Let D be a central skewfield of dimension n^2 over a field F, $L \subset D$ be a maximal subfield. Then the subalgebra $A = L$ satisfies 1 and 2 [6] and therefore defines the morphism $\mathrm{Spec} L \to X = X(D)$. Denote by $x \in X$ the image of the unique point in $\mathrm{Spec} L$. Since the field $F(x)$ splits D, we have $[F(x):F] \geq n$. On the other hand, our morphism induces the embedding $F(x)$ in L. Therefore this embedding is an isomorphism. We'll denote the point x by $[L]$. So $[L]$ is the closed point of degree n with the residue field isomorphic to L.

Let $u \in D$; the ring $F[u]$ generated by u over F is a subfield in D. We define the set $S(D)$ of all elements $u \in D^*$ such that $F[u]$ is the maximal subfield in D. Since there exists a separable over F maximal subfield [6] and this subfield is generated by one element, the set $S(D)$ is not empty. Note also that $u \in S(D)$ if and only if Cayley-Hamilton polynomial of u [4] is irreducible.

Define the following map:

$$q: S(D) \to H_0(X, K_1)$$

by the formula $q(u) = u[L]$ where $L = F[u]$ (we identify L and the residue field of the point $[L]$).

Lemma 4. For any $u \in S(D)$ the following equality holds:

$$p(q(u)) = u \bmod [D^*, D^*] \in K_1 D = D^*/[D^*, D^*].$$

Proof. Let $x = [L]$; the results of Section 2 imply that the composition $L^* = F(x)^* \to H_0(X, K_1) \to K_1(D)$ is induced by the embedding L to D. Therefore $p(q(u)) = p(ux) = u \bmod [D^*, D^*]$ in $K_1 D$.

Now consider the behavior of q under the specialization. We take an affine line $A^1 = \mathrm{Spec} F[T]$, rational point $T = t \in F$ with a local ring $R = F[T\}_{(T-t)}$ and the prime element $\pi = T - tR$. It is clear that $R/\pi R = F$ and $F(T)$ is the fraction field of R. Consider the specialization map s_π associated with the discrete valuation ring R.

Proposition 2. Let S_t be the set of all polynomials $u(T) \in D[T]$ such that $u(t) \in S(D)$. Then $S_t \subset S(D)$ and we have commutative diagram

$$\begin{array}{ccc} S_t & \longrightarrow & S(D) \\ \downarrow q_{F(T)} & & \downarrow q \\ H_0\left(X\left(D_{F(T)}\right), K_1\right) & \xrightarrow{s_\pi} & H_0(X(D), K_1) \end{array}$$

where the above homomorphism is the "value in the point $T = t$".

Proof. Let $u(T) \in S_t, P(T, X) \in F[T, X]$ be Cayley-Hamilton polynomial of $u(T)$ as an element of Azumaya algebra $D[T]$ over $F[T]$. Since the polynomial $P(t,X)$ is irreducible, $P(T,X)$ is also irreducible and $u(T) \in S(D(T))$.

The homomorphism s_π coincides with the composition

$$H_0\left(X\left(D_{F(T)}\right),K_1\right)\xrightarrow{\cdot\pi}H_0\left(X\left(D_{F(T)}\right),K_2\right)\xrightarrow{\partial}H_0(X(D),K_1).$$

Hence it is sufficient to prove the equality $\partial(\{u(T),T-t\}[E])=u(t)[L]$ where $E=F(T)[u(T)]$, $L=F[u(t)]$.

Denote the ring $R\{u(T)]$ by A. It is clear that A is a discrete valuation ring with the prime element π, fraction field E and residue field L. We consider A as a commutative subalgebra in Azumaya R-algebra $B=D\otimes_F R$ and show that the canonical homorphism $f:B\otimes_R A\to \mathrm{End}_A(B)$ is an isomorphism. L is the maximal subfield in D, hence f is an isomorphism modulo maximal ideal of R and by Lemma of Nakayama f is surjective. Since E is the maximal subfield in $D(T)$, the localization $S^{-1}f$ with respect to the multiplicative set S of nonzero elements in R is an isomorphism. Therefore f is injective and hence is an isomorphism.

Since $A/\pi A\hookrightarrow B/\pi B$ A-module B/A is torsionfree and therefore B/A is free A-module and A the direct summand in B.

So we have shown that algebra B and commutative subalgebra A satisfy the conditions 1 and 2 and define the morphism $\mathrm{Spec}A\to X(B)$.. The functional property gives us the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Spec}L= & \mathrm{Spec}\,A/\pi A & \to & X(B/\pi B)=X(D) \\
 & \downarrow & & \downarrow \\
 & \mathrm{Spec}A & \to & X(B) \\
 & \uparrow & & \uparrow \\
\mathrm{Spec}E= & \mathrm{Spec}\left(S^{-1}A\right) & \to & X\left(S^{-1}B\right)=X(D(T))
\end{array}$$

which induces the following commutative diagram

$$\begin{array}{ccc} K_2E = & H_0(\mathrm{Spec}E, K_2) \rightarrow & H_0\left(X\left(D_{F(T)}\right), K_2\right) \\ \downarrow\partial & \downarrow\partial' & \downarrow\partial'' \\ K_1L = & H_0(\mathrm{Spec}L, K_1) \rightarrow & H_0(X(D), K_1) \end{array}$$

where ∂ is the tame symbol associated to a discrete valuation ring A. In particular $\partial(\{u(T), T-t\}) = u(t)$.

Consider another example of the specialization.

Proposition 3. Let $K \subset D$ be a maximal subfield, $u(T) \in K[T], u(t) \neq 0$. Then $s_\pi(u(T)[K(T)]) = u(t)[K]$.

Proof. The functional property gives us the commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}K(T) & \rightarrow & X_{F(T)} \\ \downarrow & & \downarrow j \\ \mathrm{Spec}K & \rightarrow & X \end{array}$$

Denote $[K]$ by $x \in X$ and $[K(T)]$ by $y \in X_{F(T)}$. The projection j is decomposed into the composition $X_{F(T)} \xrightarrow{r} XxA^1 \rightarrow X$ and the closure of the point $r(y)$ in $Xx\ A^1$ equals xxA^1. Since $u(T) \in K[T] = F[xxA^1]$ is a regular functor on $xxA1$, the specification map sends the element $u(T)y$ at first by the multiplication on $\pi = T - t$ in $\{u(T), T-t\}y$ and then by ∂ to the element $u(\mathrm{t})x$.

4. The construction of the homomorphism $q: K_1(D) \rightarrow H_0(X(D), K_1)$

In this section we show how to extend the map q constructed in Section 3 from the "dense" subset $S(D)$ to the whole group D^*. This extension modulo the commutant appears to be the inverse map to $p = p_1$.

We begin with the following abstract situation. Let G be any group; a subset $S \subset G$ is called dense in G if for any elements g_1, $g_2, \ldots, g_n \in G$ the intersection $\bigcap Sg_i$ is not empty.

Lemma 5. Let S be a dense subset in group G such that $S = S^{-1}$ and $q: S \to B$ be a map to abelian group B. Suppose that

1. $q(g^{-1}) = -q(g)$ for any $g \in G$.

2. $q(g_1 g_2) = q(g_1) + q(g_2)$ for all $g_1, g_2 \in S$ such that $g_1 g_2 \in S$.

Then there exists the unique homomorphism $q': G \to B$ extending the map q.

Proof. Let $g \in G$; since $Sg \bigcap S1 \neq \varnothing$, we have: $sg = t \in S$ for some $s \in S; g = s^{-1}t$. If q' extends q then $q'(g) = -q(s) + q(t)$ which proves the uniqueness.

Now we prove the existence of the extension. Let $g \in G$; as before we find $s, t \in S$ such that $g = s^{-1}t$. We define q' by the formula $q'(g) = -q(s) + q(t)$. To prove that q' is well defined, take $g = s_1^{-1} - t_1$ where $s_1, t_1 \in S$. Choose $s_2 \in Ss \bigcap Ss_1 \bigcap Sg^{-1} \bigcap S1$ then $g = s_2^{-1} t_2$, $t_2 \in S$ and $s_2 s^{-1} = t_2 t^{-1} \in S$, $s_2 s_1^{-1} = t_2 t_1^{-1} \in S$. Therefore

$$-q(s) + q(s_2) = q(s_2 s^{-1}) = q(t_2 t^{-1}) = q(t_2) - q(t),$$
$$-q(s_1) + q(s_2) = q(s_2 s_1^{-1}) = q(t_2 t_1^{-1}) = q(t_2) - q(t_1),$$

hence $-q(s) + q(t) = -q(s_1) + q(t_1)$ which proves that q' is well defined.

If $g \in S$ and $g = s^{-1}t$ for $s, t \in S$, then $q'(g) = -q(s) + q(t) = q(s^{-1}t) = q(g)$, i.e. q' is the extension of q.

Finally we have to show that $q'(gh) = q'(g) + q'(h)$ for any g, $h \in G$. Suppose at first that $g \in S$. Choose $s \in Sg \bigcap Sh^{-1} \bigcap S1$ then

$h = s^{-1}t, t \in S$ and $gs^{-1} \in S$. We have: $q'(g) + q'(h) = q(g) - q(s) + q(t) = q(gs^{-1}) + q(t) = q'(gh)$ since $gh = (sg^{-1})^{-1}t$. Now consider the general case. Choose $t \in Sg \cap Sh^{-1} \cap S1$ i.e., $s^{-1}t = g, s \in S$ and $th \in S$. Using the first case we have: $q'(g) + q'(h) = -q(s) + q(t) + q'(h) = -q(s) + q'(th) = -q(s) + q(th) = q'(gh)$ since $gh = s^{-1}th$.

Remark. It follows from the proof that G is generated by any dense subset.

Let D be a central skewfield of dimension n^2 over a field F, $G = D^*$, $S = S(D) \subset G$.

Lemma 6. The set S satisfies the conditions of Lemma 5, i.e. $S^{-1} = S$ and S is dense in G.

Proof. Since $F[u^{-1}] = F[u]$, $S^{-1} = S$. If F is a finite field, the skewfield D is trivial [6] and therefore $S = G$ is dense in G.

Suppose now that F is an infinite field. Note that the set S is open in Zarisky topology of affine space $D = A$ dim D. Indeed, $u \in S$ iff the elements $1, u, u^2, ..., u^{n-1} \in D$ are linearly independent over F if the rank of the matrix of coefficients of these elements in some basis of D is lesser then n, i.e. the set $D-S$ is closed in D and S is open. Therefore, for any $g_1, g_2, ..., g_n$ in G the sets Sg_i are open and nonempty and since the field F is infinite, the intersection of these sets is not empty, i.e. S is dense in G.

Now consider the abelian group $B = H_0(X(D),K_1)$ and the map $q:S \to B$ defined in Section 3. We prove that q satisfies the conditions of Lemma 5. Let $u \in S$, $L = F[u]$; since $F[u^{-1}] = L$, $q(u^{-1}) = (u^{-1})[L] = -(u[L]) = -q(u)$.

Finally we have to show that $q(uv) = q(u) + q(v)$ for $u, v \in S$ such that $uv \in S$. Denote the polynomial $vT + 1 - T$ by $v(T)$. Since $v(1) = v \in S(D)$, it is clear that $uv(T) \in S\left(D_{F(T)}\right)$. Consider the element $w = q(u) + q(v(T)) - q(uv(T)) \in H_0\left(X_{F(T)}, K_1\right)$. By Lemma 4 $p(w) = uv(T)(uv(T))^{-1} = 1 \in K_1(D(T))$.

Lemma 7. Let $u \in \ker\left(H_0\left(X_{F(T)},K_1\right) \xrightarrow{p} K_1 D_{F(T)}\right)$. Then the image of the specialization $s_\pi(u) \in H_0(X,K_1)$ in the rational point $T = t \in F$ does not depend on the choice of t and π.

Proof. Let L/F be any splitting field of D. From the commutative diagram

$$\begin{array}{ccc} H_0\left(X_{F(T)},K_1\right) & \xrightarrow{p} & K_1 D_{F(T)} \\ \downarrow i & & \downarrow \\ H_0\left(X_{L(T)},K_1\right) & \rightarrow & K_1 D_{L(T)} \end{array}$$

and Lemma 3 we get that $u \in \ker i$. The exact sequence of complexes

$$0 \rightarrow \bigcup_{y \in A^1,} \bigcup_{x \in X_{F(y)}{}^{*-1}} K_* F(x) \rightarrow \bigcup_{x \in (XxA^1)_*} K_* F(x) \rightarrow \bigcup_{x \in X_{F(T)}{}^*} K_* F(x) \rightarrow 0$$

and isomorphism $H_1\left(XxA^1,K_2\right) = H_0(X,K_1)$ [8] give us the commutative diagram with the exact top row

$$\begin{array}{ccccc} H_0(X,K_1) & \xrightarrow{k} & H_0\left(X_{F(T)},K_1\right) & \rightarrow & \bigcup_{y \in A^1} H_0\left(X_{F(y)},K_0\right) \\ \downarrow i & & | & & \downarrow j \\ & & H_0\left(X_{L(T)},K_1\right) & \rightarrow & \bigcup_{y \in A^1} H_0\left(X_{L(y)},K_0\right). \end{array}$$

The homomorphism j is injective by the theorem of Panin [5]. Therefore $u \in im(k)$ and we can apply the Corollary to Lemma 1.

By Lemma 7 and Propositions 2 and 3 we have:

$$q(u) + q(v) - q(uv) = s_{T-1}(w) = s_T(w) = q(u) - q(u) = 0$$

So we can apply Lemma 5 to construct the extension of q:

$$q':D^* \to H_0(X(D),K_1)$$

which clearly factors through the homomorphism

$$K_1(D) \to H_0(X(D),K_1)$$

that we'll denote by q.

Since the set $S(D)$ generates D^*, the composition poq is identified by Lemma 4. In the rest of this section we prove that q commutes with the specialization.

Lemma 8. For any $t \in F$ the group $D_{F(T)}{}^*$ is generated by the element $T-t$ and set S_t.

Proof. It is clear that $D_{F(T)}{}^*$ is generated by $T-t$ and the set of polynomials $u(T) \in D[T]$ such that $u(t) \neq 0$. Since $S(D)$ is a dense subset in D^*, we can find $v \in S(D)$ such that $u(t)v \in S(D)$. Then $v, u(T)v \in S_t$ and $u(T) = (u(T)v)v^{-1}$.

Proposition 4. For any $t \in F$ the diagram

$$\begin{array}{ccc} K_1 D_{F(T)} & \xrightarrow{s_{T-t}} & K_1 D \\ \downarrow q_T & & \downarrow q \\ H_0(X_{F(T)},K_1) & \xrightarrow{s_{T-t}} & H_0(X,K_1) \end{array}$$

is commutative.

Proof. By Lemma 8 the group $D_{F(T)}{}^*$ is generated by $T-t$ and the set S_t. The commutativity for the elements of the set S_t was proved in Proposition 2. Instead of element $T-t$ it is sufficient to consider $(T-t)u$, where u is any element in $S(D)$. Let $L = F[u]$; then $F(T)\,[T-t)u] = L(T)$ and

$$(s_{T-t} \circ g_T)((T-t)u)) = S_{T-t}\big((T-t)u[L(T)]\big) = \partial\big(\{(T-t)u, T-t\}[L(T)]\big)$$
$$= \partial\big(\{-u, T-t\}[L(T)]\big) = (-u)[L],$$
$$\big(g \circ s_{T-t}\big)\big((T-t)u\big) = g\big(\partial\big(\{(T-t)u, T-t\}\big)\big) = g(-u) = (-u)[L].$$

5. Proof of the Theorem

We have only to show that the composition $q \circ p$ is identity. Let $x \in X$, and $u \in F(x)^*$. Consider the point $\bar{x} \in X_{F(T)}$ over x and generic point of $\operatorname{Spec} F(T)$ and an element $\bar{u} = uT + 1 - T \in F(T)(\bar{x}) = F(x)(T)$. Denote by w the element $q(p(\bar{u}\bar{x})) - \bar{u}\bar{x} \in H_0\big(X_{F(T)}, K_1\big)$. Since $p(w) = 0$, Lemma 7 all the specializations of w in rational points coincide; in particular $s_{T-1}(w) = s_T(w)$. By Propositions 1 and 4 the homomorphisms p and q commute with the specialization and we have: $s_{T-1}(w) = q\big(p(s_{T-1}(\bar{u}\bar{x})\big) - s_{T-1}(\bar{u}\bar{x}) = q\big(p(ux)\big) - ux$ since $s_{T-1}(\bar{u}\bar{x}) = ux$ and $s_T(w) = q\big(p(s_T(\bar{u}\bar{x})\big) - s_T(\bar{u}\bar{x}) = 0$ since $s_T(\bar{u}\bar{x}) = 0$. Therefore, $q\big(p(ux)\big) = ux$, i.e. $q \circ p = id$.

So we have proved the Theorem in the case when D is a skewfield. Now let A be any central simple F-algebra, $A = M_m(D)$ where D is a skewfield. Using the results of [5] one can find a closed subvariety $Z \subset X(A)$ such that $Z \cong X(D)$ and a vector bundle $X(A) - Z \to X(A')$ where $A' = M_{m-1}(D)$. Therefore, $H_0(X(A)-Z, K_1) = 0$ and the direct image

$$H_0\big(X(D), K_1\big) = H_0\big(Z, K_1\big) \xrightarrow{i_*} H_0\big(X(A), K_1\big)$$

is a surjective map. The Theorem follows from the commutative diagram

$$\begin{array}{ccc} H_0(X(D),K_1) & \to & K_1(D) \\ \downarrow i_* & & \downarrow \\ H_0(X(A),K_1) & \to & K_1(A). \end{array}$$

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