ADAMS OPERATIONS AND THE BROWN-GERSTEN-QUILLEN SPECTRAL SEQUENCE

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Abstract. By means of Adams operations in algebraic K-theory we study the order of differentials in the Brown-Gersten-Quillen spectral sequence for a scheme.

1. Introduction

Let X be a separated scheme of finite type over a field F. We write $X_{(p)}$ for the set of points in X of dimension p. There is the *niveau spectral sequence*

$$E^1_{p,q} = \coprod_{x\in X_{(p)}} K_{p+q}F(x) \Rightarrow G_{p+q}(X)$$

converging to the G-groups of X (the K-groups of the category $M(X)$ of coherent sheaves on X) with the topological filtration [10, §7, Th. 5.4]. The term $E^1_{p,-p}$ coincides with the group of algebraic cycles of dimension p on X and $E^2_{p,-p}$ with the Chow group $\mathrm{CH}_p(X)$ of classes of cycles of dimension p [10, §7, Prop. 5.14].

The topological filtration on $G_n(X)$ is defined as follows. Write $M_p(X)$ for the category of coherent sheaves on X supported on a closed subset of dimension at most p. The image the homomorphism $K_n\big(M_p(X)\big) \to G_n(X)$ induced by the inclusion functor $M_p(X) \to M(X)$ is the pth term $G_n(X)_{(p)}$ of the topological filtration on $G_n(X)$. The subsequent factors of the filtration are denoted by $G_n(X)_{(p/p-1)}$.

The spectral sequence yields surjective homomorphisms

$$\varphi_p : \mathrm{CH}_p(X) = E^2_{p,-p} \to G_0(X)_{(p/p-1)}.$$

The kernel of φ_p is detected by the differentials of the spectral sequence arriving at $E^*_{p,-p}$.

In the present paper we find some constraints on the order of the differentials in the spectral sequence arriving at the G_0- and G_1-diagonals. We show that for any prime integer l, the Adams operations in algebraic K-theory developed in [4], [5], [6] and [11], split the spectral sequence localized at l into a direct sum of $l-1$ summands. (For an analog in topology see [1, p. 91].)

In the paper the word "scheme" means a quasi-projective scheme over a field and a "variety" is an integral scheme.

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2. The category $\mathcal{A}_l$

For a prime integer l, let $\mathbb{Z}_{(l)}$ denote the localization of $\mathbb{Z}$ with respect to the prime ideal $l\mathbb{Z}$.

Lemma 2.1. *Let $f, g \in \mathbb{Z}_{(l)}[t]$ be polynomials such that f and g are coprime over $\mathbb{Q}$ and the residues $\bar{f}$ and $\bar{g}$ are coprime over $\mathbb{Z}/l\mathbb{Z}$. Then f and g are coprime over $\mathbb{Z}_{(l)}$, i.e., f and g generate the unit ideal in $\mathbb{Z}_{(l)}[t]$.*

Proof. Let I be the ideal in $\mathbb{Z}_{(l)}[t]$ generated by f and g. By assumption, I contains l^k and a polynomial $1 - lh$ for some $k > 0$ and $h \in \mathbb{Z}_{(l)}[t]$. Then $1 - l^k h^k \in I$ and hence $1 \in I$. $\square$

2.1. **Definition of $\mathcal{A}_l$.** Let l be a prime integer. We define the category $\mathcal{A}_l$ as follows. An object of $\mathcal{A}_l$ is a $\mathbb{Z}_{(l)}$-module M equipped with a filtration

$$\cdots \subset M^{(2)} \subset M^{(1)} \subset M^{(0)} \subset M^{(-1)} \subset \ldots$$

by submodules such that $M^{(n)} = 0$ for $n \gg 0$ and $M^{(n)} = M$ for $n \ll 0$, and endomorphisms $\psi^k_M \in \operatorname{End}_{\mathbb{Z}_{(l)}}(M)$ for all integers k prime to l, satisfying:

(*i*) $\psi^k_M \circ \psi^{k'}_M = \psi^{kk'}_M$ for all k and k' (in particular, ψ^k_M and $\psi^{k'}_M$ commute).

(*ii*) For any k, we have $\psi^k_M(M^{(i)}) \subset M^{(i)}$ and ψ^k_M acts on $M^{(i)}/M^{(i+1)}$ by multiplication by k^i for all i.

A morphism between two objects M and N in $\mathcal{A}_l$ is a $\mathbb{Z}_{(l)}$-module homomorphism $s : M \to N$ such that $s \circ \psi^k_M = \psi^k_N \circ s$ for all k. (We don't assume that s is compatible with the filtrations.)

Let M be an object of $\mathcal{A}_l$ and $r \in \mathbb{Z}$. The define the *shift* $M[r]$ of M as the module M together with the shifted filtration $M[r]^{(i)} = M^{(i+r)}$ and the endomorphisms defined by $\psi^k_{M[r]} = k^{-r} \cdot \psi^k_M$. Clearly, the assignment $M \mapsto M[r]$ is an auto-functor of $\mathcal{A}_l$.

2.2. **A decomposition.** Let $M \in \mathcal{A}_l$ and let $a \le b$ be two integers such that $M^{(a)} = M$ and $M^{(b)} = 0$. The set J of all integers j such that $a \le j < b$ is called an *interval of M*.

Let $k > 1$ be an integer such that the congruence class $[k]_l$ of k modulo l is a generator of $(\mathbb{Z}/l\mathbb{Z})^\times$. Write $\Lambda = \mathbb{Z}/(l-1)\mathbb{Z}$. For any congruence class $\rho \in \Lambda$, consider the polynomial

$$f_\rho = \prod_{j \in \rho \cap J} (t - k^j) \tag{1}$$

over $\mathbb{Z}_{(l)}$, where J is an interval of M, and set

$$f = \prod_{\rho \in \Lambda} f_\rho = \prod_{j \in J} (t - k^j).$$

It follows from (*ii*) that $(\psi^k_M - k^j)\big(M^{(j)}\big) \subset M^{(j+1)}$ for any j, hence

$$f(\psi^k_M) = 0. \tag{2}$$

By construction, every pair of distinct polynomials f_ρ and $f_{\rho'}$ are coprime over $\mathbb{Q}$ and the residues $\bar{f}_\rho$ and $\bar{f}_{\rho'}$ are coprime over $\mathbb{Z}/l\mathbb{Z}$. By Lemma 2.1, f_ρ and $f_{\rho'}$ are coprime over $\mathbb{Z}_{(l)}$. By the Chinese Remainder Theorem, the factor ring $\mathbb{Z}_{(l)}[t]/(f)$ is canonically isomorphic to the product of the rings $\mathbb{Z}_{(l)}[t]/(f_\rho)$ over all $\rho \in \Lambda$. Note that the image of f_ρ in $\mathbb{Z}_{(l)}[t]/(f_{\rho'})$ is zero if $\rho' = \rho$ and invertible otherwise.

It follows then from (2) that M has a natural structure of a module over $\mathbb{Z}_{(l)}[t]/(f)$, where t acts by ψ^k_M. Therefore,

$$M = \coprod_{\rho \in \Lambda} M_\rho \tag{3}$$

with $M_\rho = \operatorname{Ker} f_\rho(\psi^k_M)$.

We claim that the submodules M_ρ do not depend on the choice of the interval J of M. Indeed, let J' be another interval of M containing J, f'_ρ the polynomials constructed for J' and $M'_\rho = \operatorname{Ker} f'_\rho(\psi^k_M)$. Then f_ρ divides f'_ρ, and hence

$$M_\rho = \operatorname{Ker} f_\rho(\psi^k_M) \subset \operatorname{Ker} f'_\rho(\psi^k_M) = M'_\rho.$$

Therefore, in view of (3), both for M_ρ and M'_ρ, we deduce that $M_\rho = M'_\rho$.

For any $\rho \in \Lambda$ set $\widehat{f_\rho} = f/f_\rho \in \mathbb{Z}_{(l)}[t]$. Then the polynomials $\{\widehat{f_\rho}\}_{\rho\in\Lambda}$ are coprime.

Lemma 2.2. $\operatorname{Ker} f_\rho(\psi^k_M) = \operatorname{Im} \widehat{f_\rho}(\psi^k_M)$ *for all* $\rho \in \Lambda$.

Proof. Let $\psi = \psi^k_M$. Since $f_\rho(\psi)\widehat{f_\rho}(\psi) = f(\psi) = 0$, we have $\operatorname{Im} \widehat{f_\rho}(\psi) \subset \operatorname{Ker} f_\rho(\psi)$. As f_ρ and $\widehat{f_\rho}$ are coprime, there are polynomials g and h in $\mathbb{Z}_{(l)}[t]$ such that $f_\rho g + \widehat{f_\rho} h = 1$. Then for any $m \in \operatorname{Ker} f_\rho(\psi)$, we have $m = \widehat{f_\rho}(\psi)h(\psi)(m) \in \operatorname{Im} \widehat{f_\rho}(\psi)$, i.e., $\operatorname{Ker} f_\rho(\psi) \subset \operatorname{Im} \widehat{f_\rho}(\psi)$. □

We also prove that the decomposition (3) does not depend on the choice of the integer k. Let $k' > 1$ be an integer such that the congruence class $[k']_l$ is a generator of $(\mathbb{Z}/l\mathbb{Z})^\times$. Define the polynomials f, f'_ρ and $\widehat{f'}$ in $\mathbb{Z}_{(l)}[t]$ as above with k replaced by k'. Then M is a direct sum of the submodules $M'_\rho = \operatorname{Ker} f'_\rho(\psi^{k'}_M)$ over all $\rho \in \Lambda$.

Lemma 2.3. $M_\rho = M'_\rho$ *for all* $\rho \in \Lambda$.

Proof. Take any $\rho \in \Lambda$ and $j \in J$. If $j \in \rho$, then $t - (k')^j$ divides f'_ρ. If $j \notin \rho$, then $t - (k)^j$ divides $\widehat{f_\rho}$. It follows that $f'_\rho(\psi^{k'}_M)\widehat{f_\rho}(\psi^k_M) = 0$ and hence by Lemma 2.2,

$$M_\rho = \operatorname{Ker} f_\rho(\psi^k_M) = \operatorname{Im} \widehat{f_\rho}(\psi^k_M) \subset \operatorname{Ker} f'_\rho(\psi^{k'}_M) = M'_\rho.$$

By symmetry, $M'_\rho \subset M_\rho$. □

Thus, the submodules M_ρ in the decomposition (3) depend only on the object M in the category $\mathcal{A}_l$.

Let $s : M \to N$ be a morphism in $\mathcal{A}_l$. Choose a common interval J for both M and N and let f_ρ be the polynomials defined by (1). Then $s \circ f_\rho(\psi^k_M) = f_\rho(\psi^k_N) \circ s$, hence

$$s\big(M_\rho\big) = s\big(\operatorname{Ker} f_\rho(\psi^k_M)\big) \subset \operatorname{Ker} f_\rho(\psi^k_N) = N_\rho,$$

i.e., s induces a $\mathbb{Z}_{(l)}$-module homomorphism $M_\rho \to N_\rho$. Thus for every $\rho \in \Lambda$, we have a functor from $\mathcal{A}_l$ to the category of $\mathbb{Z}_{(l)}$-modules taking an object M to M_ρ.

Proposition 2.4. *For an object M in $\mathcal{A}_l$ and an integer r, we have $M[r]_\rho = M_{\rho+r}$ for all $\rho \in \Lambda$.*

Proof. Let J be an interval of M. Then $J' = J - r$ is an interval of $M[r]$. Let $\{f_\rho\}_{\rho\in\Lambda}$ and $\{f'_\rho\}_{\rho\in\Lambda}$ be the polynomials constructed in (1) for J and J', respectively. Then

$$f_{\rho+r}\big(\psi^k_M\big) = \prod_{j\in(\rho+r)\cap J} \big(\psi^k_M - k^j\big) = \prod_{i\in\rho\cap J'} \big(k^r\psi^k_{M[r]} - k^{i+r}\big) = k^{rd}\cdot f'_\rho\big(\psi^k_{M[r]}\big),$$

where $d = \deg(f_\rho)$. It follows that

$$M[r]_\rho = \operatorname{Ker} f'_\rho\big(\psi^k_{M[r]}\big) = \operatorname{Ker} f_{\rho+r}\big(\psi^k_M\big) = M_{\rho+r}. \qquad \square$$

3. Spectral sequences in K-theory

3.1. **Adams operations.** Let M be a scheme and $Z \subset M$ a closed subscheme. We write $K^Z_m(M)$ for the K-groups of M with support in Z and $F^i_\gamma K^Z_m(M)$ for the ith term of the (finite) gamma-filtration on $K^Z_m(M)$ (see [11, §4]). If M is a regular scheme, there exists a canonically isomorphism

$$G_m(Z) \simeq K^Z_m(M),$$

where $G_m(Z)$ is the K-group of the category of coherent sheaves on Z [3, Th.2.14].

For any integer k, there is the Adams operation ψ^k on $K^Z_m(M)$ satisfying the following properties [11], [7, §9]:

- ψ^k is a group endomorphism of $K^Z_m(M)$.
- ψ^k respects the gamma-filtration $F^i_\gamma K^Z_m(M)$.
- ψ^k acts as multiplication by k^i on the subsequent factor $F^{(i/i+1)}_\gamma K^Z_m(M)$.
- $\psi^k\psi^{k'} = \psi^{kk'}$, in particular, φ^k and $\varphi^{k'}$ commute.

Let l be a prime integer. Then the $\mathbb{Z}_{(l)}$-module $K^Z_m(X)\otimes\mathbb{Z}_{(l)}$ together with the gamma-filtration and the Adams operations ψ^k on it yield an object of the category $\mathcal{A}_l$. Therefore, $K^Z_m(X)\otimes\mathbb{Z}_{(l)}$ decomposes as in (3) into a direct sum of submodules $\big(K^Z_m(X)\otimes\mathbb{Z}_{(l)}\big)_\rho$ over all $\rho\in\Lambda = \mathbb{Z}/(l-1)\mathbb{Z}$.

Let Z be a scheme. For any integer $k > 0$, there is a well-defined map (see [11, §4]):

$$\theta^k : K_0(Z) \to K_0(Z),$$

natural in Z, satisfying:

- For an exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$ over Z, we have $\theta^k[E] = \theta^k[E'] \cdot \theta^k[E'']$.
- For a line bundle L over Z,
$$\theta^k[L] = 1 + [L^{-1}] + [L^{-2}] + \cdots + [L^{-k+1}].$$

In particular, $\operatorname{rank} \theta^k(\alpha) = k^{\operatorname{rank}(\alpha)}$ for every $\alpha \in K_0(Z)$.

The following variant of the Riemann-Roch formula was proven in [11, Th. 3] (see also [9, 2.6.2]):

Proposition 3.1. *Let M be a regular variety of dimension d and let $Z \subset M$ be a regular closed subvariety of dimension p. Let N be the normal bundle (of rank $d-p$) for the closed embedding $f : Z \to M$. Then there is an isomorphism*
$$f_* : K_*(Z) = K_*^Z(Z) \xrightarrow{\sim} K_*^Z(M)$$
such that
$$f_*\big(\psi^k(\theta^k(N) \cdot \alpha)\big) = \psi^k\big(f_*(\alpha)\big)$$
for any $\alpha \in K_*(Z)$ and any k.

Corollary 3.2. *Suppose that, in addition, N is a trivial bundle. Then $\theta(N) = k^{d-p}$ and hence*
$$f_*\big(k^{d-p} \cdot \psi^k(\alpha)\big) = \psi^k\big(f_*(\alpha)\big).$$
Thus, for a prime integer l, f_* induces an isomorphism
$$\big(K_*(Z) \otimes \mathbb{Z}_{(l)}\big)[p-d] \xrightarrow{\sim} K_*^Z(M) \otimes \mathbb{Z}_{(l)}$$
in the category $\mathcal{A}_l$.

3.2. **Localization exact sequence and the niveau spectral sequence.** Let M be a regular scheme. If $Z' \subset Z$ are closed subscheme of M, then there is an exact localization sequence [7, 9.3]:

$$\cdots \to K_m^{Z'}(M) \to K_m^Z(M) \to K_m^{Z \setminus Z'}(M \setminus Z') \to K_{m-1}^{Z'}(M) \to \dots \tag{4}$$

that is isomorphic to the localization exact sequence [10, §7, Prop. 3.2]:

$$\cdots \to G_m(Z') \to G_m(Z) \to G_m(Z \setminus Z') \to G_{m-1}(Z') \to \dots \tag{5}$$

The homomorphisms in (4) commute with the Adams operations by [7, Remark 9.6(1)]. Then, localizing at a prime integer l, we can view (4) and (5) as sequences of morphisms in the category $\mathcal{A}_l$.

Let X be a scheme. We embed X into a regular variety M of dimension d as a closed subscheme. For a pair of integers p and q set
$$E^1_{p,q} = \operatorname{colim} K_{p+q}^{Z \setminus Z'}(M \setminus Z') = \operatorname{colim} G_{p+q}(Z \setminus Z'),$$
where the colimit is taken over all pairs (Z', Z) with Z a closed subscheme of X of dimension p and Z' a closed subscheme of Z of dimension $p-1$. Note that for any such a Z one can find a Z' such that $Z \setminus Z'$ is regular and the normal bundle of $Z \setminus Z'$ in $M \setminus Z'$ is trivial. It follows that
$$E^1_{p,q} = \coprod_{x \in X_{(p)}} K_{p+q} F(x)$$

and for any prime integer l,

$$E^1_{p,q} \otimes \mathbb{Z}_{(l)} \simeq \coprod_{x \in X_{(p)}} \big(K_{p+q}F(x) \otimes \mathbb{Z}_{(l)}\big)[p-d]$$

in $\mathcal{A}_l$ by Corollary 3.2. It follows from Proposition 2.4 that for any $\rho \in \Lambda$,

$$\big(E^1_{p,q} \otimes \mathbb{Z}_{(l)}\big)_\rho = \coprod_{x \in X_{(p)}} \big(K_{p+q}F(x) \otimes \mathbb{Z}_{(l)}\big)_{\rho+p-d}. \tag{6}$$

Set

$$D^1_{p,q} = \operatorname{colim} K^Z_{p+q}(M) = \operatorname{colim} G_{p+q}(Z),$$

where the colimit is taken over all closed subschemes Z of X of dimension p. Taking colimits of the exact sequences (4) and (5) we get the exact sequences:

$$\cdots \to E^1_{p,q} \to D^1_{p-1,q} \to D^1_{p,q-1} \to E^1_{p,q-1} \to \ldots.$$

The later yields an exact couple and therefore, a (homological) *Brown-Gersten-Quillen spectral sequence*:

$$E^1_{p,q} = \coprod_{x \in X_{(p)}} K_{p+q}F(x) \Rightarrow G_{p+q}(X). \tag{7}$$

It is supported in the area $0 \le p \le \dim(X)$ and $p+q \ge 0$. In particular, all the differentials arriving at $E^s_{p,q}$ with $p+q<0$ are trivial. We shall get some information on the differentials arriving at $E^s_{p,q}$ with $p+q=0$ or 1.

Lemma 3.3. *For any prime integer l and any $\rho \in \Lambda$, we have*

$$\big(E^s_{p,q} \otimes \mathbb{Z}_{(l)}\big)_\rho = \begin{cases} E^s_{p,q} \otimes \mathbb{Z}_{(l)}, & \text{if } \rho = [d+q]_{l-1}; \\ 0, & \text{otherwise} \end{cases}$$

for any $s \ge 1$ if $p+q \le 2$.

Proof. By [11, §2], for a field L and $m \le 2$, we have $F^m_\gamma K_m(L) = K_m(L)$ and $F^{m+1}_\gamma K_m(L) = 0$, hence ψ^k acts on $K_m(L)$ by multiplication by k^m. Therefore,

$$\big(K_m(L) \otimes \mathbb{Z}_{(l)}\big)_\rho = \begin{cases} K_m(L) \otimes \mathbb{Z}_{(l)}, & \text{if } \rho = [m]_{l-1}; \\ 0, & \text{otherwise.} \end{cases}$$

The statement now follows from (6). □

Theorem 3.4. *Let X be a scheme. Let*

$$\partial : E^s_{p,q} \to E^s_{p-s,q+s-1}$$

be the differential in the spectral sequence (7) *with $p+q \le 2$. Then the order of every element $a \in \operatorname{Ker}(\partial)$ is finite and if l is a prime divisor of* $\operatorname{ord}(a)$*, then $l \le p$ and $l-1$ divides $s-1$.*

Proof. If $s > p$, then $\partial = 0$ since $E^s_{p-s,q+s-1} = 0$, so we may assume that $s \le p$.

We claim that if l is a prime integer such that $\partial \otimes \mathbb{Z}_{(l)} \ne 0$, then $l-1$ divides $s-1$. For if let $\rho = [q+d]_{l-1} \in \Lambda$. By Lemma 3.3, the (nonzero) image of $\partial \otimes \mathbb{Z}_{(l)}$ is contained in $(E^s_{p-s,q+s-1} \otimes \mathbb{Z}_{(l)})_\rho$ and therefore, $\rho + d = [q+d+s-1]_{l-1}$, i.e., $l-1$ divides $s-1$. The claim is proved.

Taking $l > s$, we get $\partial \otimes \mathbb{Z}_{(l)} = 0$ from the claim, i.e., ∂ has finite order.

Let l be a prime divisor of $\mathrm{ord}(\partial)$. Then $\partial \otimes \mathbb{Z}_{(l)} \neq 0$ and hence by the claim, $l - 1$ divides $s - 1$. In particular, $l \leq s \leq p$. □

Example 3.5. Let X be the Severi-Brauer variety of right ideals of dimension l of a central simple algebra of a prime degree l over F. Since over a splitting field extension of degree l the variety X is isomorphic to the projective space $\mathbb{P}^{l-1}$, and the BGQ spectral sequence for a projective space degenerates at E_2, all the differentials of the BGQ spectral sequence for X are l-torsion. Since $\dim(X) = l - 1$, it follows from Theorem 3.4 that all the differentials arriving at the G_0- and G_1-diagonals are trivial. This result was proved in [8] with the help of higher Chern classes.

3.3. **Motivic spectral sequence.** Let X be a smooth scheme. We write $H^i(X, \mathbb{Z}(j))$ for the motivic cohomology groups [12].

The following (cohomological) motivic spectral sequence was constructed in [2]:

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X) \tag{8}$$

The spectral sequence in compatible with the Adams operations ψ^k and ψ^k acts by multiplication by k^q on $E_2^{p,q}$ [7, Th. 9.7].

Theorem 3.6. *Let X be a smooth scheme. Let*

$$\partial : E_s^{p,q} \to E_s^{p+s,q-s+1}$$

be the differential in the spectral sequence (8). *Then the order of every element $a \in \mathrm{Ker}(\partial)$ is finite and if l is a prime divisor of* $\mathrm{ord}(a)$, *then $l \leq \dim(X) - p$ and $l - 1$ divides $s - 1$.*

Proof. The proof is parallel to the one of Theorem 3.4. One remarks that $\partial = 0$ if $s > \dim(X) - p$. □

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