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MOTIVIC DECOMPOSITION OF PROJECTIVE HOMOGENEOUS VARIETIES AND THE KRULL-SCHMIDT THEOREM

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Abstract. Given an algebraic group G defined over a (not necessarily algebraically closed) field F and a commutative ring R we associate the subcategory $\mathcal{M}(G,R)$ of the category of Chow motives with coefficients in R that is the Tate pseudo-abelian closure of the category of motives of projective homogeneous G-varieties. We show that $\mathcal{M}(G,R)$ is a symmetric tensor category, i.e., the motive of the product of two projective homogeneous G-varieties is a direct sum of twisted motives of projective homogeneous G-varieties. We also study the problem of uniqueness of a direct sum decomposition of objects in $\mathcal{M}(G,R)$. We prove that the Krull-Schmidt theorem holds in many cases.

1. Introduction

V. Voevodsky has introduced the notion of an Artin-Tate motive. A Chow motive M is called an *Artin-Tate motive* if there exists a finite collections of motives M_i such that any tensor power of M is a direct sum of Tate twists of the motives M_i. The subcategory of Artin-Tate motives contains many important motives such as Tate motives, motives of finite separable extensions of the base field, motives of quadrics, motives of Severi-Brauer varieties and generalized Rost motives. One of the aims of this paper is to show that the motive of any projective homogeneous variety is an Artin-Tate motive.

In fact we prove more than that. Let X and X' be two projective homogeneous varieties of a semisimple group G defined over a field F. In general, the diagonal action of G on the product $X \times X'$ is not transitive, hence $X \times X'$ is not a homogeneous G-variety. Nevertheless we prove that the Chow motive of $X \times X'$ is a direct sum of some Tate twists of motives of projective homogeneous G-varieties (Theorem 16). The main ingredient of the proof is Proposition 13 stating that for every orbit of the G-action on $X \times X'$ there is a flat morphism of that orbit to a projective homogeneous G-variety with all fibers affine spaces. Description of all the orbits and the associated projective homogeneous G-varieties is given in the combinatorial terms of the root system of G.

Thus, given a commutative ring R we can associate the subcategory $\mathcal{M}(G,R)$ of the

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category of Chow motives with coefficients in R that is the Tate pseudo-abelian closure of the category of motives of projective homogeneous G-varieties. Our result then says that $\mathcal{M}(G, R)$ is a symmetric tensor category consisting of Artin-Tate motives.

In the second part of the paper we address the problem of uniqueness of a direct sum decomposition of the motive of a projective homogeneous variety into indecomposable objects in the category of Chow motives (the Krull-Schmidt theorem). Our category $\mathcal{M}(G, R)$ contains a subcategory generated by some finite separable extensions of the base field. This subcategory of zero-dimensional varieties is equivalent to the category of permutation $R\Gamma$-lattices where Γ is the finite group canonically associated to the so-called $\star$-action on G. The question whether the Krull-Schmidt theorem holds for $R\Gamma$-lattices was studied intensively in the context of the representation theory of finite groups and orders. We refer for the results and history of this important question to the book of C. Curtis and I. Reiner [7].

Example 32 shows that the Krull-Schmidt theorem does not hold in general for integral motives of projective homogeneous varieties. On the other hand, when dealing with a particular motive (such as a projective quadric or a Severi-Brauer variety) the coefficient ring $\mathbb{Z}$ can be replaced by a localization when all "insufficient" integers are inverted. For example, in the case of a quadric one can invert all odd integers, i.e., replace the ring $\mathbb{Z}$ by its localization $\mathbb{Z}_{(p)}$ for $p = 2$ not losing essential information. The situation with the motives of projective homogeneous varieties is better over the ring $\mathbb{Z}_{(p)}$: we prove the Krull-Schmidt theorem in many cases (Corollary 35), for example when G is a simple group or a group of inner type. If one replaces the coefficient ring $\mathbb{Z}_{(p)}$ by its completion, the Krull-Schmidt theorem holds without restrictions.

1.1. Notation

Basic reference are [2] and [12]. Let G^{qs} be an adjoint semisimple quasi-split algebraic group defined over a field F. Choose a maximal quasi-split torus T of G^{qs} and a Borel subgroup B defined over F such that $T \subset B$. Let Σ be the root system of G^{qs} with respect to T and $\Pi \subset \Sigma$ the system of simple roots corresponding to B. Let Σ^+ (resp. Σ^-) be the set of positive (resp. negative) roots with respect to the basis Π. We write U for the unipotent radical of B and U_α for the root subgroup of $\alpha \in \Sigma$.

Denote by W the Weyl group of Σ. The group W is generated by the set V of reflections with respect to all roots in Π.

Let F_{sep} be a separable closure of F and set $\Gamma_F = \mathrm{Gal}(F_{sep}/F)$. The group Γ_F acts on T, B, W, Σ, Π. This action is called the *$\star$-action*.

If S is a subset of Π, we denote by P_S the standard parabolic subgroup of G^{qs} corresponding to S. Recall that P_S is generated by B and the root subgroups $U_{-\alpha}$ for all $\alpha \in S$. We will say that *P_S is a parabolic subgroup of type S*. Any parabolic subgroup Q of G^{qs} is conjugate to P_S for a unique $S \subset \Pi$. We will say that Q has *type* S. If P is a parabolic subgroup we denote by $R_u P$ the unipotent radical of P.

We denote by $\mathcal{H}_S$ the variety of all parabolic subgroups of G^{qs} of type S, so that $\mathcal{H}_S \simeq G^{qs}/P_S$. The variety $\mathcal{H}_S$ is defined over F if and only if S is invariant under the $\star$-action. For example, $\mathcal{H}_\Pi = \mathrm{Spec} F$ and $\mathcal{H}_\emptyset$ is the variety of all Borel subgroups of G^{qs}. Any projective homogeneous G^{qs}-variety defined over F is isomorphic to $\mathcal{H}_S$ for a unique subset $S \subset \Pi$ invariant under the $\star$-action. If $S \subset S' \subset \Pi$ are two $\star$-invariant subsets then there is a canonical morphism $\mathcal{H}_S \to \mathcal{H}_{S'}$ defined over F.

Let G be an arbitrary adjoint semisimple group. It is an inner form of a quasi-split group G^{qs} unique up to isomorphism. That is the groups $G(F_{sep})$ and $G^{qs}(F_{sep})$ can be identified and Γ_F acts on $G(F_{sep}) = G^{qs}(F_{sep})$ by the formula $\sigma(g) = g_\sigma \sigma^\star(g) g_\sigma^{-1}$ where (g_σ) is a 1-cocycle of Γ_F with values in $G^{qs}(F_{sep})$.

Let F_G be the subfield of F_{sep} corresponding to the kernel of the $\star$-action on Π. Thus, F_G/F is a finite Galois extension and F_G is the smallest field extension of F such that the group G_{F_G} is of inner type. In particular G is of inner type if and only if $F_G = F$.

For a subgroup $H \subset G$ and an element g of G we denote by ${}^gH$ the subgroup gHg^{-1}.

We write $\mathrm{CH}_p(X)$ for the Chow group of classes of dimension p algebraic cycles on an algebraic variety X.

2. Product of two homogeneous projective varieties

Let S, S' be two subsets of Π invariant with respect to the $\star$-action of Γ_F. We denote by P and P' the standard parabolic subgroups of G corresponding to S and S'. Let G_P (resp. $G_{P'}$) be the subgroup of G generated by $U_{\pm\alpha}$, $\alpha \in S$ (resp. S'). The groups G_P and $G_{P'}$ are the semisimple parts of Levi subgroups of P and P'. The root systems Σ_P and $\Sigma_{P'}$ of G_P and $G_{P'}$ with respect to the tori $T_P = T \cap G_P$ and $T_{P'} = T \cap G_{P'}$ are the root subsystems in Σ generated by S and S'. The Weyl groups W_P and $W_{P'}$ of G_P and $G_{P'}$ are the subgroups of W generated by the sets V_S and $V_{S'}$ of reflections with respect to the roots in S and S'.

Let $X = \mathsf{H}_S$ and $X' = \mathsf{H}_{S'}$ be the projective homogeneous G-varieties. Since S, S' are stable with respect to the $\star$-action, the varieties X, X' are defined over F . We consider the diagonal action of G on $X \times X'$. The orbits of this action are defined over F_{sep} but may not be defined over F. The Galois group Γ_F acts on the set of orbits. Note that this action coincides with the $\star$-action.

Lemma 1. (cf. [3], Corollary 5.20) *The assignment $w \mapsto (P, {}^wP')$ gives rise to a Γ_F-equivariant bijection between the set of double cosets $W_P\backslash W/W_{P'}$ and the set of G-orbits in $X \times X'$.*

Proof. Since $X = G/P$ and $X' = G/P'$ the assignment $g \mapsto (P, {}^gP')$ induces a Γ_F-equivariant bijection between $P\backslash G/P'$ and the set of G-orbits in $X \times X'$.

Consider the natural map

$$\gamma : W_P\backslash W/W_{P'} \to P\backslash G/P', \quad \gamma(W_P\, w\, W_{P'}) = P\, w\, P'.$$

Since G is a disjoint union of Bruhat cells BwB, it follows that γ is surjective. Assume that $Pw_1P' = Pw_2P'$ for some $w_1, w_2 \in W$. Then we have

$$w_1 \in Pw_2P' \subset \bigcup_{(w_p, w_{p'}) \in W_P \times W_{P'}} Bw_pBw_2Bw_{p'}B. \tag{1}$$

Recall that for every reflection w_α with respect to a simple root $\alpha \in \Pi$ and every $w \in W$ one has

$$w_\alpha Bw \subset BwB \cup Bw_\alpha wB \quad \text{and} \quad wBw_\alpha \subset BwB \cup Bww_\alpha B.$$

Therefore there exist $w_p \in W_P$, $w_{p'} \in W_{P'}$ such that $w_1 \in Bw'_pw_2w'_{p'}B$ where $w'_p, w'_{p'}$ are subwords of $w_p, w_{p'}$ respectively. Since any subword of w_p (resp. $w_{p'}$) belongs to W_P (resp. $W_{P'}$), we deduce that $w_1 \in W_Pw_2W_{P'}$. This implies that γ is injective.

In the following statement we give necessary and sufficient conditions of a transitive G-action on $X \times X'$.

Corollary 2. *Let $\Sigma_1, \dots, \Sigma_m$ be simple components of Σ and let $\Pi_i = \Pi \cap \Sigma_i$ for $i = 1, 2, \dots, m$ be their bases. Then G acts transitively on $X \times X'$ if and only if each Π_i is contained in either S or S'.*

Proof. By Lemma 1, the number of G-orbits in $X \times X'$ is 1 if and only if $W = W_P\, W_{P'}$. Assume that Π_i is contained in S or S'. Then the corresponding Weyl group W_{Π_i} is contained in W_P or $W_{P'}$. Since W is directly generated by its subgroups $W_{\Pi_1}, \dots, W_{\Pi_m}$ we easily get $W = W_P\, W_{P'}$.

Conversely, let $W = W_P W_{P'}$. Assume that $\Pi_i \cap S \neq \Pi_i$ and $\Pi_i \cap S' \neq \Pi_i$ for some i. Replacing Π by Π_i, S and S' by $\Pi_i \cap S$ and $\Pi_i \cap S'$ respectively we may assume that Σ is irreducible and that $S \neq \Pi$, $S' \neq \Pi$. Consider a positive root α such that its decomposition $\alpha = n_1\alpha_1 + \dots + n_s\alpha_s$, where $\alpha_1, \dots, \alpha_s \in \Pi$ and $n_1, \dots, n_s$ are positive integers, contains simple roots from $\Pi \setminus S$ and $\Pi \setminus S'$.

Let $w \in W$ be the reflection with respect to α and let $w = w_1 w_2$ where $w_1 \in W_P$, $w_2 \in W_{P'}$. Then $w(\alpha) = -\alpha$ is a negative root. On the other hand, by [5, Lemma 5.2], $w_2(\Sigma^+ \setminus \Sigma^+_{S'}) \subset \Sigma^+$. In particular, since the decomposition of α contains a simple root from $\Pi \setminus S'$, the root $w_2(\alpha)$ is positive. Analogously, $w_1^{-1}(\Sigma^- \setminus \Sigma^-_S) \subset \Sigma^-$ and hence $w_1^{-1}(-\alpha)$ is a negative root. This contradicts the equality $w_2(\alpha) = w_1^{-1}(-\alpha)$.

Corollary 3. *If G is a simple group and none of X and X' is a point, then G does not act transitively on $X \times X'$.*

3. The type of the parabolic subgroup $R_uP \cdot (P \cap {}^wP')$

The main result in this section, Proposition 8, is standard and can be found in [4], Theorem 2.7.4 and Proposition 2.8.4. For the reader's convenience we include the proof. We keep the notation of the previous section. In particular, P and P' denote the standard parabolic subgroups of types S and S'. By [2, Proposition 14.22], for every $w \in W$ the subgroup $R_uP \cdot (P \cap {}^wP')$ of G is parabolic and we are going to determine its type. If $w \in W$, we let $l(w)$ denote the length of w with respect to the set of generators V of W.

Lemma 4. *Assume that w is an element of minimal length in the double coset $D = W_P w W_{P'} \subset W$. Then any element $w_1 \in W_P w W_{P'}$ can be written in the form $w_1 = awa'$ with $a \in W_P, a' \in W_{P'}$ such that $l(w_1) = l(a) + l(w) + l(a')$. In particular, D contains a unique element of minimal length.*

Proof. Apply the same argument as in [5, Proposition 3.4].

Lemma 5. *In the notation above we have $w(\Sigma^+_{S'}) \subset \Sigma^+$ and $w^{-1}(\Sigma^+_S) \subset \Sigma^+$.*

Proof. Apply the same argument as in [5, Lemma 5.1].

Remark 6. In the notation of the previous section, every G-orbit of $X \times X'$ contains a unique element of the form $(P, {}^wP')$ where $w \in W$ is the element of minimal length in the double coset $W_P w W_{P'}$.

Let $D \in W_P \backslash W / W_{P'}$ be a double coset and let $w \in D$ be the element of minimal length. Denote by R the parabolic subgroup $R_uP \cdot (P \cap {}^wP')$.

Lemma 7. *One has $B \subset R$, in particular R is a standard parabolic subgroup.*

Proof. Since $R_uP \subset R$, it suffices to show that for every root $\alpha \in S$ one has $U_\alpha \subset {}^wP'$ or equivalently ${}^{w^{-1}}U_\alpha \subset P'$. Since ${}^{w^{-1}}U_\alpha = U_{w^{-1}(\alpha)}$, the required result follows from Lemma 5.

Consider the following subset of the set of simple roots:

$$R_D = \{\alpha \in S \mid w^{-1}(\alpha) \in \Sigma^+_{S'}\} \subset \Pi.$$

Note that in the case $w = 1$, i.e., $D = W_S W_{S'}$, we have $R_D = S \cap S'$.

Proposition 8. *The parabolic subgroup R of G has type R_D.*

Proof. Since R is standard, it suffices to show that the set consisting of simple roots $\alpha \in \Pi$ such that $U_{\pm\alpha} \subset R$ coincides with R_D.

Clearly, for every $\alpha \in R_D$ we have $U_{\pm\alpha} \subset R$. Conversely, let α be a root with this property. Since R_uP is contained in the unipotent radical of R we first conclude that $U_{\pm\alpha} \in G_P \cap {}^wP'$, in particular $\alpha \in S$. Since

$$U_{\pm w^{-1}(\alpha)} = {}^{w^{-1}}U_{\pm\alpha} \subset P',$$

we have $w^{-1}(\alpha) \in \Sigma_{S'}$. By Lemma 5, $w^{-1}(\alpha) \in \Sigma^+ \cap \Sigma_{S'} = \Sigma^+_{S'}$ implying $\alpha \in R_D$.

4. Structure of G-orbits on $X \times X'$

Since S, S' are $\star$-stable, the $\star$-action on W extends to the set of double cosets $W_P\backslash W/W_{P'}$. Let $D \in W_P\backslash W/W_{P'}$ and let O_D be the corresponding G-orbit in $X \times X'$ (Lemma 1). Denote by Z_D the projective homogeneous G-variety H_{R_D} and consider the G-equivariant map

$$\lambda_D : O_D \to Z_D, \quad \lambda_D(Q, Q') = R_uQ \cdot (Q \cap Q').$$

Clearly, λ_D is induced by the canonical morphism $G/(P \cap {}^wP') \to G/R$, where $w \in D$ is the element of minimal length, and therefore, λ_D is a flat morphism of varieties.

Proposition 9. *Assume that the double coset D is $\star$-stable. Then the varieties O_D, Z_D and the morphism λ_D are defined over F.*

Proof. The element $\star(w)$ belongs to D and has the same length as w. It follows that $\star(w) = w$. Hence the set R_D is $\star$-stable and therefore, Z_D is defined over F. By Lemma 1, the orbit O_D is Γ_F-stable, hence O_D is defined over F. It follows from the definition that λ_D is Γ_F-equivariant and hence is also defined over F.

Remark 10. Since $R_D \subset S$, there is a canonical morphism $Z_D \to X$.

Let $D \in W_P\backslash W/W_{P'}$ be an arbitrary double coset. Let F_D be the finite separable field extension of F corresponding to the stabilizer of D in Γ_F (with respect to the $\star$-action). By Proposition 9, the varieties O_D and Z_D and the morphism λ_D are defined over F_D. Let $O_D \to \mathrm{Spec} F_D$ and $Z_D \to \mathrm{Spec} F_D$ be the corresponding structure morphisms. We consider O_D and Z_D as schemes over F with respect to the composites $O_D \to \mathrm{Spec} F_D \to \mathrm{Spec} F$ and $Z_D \to \mathrm{Spec} F_D \to \mathrm{Spec} F$. By Proposition 9, we can view $\lambda_D : O_D \to Z_D$ as a morphism of schemes defined over F.

Proposition 11. *Let $z \in Z_D$ be a point (not necessary closed). Then $\lambda_D^{-1}(z)$ is isomorphic to the affine space over $F(z)$ of dimension $l(w)$.*

Proof. Replacing F by $F(z)$ we may assume that Z_D has a point over F. This point corresponds to a parabolic subgroup in G conjugate to R. Replacing T, B, U and etc. by the corresponding conjugate objects we may assume that $z = R$. Since R is defined over F , we have $g_\sigma \sigma^\star(R) g_\sigma^{-1} = g_\sigma R g_\sigma^{-1} = R$, hence $g_\sigma \in R \subset P$ for all $\sigma \in \Gamma_F$. It follows that P is defined over F . Since D and w are Γ_F-invariant and $R \subset {}^wP'$, analogously, ${}^wP'$ is defined over F.

Since λ_D is G-equivariant, by definition of R,

$$\overset{-1}{\lambda_D}(z) \simeq R/(P \cap {}^wP') \simeq R_uP/(R_uP \cap {}^wP').$$

The next step is to show that

$$\dim R_uP/(R_uP \cap {}^wP') = l(w).$$

Set

$$\Sigma' = (\Sigma^+ \setminus \Sigma_S^+) \cap w(\Sigma^- \setminus \Sigma_{S'}^-), \quad \Sigma'_w = \{\alpha \in \Sigma^+ \mid w^{-1}(\alpha) \in \Sigma^-\}.$$

Note that

$$|\Sigma'_w| = l(w) \text{ and } |\Sigma'| = \dim R_uP/(R_uP \cap {}^wP').$$

We claim that $\Sigma' = \Sigma'_w$. The inclusion $\Sigma' \subset \Sigma'_w$ follows from our construction. Conversely, let $\alpha \in \Sigma'_w$. By Lemma 5, $\alpha \in \Sigma^+ \setminus \Sigma_S^+$. Since $w^{-1}(\alpha)$ is a negative root and since $w(-S') \subset \Sigma^-$ (Lemma 5) we also have $w^{-1}(\alpha) \in \Sigma^- \setminus \Sigma_{S'}^-$ that implies $\alpha \in w(\Sigma^- \setminus \Sigma_{S'}^-)$. Thus $\alpha \in \Sigma'$ as required.

It remains to show that $R_uP/(R_uP \cap {}^wP')$ is isomorphic to an affine space over F. We set $Q' = {}^wP'$ and $\Psi = \Sigma^+ \setminus \Sigma_S^+$. Let Ψ_1 be the subset in Ψ consisting of all roots α such that the corresponding root subgroup U_α is contained in $R_uP \cap Q'$. Clearly, Ψ_1 is closed, i.e., if $\alpha, \beta \in \Psi_1$ and $\alpha + \beta$ is a root, then $\alpha + \beta \in \Psi_1$. Since R_uP and Q' are defined over F, Ψ_1 is Γ_F-stable. Hence the subgroup U_1 generated by U_α, $\alpha \in \Psi_1$ (it coincides with $R_uP \cap Q'$) is defined over F.

Let $R_u^- Q'$ be the negative unipotent radical of Q'. Let Ψ_2 be the subset in Ψ consisting of all roots α such that U_α is not contained in Q' or equivalently $U_\alpha \subset R_uP \cap R_u^- Q'$. As above, Ψ_2 is closed and Γ_F-stable. Let U_2 be a subgroup generated by U_α, $\alpha \in \Psi_2$ (it coincides with $R_uP \cap R_u^- Q'$).

Since $\Psi_1 \cap \Psi_2 = \emptyset$ and $\Psi = \Psi_1 \cup \Psi_2$ the product morphism $U_1 \times U_2 \to R_uP$ is an F-isomorphism. It follows that the variety $R_uP/(R_uP \cap Q') = R_uP/U_1$ is isomorphic to U_2. By [3, Cor. 3.18], U_2 is an F-split subgroup of G. In view of [2, Rem. 15.13], any F-split subgroup as a variety is isomorphic to an affine space over F.

Corollary 12. *There is only one closed orbit in $X \times X'$, namely O_D where $D = W_S W_{S'}$.*

Proof. The orbit is closed if and only if the affine fibers of the projection $O_D \to Z_D$ are trivial, or equivalently, $l(w) = 0$, i.e., $w = 1$ and $D = W_S W_{S'}$.

5. Filtration on $X \times X'$

Recall that we fix two subsets S, S' in Π invariant with respect to the $\star$-action of Γ_F. They determine the standard parabolic subgroups P, P' of G and the projective homogeneous G-varieties X, X'.

Consider the $\star$-action on the set of double cosets $W_P \backslash W / W_{P'}$. Let Δ be the set of all $\star$-orbits in $W_P \backslash W / W_{P'}$. For an orbit $\delta \in \Delta$, the length $l(D)$ does not depend on the choice of a representative $D \in \delta$. We denote this number by l_δ. The varieties O_D and Z_D considered as schemes over F also do not depend on $D \in \delta$. We denote them by O_δ and Z_δ respectively.

Proposition 13. *One can number all elements $\delta_1, \delta_2, \ldots, \delta_n$ of Δ so that there exists a filtration by closed subvarieties*

$$\emptyset = V_0 \subset V_1 \subset \ldots \subset V_n = X \times X'$$

together with flat morphisms $f_i : V_i \setminus V_{i-1} \to Z_{\delta_i}$ of constant relative dimension l_{δ_i} for every $i = 1, 2, \ldots, n$, with the property that the fiber of every f_i over any point $s \in Z_{\delta_i}$ is isomorphic to the affine space over $F(s)$ of dimension l_{δ_i}.

Proof. Recall that we consider the diagonal G-action in $X \times X'$. For every $j \geq 0$ let V'_j be the union of all orbits of dimension at most j. Clearly V'_j is closed in $X \times X'$ and defined over F. Thus we have a filtration by closed subvarieties

$$\emptyset = V'_0 \subset V'_1 \subset \ldots \subset V'_m = X \times X'$$

such that for every $j > 0$, the variety $V'_j \setminus V'_{j-1}$ is the union of G-orbits of the dimension j. The Galois group Γ_F permutes the orbits in $V'_j \setminus V'_{j-1}$ according to the $\star$-action on the set of double cosets $W_P \backslash W / W_{P'}$ in view of Lemma 1. We can then rearrange the filtration:

$$\emptyset = V_0 \subset V_1 \subset \ldots \subset V_n = X \times X'$$

in such a way that every $V_i \setminus V_{i-1}$ coincides with O_δ for an appropriate $\delta \in \Delta$. By Proposition 9, there is a flat morphism $\lambda_\delta : O_\delta \to Z_\delta$ of constant relative dimension l_δ. In view of Proposition 11, every fiber of λ_δ is an affine space.

6. Category of Chow motives

Let J be a class of objects of an additive category $\mathcal{A}$. The *subcategory $\mathcal{A}'$ generated by J* is the smallest full additive subcategory $\mathcal{A}' \subset \mathcal{A}$ containing all objects of J. Objects of $\mathcal{A}'$ are finite direct sums of objects from J.

An additive category $\mathcal{A}$ is called *pseudo-abelian* if every projector in $\mathcal{A}$ splits [1, Ch. I, §3]. Let J be a class of objects of a pseudo–abelian category $\mathcal{A}$. The *pseudo–abelian subcategory $\mathcal{A}'$ generated by J* is the smallest full pseudo–abelian subcategory $\mathcal{A}' \subset \mathcal{A}$ containing all objects of J. Objects of $\mathcal{A}'$ are direct summands of finite direct sums of objects from J.

We say that an additive category $\mathcal{A}$ is a *Tate category* if an additive endo-functor $T : \mathcal{A} \to \mathcal{A}$ is given (called the *Tate twist* functor). We write $A(n) = T^n(A)$ for every $A \in \mathcal{A}$ and $n \geq 0$. Let $\mathcal{A}$ be a Tate category. We say that a full subcategory $\mathcal{A}' \subset \mathcal{A}$

is a *Tate subcategory* if $\mathcal{A}'$ is additive and closed under the Tate twist functor, that is, if $A \in \mathcal{A}'$ then $A(n) \in \mathcal{A}'$ for every $n \geq 0$. Let J be a class of objects of $\mathcal{A}$. The *Tate subcategory of $\mathcal{A}$ generated by J* is the smallest Tate subcategory of $\mathcal{A}$ containing all objects of J. Objects of this subcategory are direct sums of objects of the form $A(n)$ where $A \in J$. A Tate category $\mathcal{A}$ is *finitely generated* if it is generated by a finite set of objects as a Tate category.

An object A of an additive category is called *indecomposable* if A is not isomorphic to the direct sum of two nonzero objects. If $\mathcal{A}$ is pseudo-abelian, A is indecomposable if and only if the ring $\operatorname{End}_{\mathcal{A}}(A)$ has no nontrivial idempotents.

We say that *the Krull-Schmidt theorem* holds for an additive category $\mathcal{A}$ if any two finite direct sum decompositions of an object of $\mathcal{A}$ into indecomposable objects are isomorphic. Recall that a ring S is called *local* if whenever $a + b = 1$ for $a, b \in S$ then either a or b is invertible in S. We will be using the following

Theorem 14. [1, Ch. I, Th. 3.6] *Let $\mathcal{A}$ be a pseudo-abelian category. Suppose that the endomorphism ring of every indecomposable object is local. Then the Krull-Schmidt theorem holds for $\mathcal{A}$.*

Let F be a field and let $\mathcal{V}ar$ be the category of smooth complete varieties over F. For any commutative ring R we define the additive *category $\mathcal{C}(R)$ of correspondences with coefficients in R* as follows. The objects of $\mathcal{C}(R)$ are the same as in $\mathcal{V}ar$. The group of morphisms between X and Y in the case when X is integral of dimension d is equal to

$$\operatorname{Mor}_{\mathcal{C}(R)}(X, Y) = \operatorname{CH}_d(X \times Y) \otimes_{\mathbb{Z}} R.$$

In general, $\operatorname{Mor}_{\mathcal{C}(R)}(X, Y)$ is the direct sum of $\operatorname{Mor}_{\mathcal{C}(R)}(X_i, Y)$ over all irreducible (connected) components X_i of X. The composition of $\alpha \in \operatorname{Mor}_{\mathcal{C}(R)}(X, Y)$ and $\beta \in \operatorname{Mor}_{\mathcal{C}(R)}(Y, Z)$ is defined as

$$\beta \circ \alpha = p_{13*}\big(p_{12}^*(\alpha) \cdot p_{23}^*(\beta)\big),$$

where p_{12}, p_{23} and p_{13} are the projections of $X \times Y \times Z$ on $X \times Y$, $Y \times Z$ and $X \times Z$ respectively.

The direct sum of X and Y in $\mathcal{C}(R)$ is the disjoint union $X + Y$.

We have the functor $\mathcal{V}ar \to \mathcal{C}(R)$ that is the identity on objects and takes a morphism f to the class of the cycle $[\Gamma_f] \otimes 1_R$ where $\Gamma_f \subset X \times Y$ is the graph of f.

The *category of (effective) Chow motives $\mathcal{M}(R)$ with coefficients in* R is the pseudo-abelian closure of $\mathcal{C}(R)$. The objects of $\mathcal{M}(R)$ (called the *motives*) are the pairs (X, ρ) where X is a smooth complete variety over F and $\rho \in \operatorname{End}_{\mathcal{C}(R)}(X)$ is an idempotent (projector). The R-module of all morphisms between (X, ρ) and (X', ρ') is the submodule

$$\rho' \circ \operatorname{Mor}_{\mathcal{C}(R)}(X, X') \circ \rho \subset \operatorname{Mor}_{\mathcal{C}(R)}(X, X').$$

The direct sum operation is given by the disjoint union of varieties:

$$(X, \rho) \oplus (X', \rho') = (X + X', \rho + \rho').$$

The category $\mathcal{M}(R)$ is a symmetric tensor additive category with respect to the tensor product defined by

$$(X, \rho) \otimes (X', \rho') = (X \times X', \rho \times \rho').$$

The composition of functors $\mathcal{V}ar \to \mathcal{C}(R) \to \mathcal{M}(R)$ takes any X to the *motive* $M(X) = (X, \mathrm{id}_X)$ *of* X.

The motive $(\mathbb{P}^1, \rho)$ in $\mathcal{M}(R)$, where ρ is given by the cycle $x \times \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$ for a rational point $x \in \mathbb{P}^1$, is denoted by $R(1)$ and called the *Tate motive.* Let $R(n)$ for $n \geq 0$ be the n-th tensor power of $R(1)$. In particular, $R(0) = M(\mathrm{Spec}F)$. We have

$$\mathrm{Hom}_{\mathcal{M}(R)}\big(R(i), R(j)\big) = \begin{cases} R, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

For a motive M in $\mathcal{M}(R)$ we set $M(n) = M \otimes R(n)$. In particular, $M(0) = M$. Clearly, $\mathcal{M}(R)$ is a Tate category with respect to the Tate twist functor $M \mapsto M(1)$.

Example 15. (Projective bundle theorem [11, §7]) Let E be a vector bundle of rank $r+1$ over X in $\mathcal{V}ar$ and let $\mathbb{P}(E)$ be the associated projective bundle. Then in $\mathcal{M}(R)$:

$$M(P(E)) \simeq M(X) \oplus M(X)(1) \oplus \ldots \oplus M(X)(r).$$

For any two motives M and M' the natural homomorphism

$$\mathrm{Mor}_{\mathcal{M}(R)}(M, M') \to \mathrm{Mor}_{\mathcal{M}(R)}\big(M(n), M'(n)\big)$$

is an isomorphism for every $n \geq 0$. In particular, the endomorphism rings of M and $M(n)$ are isomorphic for every $n \geq 0$. Hence the motive M is indecomposable if and only if $M(n)$ is indecomposable.

We keep the notation of Section 5. The theorem below is a corollary of Proposition 13 and results of N. Karpenko [9, Th. 6.5, Cor. 6.11] (see also [5, Th. 7.2, Rem. 7.3]).

Theorem 16. *Let X and X' be projective homogeneous varieties of a semisimple group G over F. There is an isomorphism in $\mathcal{M}(R)$:*

$$M(X \times X') = \coprod_{\delta \in \Delta} M(Z_\delta)(l_\delta),$$

where each Z_δ is a projective homogeneous G-variety.

Remark 17. The variety Z_δ is defined over the field F_D for $D \in \delta$. Note that $F_D \subset F_G$.

Remark 18. By Corollary 12, the direct sum contains exactly one term with the zero Tate twist - the motive of the closed orbit $O_D = Z_D$ for $D = W_S W_{S'}$. We have $O_D \simeq Z_D = \mathsf{H}_{S \cap S'}$. Moreover, the canonical morphism $\mathsf{H}_{S\cap S'} \to \mathsf{H}_S \times \mathsf{H}_{S'} = X \times X'$ induces an isomorphism between $\mathsf{H}_{S\cap S'}$ and the orbit O_D.

Remark 19. There are canonical morphisms $Z_\delta \to X$ over F (Remark 10). Interchanging the roles of X and X' we can get another decomposition

$$M(X \times X') = \coprod_{\delta \in \Delta} M(Z'_\delta)(l_\delta),$$

where Z'_δ are projective homogeneous G-varieties equipped with the morphisms $Z'_\delta \to X'$ (see Remark 10).

7. Motives of zero-dimensional varieties

Let L/F be a finite Galois field extension with the Galois group Γ. Denote by $\mathcal{C}(R)^L$ the additive subcategory of $\mathcal{C}(R)$ generated by all zero-dimensional varieties $X = \mathrm{Spec}E$ where E is a field such that $F \subset E \subset L$.

Recall that a $R\Gamma$-module P is called a *permutation module* if there is an R-basis of P permuted by Γ. For a variety $X \in \mathcal{C}(R)^L$, the R-module

$$\mathrm{Hom}_{\mathcal{C}(R)}(\mathrm{Spec}L, X) = \mathrm{CH}_0(X_L) \otimes R$$

has a natural structure of a permutation $R\Gamma$-module. For another variety $Y \in \mathcal{C}(R)^L$, the composition law yields a homomorphism

$$\mathrm{Hom}_{\mathcal{C}(R)}(X, Y) \to \mathrm{Hom}_\Gamma\big(\mathrm{CH}_0(X_L) \otimes R, \mathrm{CH}_0(Y_L) \otimes R\big).$$

This is an isomorphism since it can be obtained by taking the $R\Gamma$-invariant elements of the two Γ-modules in the isomorphism (over L)

$$\mathrm{Hom}_{\mathcal{C}(R)}(X_L, Y_L) \to \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{CH}_0(X_L) \otimes R, \mathrm{CH}_0(Y_L) \otimes R\big).$$

Thus, the subcategory $\mathcal{C}(R)^L$ is equivalent to the category of permutation $R\Gamma$-lattices.

An $R\Gamma$-module I is called *invertible* (see [6]) if it is a direct summand of a permutation $R\Gamma$-module. The category $\mathrm{Inv}(R\Gamma)$ of invertible $R\Gamma$-modules is pseudo-abelian. It is equivalent to the pseudo-abelian closure $\mathcal{M}(R)^L$ of the category $\mathcal{C}(R)^L$.

If Γ is trivial, the category $\mathrm{Inv}(R\Gamma)$ coincides with the category $\mathcal{P}(R)$ of finitely generated projective R-modules. Thus the category $\mathcal{M}(R)^F$ is equivalent to $\mathcal{P}(R)$.

If $R = \mathbb{Z}$ and Γ is a cyclic group of prime order ≤ 19, the Krull-Schmidt theorem holds for the category $\mathrm{Inv}(\mathbb{Z}\Gamma)$ [7, Th. 34.31].

Consider the case of a local ring. Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and quotient field K. Denote by $\widehat{R}$ and $\widehat{K}$ the completions of R and K respectively. Let Γ be a finite group of order relatively prime to $\mathrm{char}K$, so that the group algebra $K\Gamma$ is semisimple. Let $D_1, D_2, \ldots, D_s$ be the endomorphism division K-algebras of all simple $K[\Gamma]$-modules. The group Γ is called *split* if $D_i = K$ for all i.

We will be considering the following condition on Γ:

($*$) For all i, $D_i \otimes_K \widehat{K}$ is a division $\widehat{K}$-algebra.

The condition ($*$) implies that if M is a simple $K[\Gamma]$-module, then $M \otimes_K \widehat{K}$ is a simple $\widehat{K}[\Gamma]$-module. The following statement was proved in [7, Th. 30.18]:

Proposition 20. *Let Γ satisfy the condition ($*$). Then the Krull-Schmidt theorem holds for the category* $\mathrm{Inv}(R\Gamma)$.

Remark 21. The condition ($*$) holds in the following cases:

(1) Γ is a split group. For example, $\Gamma = S_n$ the symmetric group;
(2) $R = \mathbb{Z}_{(p)}$ and Γ is a p-group [7, Th. 36.1];
(3) R is complete.

Remark 22. The automorphism group Γ of a connected Dynkin diagram satisfies $(*)$ over $R = \mathbb{Z}_{(p)}$. Indeed, Γ is either a cyclic group of order ≤ 3 or the symmetric group S_3.

Remark 23. In general, the Krull-Schmidt theorem does not hold for $\operatorname{Inv}(R\Gamma)$ over a discrete valuation ring R. The following example is due to E. Beneish.

Let $R = \mathbb{Z}_{(p)}$ for a prime $p \geq 5$ and let Γ be the semi-direct product of a cyclic group Λ of order p with its automorphism group Φ. Let $\tau \in \Phi$ be a generator (of order $p-1$). Consider the ring $L = R[\xi]$ where ξ is a primitive $(p-1)$-th root of unity. We regard L as a $R\Gamma$-module of rank $\varphi(p-1)$ via the composition $R\Gamma \to R\Phi \to L$ where the first map is induced by the canonical homomorphism $\Gamma \to \Phi$ and the second one takes τ to ξ. Since $L_K = L \otimes_R K$ is a field, the $R\Gamma$-module L is indecomposable. Since the homomorphism $R\Phi \to L$ splits, L is a direct summand of $R\Phi$ and therefore is invertible.

Let $\theta \in \widehat{R}$ be a primitive root of unity of degree $p-1$. For every k with $(k, p-1) = 1$ let L^k be the $\widehat{R}\Gamma$-module of rank 1 so that Λ acts trivially on L^k and τ acts by the multiplication by θ^k. We have

$$L_{\widehat{R}} = L \otimes_R \widehat{R} = \coprod_{\substack{1 \leq k < p-1, \\ (k,p-1)=1}} L^k.$$

Consider $M = R[\Gamma/\Phi]$ as a (permutation) $R\Gamma$-module of rank p and let V be the kernel of the augmentation Γ-homomorphism $M \to R$. Over the field $\widehat{K}$,

$$M_{\widehat{K}} = M \otimes_R \widehat{K} = V_{\widehat{K}} \oplus \widehat{K}. \tag{3}$$

The dimension count shows that $V_{\widehat{K}}$ is the only (up to isomorphism) indecomposable (irreducible) $\widehat{K}\Gamma$-module of rank $p-1$. Therefore, $V_{\widehat{K}} \otimes L^k_{\widehat{K}} \simeq V_{\widehat{K}}$ for every k with $(k, p-1) = 1$. The decomposition (3) does not descent to $\widehat{R}$, hence $M_{\widehat{R}}$ is indecomposable over $\widehat{R}$.

Consider the following direct sums of indecomposable $\widehat{R}\Gamma$-modules:

$$C = L^1 \oplus \coprod_{\substack{1 < k < p-1, \\ (k,p-1)=1}} (M \otimes_R L^k), \quad C' = (M \otimes_R L^1) \oplus \coprod_{\substack{1 < k < p-1, \\ (k,p-1)=1}} L^k.$$

Since

$$C_{\widehat{K}} \simeq V_{\widehat{K}}^{\varphi(p-1)-1} \oplus L_{\widehat{K}}, \quad C'_{\widehat{K}} \simeq V_{\widehat{K}} \oplus L_{\widehat{K}},$$

the modules $C_{\widehat{K}}$ and $C'_{\widehat{K}}$ are defined over K. By [7, Cor. 30.10], there are $R\Gamma$-modules D and D' such that $D_{\widehat{R}} \simeq C$ and $D'_{\widehat{R}} \simeq C'$. We have

$$D_{\widehat{R}} \oplus D'_{\widehat{R}} \simeq C \oplus C' \simeq (M_{\widehat{R}} \otimes L_{\widehat{R}}) \oplus L_{\widehat{R}},$$

By [7, Prop. 30.17],

$$D \oplus D' \simeq (M \otimes L) \oplus L.$$

Note that since $p \geq 5$, we have $L_{\widehat{R}} \neq L^1$, hence L is not a direct summand of D or D' even over $\widehat{R}$ since the Krull-Schmidt theorem holds over $\widehat{R}\Gamma$ by [7, Th. 30.18].

8. Category $\mathcal{M}(G,R)$

Let G be a semisimple algebraic group defined over F. Recall that F_G is the subfield of F_{sep} corresponding to the kernel of the $\star$-action.

Let J be the class of all projective homogeneous varieties X over L of the group G_L for all fields L such that $F \subset L \subset F_G$, where X is considered as a variety over F via the natural morphism $\mathrm{Spec}L \to \mathrm{Spec}F$. Denote by $\mathcal{M}(G,R)$ the Tate pseudo-abelian subcategory of $\mathcal{M}(R)$ generated by the motives of varieties from J. An object of $\mathcal{M}(G,R)$ is a direct summand of a motive of the form $\coprod M(X_i)(n_i)$ where all $X_i \in J$ and $n_i \geq 0$.

Theorem 16 and Remark 17 imply that the tensor product of two motives $M(X)$ and $M(Y)$ for two projective homogeneous varieties X and Y is an object of $\mathcal{M}(G,R)$. In other words the following holds:

Theorem 24. *The category $\mathcal{M}(G,R)$ is a symmetric tensor category.*

We will be using the following generalization of the Nilpotence theorem [5, Th. 8.2]:

Theorem 25. *Let M be an object of $\mathcal{M}(G,R)$. Then for every field extension L/F, the kernel of the natural ring homomorphism $\mathrm{End}(M) \to \mathrm{End}(M_L)$ is a nil ideal.*

Proof. We may assume that M is a direct sum of motives of the form $M(Y)(i)$ for a projective homogeneous G-variety Y. The motive $M(Y)(i)$ is a direct summand of $M(Y \times \mathbb{P}^i)$ (Example 15). Hence we are reduced to the case when $M = M(X)$ where X is the disjoint union of varieties of the form $Y \times \mathbb{P}^i$. Note that $Y \times \mathbb{P}^i$ is a projective homogeneous variety of $G \times \mathbf{SL}_{i+1}$. The statement follows from [5, Th. 8.2] (note that the proof goes through if X is not connected).

Corollary 26. *Let M be an object of $\mathcal{M}(G,R)$ and let L/F be a field extension such that $M_L = 0$. Then $M = 0$.*

Proof. The identity endomorphism id_M is zero over L, hence it is nilpotent, and therefore zero.

The category $\mathcal{M}(G,R)$ contains $R(0) = M(\mathrm{Spec}F)$. Let $\mathcal{M}(G,R)_s$ be the Tate pseudo-abelian subcategory of $\mathcal{M}(G,R)$ generated by $R(0)$. By (2), $\mathcal{M}(G,R)_s$ is equivalent to the direct sum of the full subcategories $\mathcal{M}(R)^F(n)$ over all $n \geq 0$ and hence it is equivalent to the direct sum of countably many copies of the category $\mathcal{P}(R)$:

$$\mathcal{M}(G,R)_s \simeq \mathcal{P}(R)^{\mathbb{N}}. \tag{4}$$

where $\mathbb{N} = \{0,1,2,\dots\}$.

If G is split, the inclusion $\mathcal{M}(G,R)_s \hookrightarrow \mathcal{M}(G,R)$ is an equivalence of categories (see Lemma 29 below). If G is arbitrary, for any motive $M \in \mathcal{M}(G,R)$, the motive $M_{sep} = M \otimes_F F_{sep}$ of the split group G_{sep} defines then a sequence of projective modules $P_n \in \mathcal{P}(R)$, $n \geq 0$, almost all zero. Suppose that $\mathrm{Spec}R$ is connected. Then $\mathrm{rank}(P_n) \in \mathbb{N}$ of every P_n is well defined. For every $n \geq 0$ set

$$\mathrm{rank}_n(M) = \mathrm{rank}(P_n) \in \mathbb{Z}.$$

We have $\mathrm{rank}_n(M \oplus M') = \mathrm{rank}_n(M) + \mathrm{rank}_n(M')$ for any two motives M and M'.

Proposition 27. *Let* Spec(R) *be connected. A motive* $M \in \mathcal{M}(G,R)$ *is zero if and only if* $\operatorname{rank}_n(M) = 0$ *for all* $n \geq 0$.

Proof. Let $\operatorname{rank}_n(M) = 0$. Since the category $\mathcal{M}(G_{sep},R)$ is equivalent to $\mathcal{M}(G_{sep},R)_s$, in view of the equivalence (4) we get $M_{sep} = 0$, hence $M = 0$ by Corollary 26.

Theorem 28. *Let* Spec(R) *be connected. Then*

(1) Every motive in $\mathcal{M}(G,R)$ *is isomorphic to a finite direct sum of indecomposable motives.*
(2) Suppose the Krull-Schmidt theorem holds for $\mathcal{M}(G,R)$. *Then the category* $\mathcal{M}(G,R)$ *is finitely generated as a Tate category.*

Proof. (1) Let $M \in \mathcal{M}(G,R)$. We proceed by induction on

$$r(M) = \sum_{n\geq 0} \operatorname{rank} M_n.$$

If N is a nontrivial direct summand of a motive M then $\operatorname{rank}_n(N) \leq \operatorname{rank}_n(M)$ for all n, and it follows from Proposition 27 that the strict inequality holds for at least one n. Hence $r(N) < r(M)$ and the first statement holds for N by induction.

(2) Let J be the finite set of indecomposable direct summands of $M(X)$ for all (finitely many up to isomorphism) projective homogeneous varieties of G over (finitely many) fields between F and F_G. The set J generates $\mathcal{M}(G,R)$. Indeed, an indecomposable object M is a direct summand of a motive of the form $\coprod_i^m M(X_i)(n_i)$ where X_i are projective homogeneous G-varieties. By the Krull-Schmidt theorem, M is isomorphic to $N(n_i)$ for some $N \in J$ and $n_i \geq 0$.

9. Krull-Schmidt theorem

Let $\mathcal{M}(G,R)_{qs}$ be the Tate subcategory of $\mathcal{M}(G,R)$ generated by the motives of zero-dimensional varieties Spec(L) where L is a field such that $F \subset L \subset F_G$. Set $\Gamma = \operatorname{Gal}(F_G/F)$.

From the discussion in Sections 6 and 7 we get an equivalence of categories

$$\mathcal{M}(G,R)_{qs} \simeq \operatorname{Inv}(R\Gamma)^{\mathbb{N}}. \tag{5}$$

Lemma 29. *Let* G *be a quasi-split group. Then the inclusion* $\mathcal{M}(G,R)_{qs} \hookrightarrow \mathcal{M}(G,R)$ *is an equivalence of categories.*

Proof. Let L be a field such that $F \subset L \subset F_G$. By [5, Th. 7.5], the motive of a projective homogeneous G_L-variety over L is a direct sum of the twisted motives of anisotropic projective homogeneous varieties defined over fields E between L and F_G. Since G_E is quasi-split, we have $X(E) \neq \emptyset$ for any projective homogeneous G-variety X. Hence every anisotropic homogeneous variety X over E is the point $SpecE$.

Proposition 30. *Let* G *become quasi-split over a separable field extension of degree* n. *If* n *is invertible in* R *then the category* $\mathcal{M}(G,R)$ *is equivalent to* $\operatorname{Inv}(R\Gamma)^{\mathbb{N}}$.

Proof. Consider first the case when G is quasi-split. By Lemma 29, the category $\mathcal{M}(G,R)$ is equivalent to $\mathcal{M}(G,R)_{qs}$ and therefore to $\operatorname{Inv}(R\Gamma)^{\mathbb{N}}$ by (5).

In the general case, let L/F be a separable field extension of degree n such that G_L is quasi-split. For any two projective homogeneous G-varieties X and Y, the homology groups of the Čech complex

$$0 \to \mathrm{CH}_*(X \times Y) \otimes R \to \mathrm{CH}_*(X_L \times Y_L) \otimes R \to \mathrm{CH}_*(X_{L\otimes L} \times Y_{L\otimes L}) \otimes R$$

have exponent n. Since n is invertible in R, the complex is acyclic, hence the group $\operatorname{Mor}\big(M(X), M(Y)\big)$ is isomorphic to the kernel of

$$\operatorname{Mor}\big(M(X_L), M(Y_L)\big) \to \operatorname{Mor}\big(M(X_{L\otimes L}), M(Y_{L\otimes L})\big).$$

Let G^{qs} be a quasi-split twisted form of G. Recall that a quasi-split form of G is unique up to an isomorphism and that two groups G and G^{qs} have the same $\star$-action of Γ_F on the root systems of G and G^{qs}. The projective homogeneous varieties X, Y determine uniquely the subsets S_X and S_Y in Π. These subsets in turn determine uniquely the projective homogeneous varieties X^{qs} and Y^{qs} of G^{qs}. Since the groups G^{qs}_L and G_L are isomorphic, then the group $\operatorname{Mor}\big(M(X^{qs}), M(Y^{qs})\big)$ is isomorphic to the same kernel. Thus, the categories $\mathcal{M}(G,R)$ and $\mathcal{M}(G^{qs},R)$ are equivalent. The result follows then from the first part of the proof.

Corollary 31. *The Krull-Schmidt theorem holds for $\mathcal{M}(G,R)$ in the following cases:*

1. *G is split and every finitely generated projective R-module is free;*
2. *R is a discrete valuation ring and G is quasi-split such that Γ satisfies condition $(*)$. For example, G is a simple quasi-split group.*

Proof. The first statement follows from Proposition 30 since $n = 1$. The second follows from Proposition 20.

The following example shows that the Krull-Schmidt theorem does not hold over $R = \mathbb{Z}$ even for a group G of inner type.

Example 32. Let A and B be two central simple F-algebras generating the same subgroup in the Brauer group $\operatorname{Br}(F)$. Let X and Y be the corresponding Severi-Brauer varieties. We can view X and Y as projective homogeneous varieties of the group $G = \mathbf{PGL}_1(A) \times \mathbf{PGL}_1(B)$ of inner type. By assumption, the projections of $X \times Y$ to X and Y are projective vector bundles. By the projective bundle theorem (Example 15),

$$M(X \times Y) \simeq M(X) \oplus M(X)(1) \oplus \ldots \oplus M(X)(m),$$
$$M(X \times Y) \simeq M(Y) \oplus M(Y)(1) \oplus \ldots \oplus M(Y)(n),$$

where $n = \dim X$ and $m = \dim Y$. If A and B are division algebras then the motives $M(X)$ and $M(Y)$ are indecomposable by [8, Th. 2.2.1]. On the other hand, the integral motives of X and Y are not isomorphic if A is not isomorphic to B or the opposite algebra B^{op} [10, Criterion 7.1].

Remark 33. A. Vishik proved in [13] that the Krull-Schmidt theorem holds for the integral motives of projective quadrics.

We use the notation of Section 7.

Theorem 34. *Let R be a discrete valuation ring and let G be a semisimple group such that the group Γ satisfies the condition $(*)$. Then*

(1) The Krull-Schmidt theorem holds for $\mathcal{M}(G,R)$.
(2) The category $\mathcal{M}(G,R)$ is finitely generated as a Tate category.

Proof. The second statement follows from (1) and Theorem 28. Let M be an indecomposable motive in $\mathcal{M}(G,R)$. By Theorem 14, it suffices to show that the ring $\mathrm{End}(M)$ is local. Let S be the image of $\mathrm{End}(M)$ under the R-algebra homomorphism $f : \mathrm{End}(M) \to \mathrm{End}(M_{sep})$. Since G is split over F_{sep}, the R-module $\mathrm{End}(M_{sep})$ is finitely generated, hence S is an R-order in $S \otimes_R K$. By Theorem 25, the kernel of f is a nil ideal. Therefore, it is sufficient to prove that S is a local ring.

The K-algebra $S \otimes_R K$ is the endomorphism ring of M_K in $\mathcal{M}(G,K)$. By Proposition 30 the category $\mathcal{M}(G,K)$ is equivalent to $\mathrm{Inv}(K\Gamma)^{\mathbb{N}}$, hence $S \otimes_R K$ is isomorphic to a product of rings of the form $\mathrm{End}_{K\Gamma}(N)$ for some finitely generated $K\Gamma$-module N. It follows that the endomorphism ring of a simple $S \otimes_R K$-module is isomorphic to the K-algebra D_i for some i. By the condition $(*)$, $D_i \otimes \widehat{K}$ is a division $\widehat{K}$-algebra. It is proven in [7, Th. 30.18] that for every indecomposable S-lattice T, the $\widehat{S}$-lattice $\widehat{T}$ is also indecomposable. Applying this statement to $T = S$ we see that the ring $\widehat{S}$ has no nontrivial idempotents. The lifting property for the idempotents [1, Ch. III, Prop. 2.10] shows that the factor ring $S/\mathfrak{m}S = \widehat{S}/\mathfrak{m}\widehat{S}$ has no nontrivial idempotents. The ring $S/\mathfrak{m}S$ is a finite dimensional algebra over the field $R/\mathfrak{m}R$ and hence is local.

We are finally ready to prove that S is a local ring. Let $a + b = 1$ for $a, b \in S$. Since the ring $S/\mathfrak{m}S$ is local one of the residues $\bar{a}$ or $\bar{b}$ in $S/\mathfrak{m}S$, say $\bar{a}$, is invertible. By Nakayama Lemma, the left and right multiplication endomorphisms by a of the finitely generated R-module S are surjective, therefore, a is invertible in S and hence S is a local ring.

Remarks 21 and 22 imply

Corollary 35. *Let R be a discrete valuation ring. The Krull-Schmidt theorem holds for $\mathcal{M}(G,R)$ in the following cases:*

(1) $R = \mathbb{Z}_{(p)}$ and G is a simple group.
(2) G has inner type.
(3) The group $\mathrm{Gal}(F_G/F)$ is split.
(4) R is complete.

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