DEGREE THREE UNRAMIFIED COHOMOLOGY OF ADJOINT SEMISIMPLE GROUPS

ALEXANDER S. MERKURJEV

Abstract. We compute degree 3 unramified cohomology of adjoint semisimple group. New classes of non-rational of adjoint semisimple groups are found.

1. Introduction

Let X be an integral variety over a field F. We write $H^d(F(X))$ for the degree d Galois cohomology group $H^d(F(X), \mu^{\otimes(d-1)})$ of the function field $F(X)$, where $\mu^{\otimes(d-1)} = \operatorname{colim}\big(\mu_n^{\otimes(d-1)}\big)$. The subgroup $H^d(F(X))_{\mathrm{nr}}$ of *unramified* cohomology is a birational invariant of X. In particular, if X is a rational variety, i.e., the function field $F(X)$ is a purely transcendental extension of F, then the *reduced* unramified cohomology group $\overline{H}^d(F(X))_{\mathrm{nr}} := H^d(F(X))_{\mathrm{nr}}/H^d(F)$ is trivial.

Let G be a connected algebraic group over a field F. If F is algebraically closed, the variety of G is rational. Over arbitrary fields, the first example of a nonrational group was constructed by Chevalley. He proved that the norm one torus for a biquadratic field extension is never rational. The obstruction to rationality is the reduced unramified Brauer group $\overline{H}^2(F(G))_{\mathrm{nr}}$ which is a cyclic group of order 2.

The unramified Brauer group of a semisimple group G was computed in [4]. If Z is the kernel of a simply connected cover $\widetilde{G} \to G$ and Z^* is the character Galois module of Z, then the group $\overline{H}^2(F(G))_{\mathrm{nr}}$ is isomorphic to the subgroup of $H^1(F, Z^*)$, consisting of all elements that are split over a field extension with pro-cyclic absolute Galois group. There are examples of semisimple groups G such that the group $\overline{H}^2(F(G))_{\mathrm{nr}}$ is not trivial (see [29, Chapter III, §4.7]), and therefore, G is not rational.

The group $\overline{H}^2(F(G))_{\mathrm{nr}}$ depends only on Z, hence a quasi-split inner twisted form G^{qs} has the same unramified Brauer group as G. In particular, since the variety of the group G^{qs} is rational when G is adjoint (Lemma 6.6), the group $\overline{H}^2(F(G))_{\mathrm{nr}}$ is trivial when G is adjoint.

In the present paper we compute the third unramified group $H^3(F(G))_{\mathrm{nr}}$ for G an adjoint semisimple group. If Z is the kernel of the simply connected cover $\widetilde{G} \to G$, we assume throughout the paper that the order $|Z|$ of Z is not divisible by $\operatorname{char}(F)$.

We use the language of cohomological invariants as follows. Let $\mathcal{B} : \mathsf{Fields}(F) \to \mathsf{Groups}$ be a functor from the category of field extensions of F to the category of groups. A degree d *homomorphic cohomological invariant of* $\mathcal{B}$ is a collection of group homomorphisms $\mathcal{B}(K) \to H^d(K)$ for all field extensions K/F, functorial in K (see [24]). We write $\operatorname{Inv}_h^d(\mathcal{B})$ for the group of all degree d cohomological invariants of $\mathcal{B}$.

An algebraic group G can be viewed as a functor $\mathit{Fields}(F) \to \mathit{Groups}$ taking a field K to the group $G(K)$ of K-points of G. The evaluation of an invariant at the generic point in $G(F(G))$ identifies the group $\mathrm{Inv}^d_h(G)$ with a subgroup of $\overline{H}^d(F(G))$. We prove the following theorem in Section 8.

Theorem A. *Let G be an adjoint semisimple group. Then the group of unramified cohomology $\overline{H}^3(F(G))_{\mathrm{nr}}$ is contained in the subgroup $\mathrm{Inv}^3_h(G)$ of $\overline{H}^3(F(G))$.*

A homomorphic invariant I in $\mathrm{Inv}^3_h(G)$ is called *unramified* if all values of I over all field extensions K/F are unramified with respect to all discrete valuations on K over F. The group of homomorphic unramified invariants is identified with $\overline{H}^3(F(G))_{\mathrm{nr}}$ (Lemma 6.1).

In order to compute the group $\overline{H}^3(F(G))_{\mathrm{nr}}$, we proceed in two steps. First, we determine the group of homomorphic invariants $\mathrm{Inv}^3_h(G)$ and then determine those invariants that are unramified.

To compute $\mathrm{Inv}^3_h(G)$, we compare invariants of G with the invariants of the functor $\mathrm{B}Z : \mathit{Fields}(F) \to \mathit{Groups}$ taking a field extension K/F to the group of isomorphism classes of principal homogeneous spaces of Z over K. The pre-image under $\widetilde{G} \to G$ of a point in $G(K)$ is a principal homogeneous Z-space over K, i.e., we have a morphism of functors $G \to \mathrm{B}Z$ and therefore, a homomorphism $\mathrm{Inv}^3_h(\mathrm{B}Z) \to \mathrm{Inv}^3_h(G)$. The latter homomorphism is almost isomorphism: it is surjective and its kernel is a finite group (Theorem 4.20). The reason is that the morphism of stacks $G \to \mathrm{B}Z$ is a $\widetilde{G}$-torsor and the simply connected group $\widetilde{G}$ is "2-connected", i.e., various cohomology groups of $\widetilde{G}$ (such as the Picard and Brauer groups) are trivial up to degree 2.

An element $\alpha \in H^2(F, Z^\circ)$, where Z° is the group dual to Z, yields an invariant $I_\alpha \in \mathrm{Inv}^3_h(\mathrm{B}Z)$ taking the class $\beta \in H^1(K, Z)$ of a principal homogeneous Z-space over K to the cup-product $\alpha_K \cup \beta \in H^3(K)$. In general, the homomorphism

$$\nu : H^2(F, Z^\circ) \to \mathrm{Inv}^3_h(\mathrm{B}Z), \quad \alpha \mapsto I_\alpha$$

is neither injective nor surjective. But if G is an adjoint group, Z is the kernel of a surjective homomorphism of quasi-trivial tori. In this case, ν is an isomorphism (Proposition 3.4).

The composition $H^2(F, Z^\circ) \xrightarrow{\nu} \mathrm{Inv}^3_h(\mathrm{B}Z) \to \mathrm{Inv}^3_h(G)$ takes an element $\alpha \in H^2(F, Z^\circ)$ to an invariant in $\mathrm{Inv}^3_h(G)$ given by the cup-product

$$g \in G(K) \mapsto \alpha_K \cup \theta_K(g) \in H^3(K)$$

for a field extension K/F, where $\theta_K : G(K) \to H^1(K, Z)$ is the connecting homomorphism. We prove (Theorem 5.3):

Theorem B. *Let G be an adjoint semisimple group, Z the kernel of a simply connected cover $\widetilde{G} \to G$. Assume that $|Z|$ is not divisible by $\mathrm{char}(F)$. Then the homomorphism $\nu : H^2(F, Z^\circ) \to \mathrm{Inv}^3_h(G)$ yields an isomorphism $H^2(F, Z^\circ)/\Phi_G \simeq \mathrm{Inv}^3_h(G)$, where Φ_G is the finite subgroup of $H^2(F, Z^\circ)$ generated by the components of the dual Tits class.*

Now we determine which degree 3 homomorphic invariants of G are unramified, i.e., we determine the group $\overline{H}^3(F(G))_{\mathrm{nr}}$. Proposition 6.2 reduces the problem to the case of an absolutely simple group.

In Section 6, using case-by-case considerations, we prove the following theorem.

Theorem C. *Let G be an absolutely simple adjoint group over F. Assume that $|Z|$ is not divisible by* $\mathrm{char}(F)$*. Then the group $\overline{H}^3(F(G))_{\mathrm{nr}}$ is trivial except for the following two cases where this group is cyclic of order 2:*

(1) *Type ${}^2\mathsf{A}_n$, $n \equiv 3$ modulo 4. The group G is the projective unitary group $\mathbf{PGU}(B,\tau)$ of a central simple K-algebra B of degree $n+1$ with an involution τ of the second kind over a quadratic field extension K/F such that the exponent of B is even and the discriminant algebra D of (B,τ) is not split but D_K is split.*
(2) *Type ${}^2\mathsf{D}_n$, $n \geq 3$ is odd. The group G is the projective orthogonal group $\mathbf{PGO}^+(A,\sigma)$ of a central simple F-algebra A of degree $2n$ with an orthogonal involution σ of nontrivial discriminant quadratic field extension K/Fsuch that A and the Clifford algebra $C(A,\sigma)$ are not split but A_K is split.*

As an application, we derive classes of non-rational adjoint groups.

We use the following notation in the paper.

F is the base field, F_{sep} is a separable closure of F, $\Gamma_F := \mathrm{Gal}(F_{\mathrm{sep}}/F)$ the absolute Galois group. A *Galois module* over F is a continuous Γ_F-module.

If X is a smooth algebraic variety over F and M_* is Rost's *cycle module* over X (see [28]), then for any $i \geq 0$, the homology group of the complex

$$\ldots \to \coprod_{x\in X^{(i-1)}} M_{d-i+1}\big(F(x)\big) \xrightarrow{\partial} \coprod_{x\in X^{(i)}} M_{d-i}\big(F(x)\big) \xrightarrow{\partial} \coprod_{x\in X^{(i+1)}} M_{d-i-1}\big(F(x)\big) \to \ldots,$$

where $X^{(i)}$ is the set of point in X of codimension i, will be denoted by $A^i(X,M_d)$. In particular, if $M = K$ is given by Milnor's K-groups, we have $A^i(X,K_i) = \mathrm{CH}^i(X)$ is the Chow group of classes of algebraic cycles on X of codimension i.

If G is an algebraic group (a smooth affine group scheme of finite type) over F, we write G^* for the Galois module $\mathrm{Hom}(G_{\mathrm{sep}},\mathbf{G}_{\mathrm{m}})$ of characters and $G^*(F)$ for the subgroup $(G^*)^{\Gamma_F}$ of characters defined over F.

2. The Tits class

Let G be an adjoint semisimple algebraic group over a field F and let Z be the kernel of a simply connected cover $\widetilde{G} \to G$.

2a. **Tits classes and Tits algebras.** Let $\xi \in H^1(F,G)$ be (a unique) class such that the twisted group ${}^{\xi}G$ is quasi-split (see [14, §31]). The *Tits class* t_G of G is negative of the image of ξ under the connecting map

$$H^1(F,G) \to H^2(F,Z).$$

The *Tits homomorphism* (see [31, §3])

$$\beta_G : Z^*(F) \to \mathrm{Br}(F)$$

takes a character $\chi \in Z^*(F)$ to the image of the Tits class t_G under the homomorphism

$$\chi_* : H^2(F, Z) \to H^2(F, \mathbf{G}_\mathrm{m}) = \mathrm{Br}(F).$$

A central simple F-algebra A_χ representing the class $\beta_G(\chi) = \chi_*(t_G)$ is called a *Tits algebra of* χ (see [14, §27]).

By [30, §3.1.2], there are finite separable field extensions $L_i/F, i \in \Omega$, and absolutely simple simply connected groups $\widetilde{G}_i$ over L_i such that

$$\widetilde{G} \simeq \prod_{i\in\Omega} R_{L_i/F}(\widetilde{G}_i) \quad \text{and} \quad Z \simeq \prod_{i\in\Omega} R_{L_i/F}(Z_i),$$

where Ω is the set of Γ_F-orbits in the set of irreducible components of the Dynkin diagram of G and Z_i is the center of $\widetilde{G}_i$.

We have

$$H^2(F, Z) \simeq \coprod_{i\in\Omega} H^2(L_i, Z_i).$$

The components $t_i \in H^2(L_i, Z_i)$ of the Tits class t_G are called *Tits components of* G.

Choose a system of simple roots Π of the root system of G and consider the Galois $*$-*action* on Π (see [30, §2.3]). Let $\Sigma \subset \Pi$ be a subset preserved by the $*$-action. We denote by X_Σ the projective homogeneous variety of parabolic subgroups in G of type Σ (see [30, §2.5.4]).

For each orbit ω of $\Pi \setminus \Sigma$ under the $*$-action, we restrict to Z the sum of all the fundamental weights λ_α associated to the roots $\alpha \in \omega$ and set

$$\chi_\omega = \sum_{\alpha\in\omega} \lambda_\alpha \in Z^*(F) \quad \text{for} \quad \omega \in (\Pi \setminus \Sigma)/\Gamma_F.$$

Theorem 2.1. [23, Theorem B] 1. *The kernel of the scalar extension map* $\mathrm{Br}(F) \to \mathrm{Br}\big(F(X_\Sigma)\big)$ *is generated by the classes of Tits algebras* $\beta_G(\chi_\omega)$ *for* $\omega \in (\Pi \setminus \Sigma)/\Gamma_F$.
2. *The variety* $X_\emptyset$ *is the variety of Borel subgroups of* G. *The characters* $\chi_\omega, \omega \in \Pi/\Gamma_F$, *generate* $Z^*(F)$ *and* $\mathrm{Ker}\big(\mathrm{Br}(F) \to \mathrm{Br}\, F(X_\emptyset)\big) = \mathrm{Im}(\beta_G)$.

2b. **Dual Tits classes.** Let $R \subset V$ be the root system of G (here V is a real vector space of finite dimension) and $R^\vee \subset V^\vee$ the dual root system. Choose a (unique) $\mathrm{Aut}(R)$-invariant scalar product on $V^\vee$ such that the length of all short co-roots in every irreducible component of $R^\vee$ is equal to 1.

Write $\Lambda_r \subset \Lambda_w$ for the *root* and *weight lattices* in V respectively and $\Lambda_r^\vee \subset \Lambda_w^\vee$ for the corresponding lattices in $V^\vee$.

Let d_α be the square-length of the co-root $\alpha^\vee$ for all $\alpha \in R$. Then d_α takes one of the three values: 1, 2 or 3.

Proposition 2.2. [10, §5b] *There is a unique* $\mathrm{Aut}(R)$*-equivariant linear isomorphism* $\varphi : V^\vee \xrightarrow{\sim} V$ *such that* $\varphi(\alpha^\vee) = d_\alpha \alpha$ *for all* $\alpha \in R$. *The map* φ *satisfies* $\varphi(\Lambda_w^\vee) \subset \Lambda_w$ *and* $\varphi(\Lambda_r^\vee) \subset \Lambda_r$.

Write $\Delta := \Lambda_w/\Lambda_r$ and $\Delta^\vee := \Lambda_w^\vee/\Lambda_r^\vee$. The natural non-degenerate pairing

$$\Delta \otimes \Delta^\vee \to \mathbb{Q}/\mathbb{Z}$$

identifies $\Delta^\vee$ with the dual $\mathrm{Hom}(\Delta, \mathbb{Q}/\mathbb{Z})$ of Δ.

The homomorphism φ yields a homomorphism

$$\rho : \Delta^\vee \to \Delta$$

that is equivariant under the automorphism group of the Dynkin diagram.

Example 2.3. (see [10, §5.c]) If an irreducible root system R is of type B_n for some $n \geq 2$, or of type C_n for some even n, then $\rho = 0$. For other types, ρ is an isomorphism.

The character Galois module Z^* of the center $Z \subset \widetilde{G}$ is isomorphic to Δ, $Z_* := \operatorname{Hom}(Z^*, \mathbb{Q}/\mathbb{Z}) = \Delta^\vee$ and we have the homomorphisms

$$\rho : Z_* \to Z^* \quad \text{and} \quad \hat{\rho} : Z \to Z^\circ$$

of Galois modules and algebraic groups respectively, where Z° is dual to Z, i.e., $(Z^\circ)^* = Z_*$.

The image t_G° of the Tits class t_G under the homomorphism $\hat{\rho}_* : H^2(F, Z) \to H^2(F, Z^\circ)$ is called the *dual Tits class of* G. We write $t_i^\circ, i \in \Omega$, for the components of t_G°.

3. Invariants

We define two types of invariants in this section.

3a. **Torsors.** Let G be an algebraic group over a field F. There is a natural bijection between the pointed set of (non-abelian) Galois cohomology

$$H^1(F, G) := H^1\big(\Gamma_F, G(F_{\mathrm{sep}})\big)$$

and the set of isomorphism classes of principal homogeneous spaces of G over $\operatorname{Spec} F$. The distinguished point of $H^1(F, G)$ is the class of a trivial G-space.

A G-torsor $f : E \to X$ over a non-empty smooth variety X over F is called *versal* if for any principal homogeneous space J of G over a field extension K/F with K infinite and for every non-empty open subset $U \subset X$, there is a morphism $x : \operatorname{Spec}(K) \to U$ (i.e., x is a point of U over K) such that J is isomorphic to the pull-back of f with respect to x (see [8]).

Versal torsors exist. More precisely, for every integer i, there is a finite dimensional representation V of G, a nonempty G-invariant Zariski open subset $V' \subset V$ and a G-torsor $V' \to V'/G$ for some smooth variety V'/G over F such the codimension in V of the complement of V' is at least i (see [32, Remark 1.4]). We will write $\mathrm{E}G \to \mathrm{B}G$ for the G-torsor $V' \to V'/G$ and call it a *universal* G*-torsor* and the variety $\mathrm{B}G = V'/G$ a *classifying space of* G. The G-torsor $\mathrm{E}G \to \mathrm{B}G$ is versal.

We can define various cohomology groups of $\mathrm{B}G$ by using an appropriate approximation of $\mathrm{B}G$. For example, the Chow group $\mathrm{CH}^j(\mathrm{B}G)$ is defined as $\mathrm{CH}^j(V'/G)$, where the codimension of $V \setminus V'$ in V is larger than j (see [32, Theorem 1.1]).

3b. **Invariants of the first type.** Let $\mu_n \subset F_{\mathrm{sep}}$ denote the Galois module of nth roots of unity and let $\mu_n^{\otimes i}$ be the ith tensor power of μ_n. Write

$$H^d(F) := \operatorname{colim} H^d(F, \mu_n^{\otimes(d-1)}),$$

where the colimit is taken over all n prime to $\operatorname{char}(F)$. In particular, $H^2(F) := \operatorname{Br}(F)'$ is "out of $p = \operatorname{char}(F)$-part" of the Brauer group of F.

Note that the groups $H^d(K)$ form a *cycle module* (see [28, Remark 1.11]).

We view H^d as a functor $K \mapsto H^d(K)$ from the category $\mathit{Fields}(F)$ of field extensions of F to the category of groups Groups or to the category PSets of pointed sets.

Let $\mathcal{A} : \mathit{Fields}(F) \to \mathit{PSets}$ be a functor. A degree d *cohomological invariant of* $\mathcal{A}$ is a collection of maps of pointed sets

$$\mathcal{A}(K) \to H^d(K)$$

for all field extensions K/F, functorial in K. In other words, an invariant is a morphism of functors $\mathcal{A} \to H^d$ from $\mathit{Fields}(F)$ to PSets. All degree d cohomological invariants of $\mathcal{A}$ form an abelian group denoted by $\operatorname{Inv}^d(\mathcal{A})$.

The functor $\mathcal{A} \mapsto \operatorname{Inv}^d(\mathcal{A})$ is contravariant in $\mathcal{A}$: a morphism of functors $\mathcal{A} \to \mathcal{A}'$ yields a group homomorphism $\operatorname{Inv}^d(\mathcal{A}') \to \operatorname{Inv}^d(\mathcal{A})$.

Example 3.1. If G is an algebraic group over F, we have a functor, denoted by $\mathrm{B}G$, taking a field extension K/F to the set $H^1(K, G)$ of isomorphism classes of principal homogeneous spaces of G over $\operatorname{Spec} K$. We have then the group of invariants $\operatorname{Inv}^d(\mathrm{B}G)$ (see [8]).

The generic fiber $E \to \operatorname{Spec} F(\mathrm{B}G)$ of a universal G-torsor $\mathrm{E}G \to \mathrm{B}G$ is a principal homogeneous space over the function field $F(\mathrm{B}G)$. By Rost's theorem (see [8, Part I, Appendix C] or [18, Lemma 5.3]), the evaluation of an invariant at the generic fiber E yields an isomorphism

$$\eta_G : \operatorname{Inv}^d(\mathrm{B}G) \xrightarrow{\sim} \overline{A}^0(\mathrm{B}G, H^d) := A^0(\mathrm{B}G, H^d)/H^d(F).$$

In particular, $\operatorname{Inv}^d(\mathrm{B}G)$ is identified with a subgroup of $\overline{H}^d(F(\mathrm{B}G))$.

3c. **Invariants of the second type.** Let $\mathcal{B} : \mathit{Fields}(F) \to \mathit{Groups}$ be a functor. A degree d *homomorphic cohomological invariant of* $\mathcal{B}$ is a collection of group homomorphisms

$$\mathcal{B}(K) \to H^d(K)$$

for all field extensions K/F, functorial in K. In other words, a homomorphic invariant is a morphism of functors $\mathcal{B} \to H^d$. All degree d homomorphic cohomological invariants of $\mathcal{A}$ form an abelian group denoted by $\operatorname{Inv}^d_h(\mathcal{B})$.

Example 3.2. If G is an algebraic group over F, we can view G as a functor $\mathit{Fields}(F) \to \mathit{Groups}$ taking K to $G(K)$. Therefore, we have the group $\operatorname{Inv}^d_h(G)$ of degree d homomorphic invariants of G. If an algebraic group Z is commutative, $\mathrm{B}Z$ can also be viewed as a functor from $\mathit{Fields}(F)$ to Groups. Therefore, one can consider the group of homomorphic invariants $\operatorname{Inv}^d_h(\mathrm{B}Z)$.

Let G be a connected algebraic group over F. Consider the three maps $p_1, p_2, m : G \times G \to G$, the two projections and the multiplication. Let $A^0(G, H^d)_{\mathrm{mult}}$ be the kernel of the homomorphism

$$p_1^* + p_2^* - m^* : A^0(G, H^d) \to A^0(G \times G, H^d).$$

Evaluating a homomorphic invariant at the generic element of G gives an *isomorphism* (see [24, Theorem 2.3])

$$\lambda_G : \operatorname{Inv}^d_h(G) \xrightarrow{\sim} A^0(G, H^d)_{\mathrm{mult}}.$$

Note that the natural map $A^0(G, H^d)_{\mathrm{mult}} \to \overline{H}^d(F(G))$ is injective. Therefore, we can identify $\mathrm{Inv}^d_h(G)$ with a subgroup of $\overline{H}^d(F(G))$.

3d. **Homomorphic invariants of** $\mathrm{B}Z$**.** In this section, Z denotes a finite algebraic group of multiplicative type over F such that there exists an exact sequence

$$1 \to Z \to T \to S \to 1$$

with S and T quasi-trivial tori.

Example 3.3. This condition holds if Z is the center of a simply connected semisimple group $\widetilde{G}$ over F. Indeed, the character Galois module Z^* is isomorphic to Λ_w/Λ_r, where Λ_w and Λ_r are permutation Galois weight and root lattices respectively (with Γ_F permuting the fundamental weights and simple roots via the $*$-action).

Let n be the order of Z. We have $Z(F_{\mathrm{sep}}) = Z_* \otimes \mu_n$ and $Z^\circ(F_{\mathrm{sep}}) = Z^* \otimes \mu_n$. The natural pairing

$$Z^\circ(F_{\mathrm{sep}}) \times Z(F_{\mathrm{sep}}) \to \mu_n^{\otimes 2}$$

yields the cup-product map

$$H^2(F, Z^\circ) \otimes H^1(F, Z) \to H^3(F).$$

Therefore, we have a homomorphism

$$\nu : H^2(F, Z^\circ) \to \mathrm{Inv}^3_h(\mathrm{B}Z)$$

given by the cup-product: $\nu(\alpha)$ is an invariant taking the class $\beta \in H^1(K, Z)$ of a principal homogeneous Z-space over K to $\alpha_K \cup \beta \in H^3(K)$.

Proposition 3.4. *The homomorphism* $\nu : H^2(F, Z^\circ) \to \mathrm{Inv}^3_h(\mathrm{B}Z)$ *is an isomorphism.*

Proof. The statement is proved in [10, Proposition 4.1] provided all invariants of quasi-trivial tori T and S are split over F_{sep}. This condition is verified in [22, Theorem 1.1] since H^* is Rost's cycle module (under our assumptions on the characteristic). □

Remark 3.5. The subgroup $\mathrm{Inv}^3_a(\mathrm{B}Z)$ of all invariants in $\mathrm{Inv}^3(\mathrm{B}Z)$ split by F_{sep} can be larger than $\mathrm{Inv}^3_h(\mathrm{B}Z)$ if Z^* is not cyclic. For example, let $G = \mathbf{PGL}_1(A) \times \mathbf{PGL}_1(B)$, where A and B are two quaternion algebras. Then for $c \in F^\times$, we have an invariant $(g, g') \mapsto c \cup \mathrm{Nrd}(g) \cup \mathrm{Nrd}(g')$ that is split by F_{sep}, but not necessarily in the image of ν.

4. A relation between $\mathrm{Inv}^3_h(\mathrm{B}Z)$ and $\mathrm{Inv}^3_h(G)$

For a smooth variety X over F we write $\mathbb{Z}_X(2)$ for the motivic complex of étale sheaves defined in [15] and [16] and $H^{q,2}(X) = H^q(X, \mathbb{Z}_X(2))$ for the weight two étale motivic cohomology. By [13, Theorem 1.1], we have the following formulas for the étale motivic cohomology of weight 2:

$$H^{q,2}(X) = \begin{cases} 0, & \text{if } q \le 0; \\ K_3(F(X))_{\mathrm{ind}}, & \text{if } q = 1; \\ A^0(X, K_2), & \text{if } q = 2; \\ A^1(X, K_2), & \text{if } q = 3, \end{cases} \tag{4.1}$$

where $K_3(L)_{\mathrm{ind}} := \operatorname{Coker}\big(K_3(L) \to K_3^Q(L)\big)$ the cokernel of the natural homomorphism on Milnor's and Quillen's K-groups for a field L.

If A is a torsion abelian group, we write A' for the factor group of A modulo its p-primary part, where $p = \operatorname{char}(F) > 0$. There is a natural isomorphism

$$H^{4,2}(F)' \simeq H^3(F). \tag{4.2}$$

Let X be a smooth variety over F. By [13, Theorem 1.1(9)], there is an exact sequence

$$0 \to \mathrm{CH}^2(X) \to H^{4,2}(X) \to H^0_{Zar}(X, \mathcal{H}^3) \to 0$$

where $\mathcal{H}^3$ is the Zariski sheaf associated with the presheaf $U \to H^3(U, \mathbb{Q}/\mathbb{Z}(2))$. Note that $\operatorname{colim} \mu_n^{\otimes 2} = \mathbb{Q}/\mathbb{Z}(2)'$ and Gersten's resolution yields $H^0_{Zar}(X, \mathcal{H}^3)' \simeq A^0(X, H^3)$. Therefore, if $\mathrm{CH}^2(X)$ is a torsion group, there is an exact sequence

$$0 \to \mathrm{CH}^2(X)' \to H^{4,2}(X)' \xrightarrow{l} A^0(X, H^3) \to 0. \tag{4.3}$$

If G is an algebraic group over F, by Example 3.1, $\overline{A}^0(\mathrm{B}G, H^d) \xrightarrow{\sim} \operatorname{Inv}^d(\mathrm{B}G)$. Therefore, if $\mathrm{CH}^2(\mathrm{B}G)$ is a torsion group, we have an exact sequence

$$0 \to \mathrm{CH}^2(\mathrm{B}G)' \to \overline{H}^{4,2}(\mathrm{B}G)' \to \operatorname{Inv}^3(\mathrm{B}G) \to 0. \tag{4.4}$$

Proposition 4.5. *Let $\widetilde{G}$ be a simply connected group over F, J a $\widetilde{G}$-torsor over F. Then*

1. $\operatorname{Pic}(\widetilde{G}) = 0$*;*
2. $\operatorname{Br}(\widetilde{G})' = 0$*;*
3. *The Galois module $A^1(\widetilde{G}_{\mathrm{sep}}, K_2)$ is a lattice with a canonical permutation $\mathbb{Z}$-basis that is bijective to the set of irreducible components of the Dynkin diagram of G as a Galois set and $A^1(\widetilde{G}, K_2) \simeq A^1(\widetilde{G}_{\mathrm{sep}}, K_2)^{\Gamma_F}$;*
4. *The Galois modules $A^1(J_{\mathrm{sep}}, K_2)$ and $A^1(\widetilde{G}_{\mathrm{sep}}, K_2)$ are naturally isomorphic;*
5. $\mathrm{CH}^2(J) = 0$*;*
6. $H^{4,2}(J)' \simeq A^0(J, H^3)$*;*
7. $H^{4,2}(J_{\mathrm{sep}})' = 0$.

Proof. Statements (1), (3) and (4) are proved in [6] and [8], (2) is shown in [11], (5) follows from [26, Theorem 8.2]. In view of (5) the exact sequence (4.3) for $X = J$ reduces to an isomorphism $H^{4,2}(J)' \simeq A^0(J, H^3)$. Finally,

$$H^{4,2}(J_{\mathrm{sep}})' \simeq H^{4,2}(\widetilde{G}_{\mathrm{sep}})' = A^0(\widetilde{G}_{\mathrm{sep}}, H^3) = 0$$

by [6, Proposition 3.20]. □

Remark 4.6. If $p = \operatorname{char}(F) > 0$, the p-component of the Brauer group $\operatorname{Br}(\widetilde{G})$ is non-trivial (see [12, Example 4.8]).

Let $\widetilde{G}$ be a simply connected group over F and let J be a $\widetilde{G}$-torsor over F.

Lemma 4.7. *The natural map $H^2(F) \to \operatorname{Br}(J)' = A^0(J, H^2)$ is an isomorphism.*

Proof. The Hochschild-Serre spectral sequence

$$H^p(F, H^q(J_{\mathrm{sep}}, \mathbf{G}_{\mathrm{m}})) \Rightarrow H^{p+q}(J, \mathbf{G}_{\mathrm{m}})$$

yields an exact sequence

$$\operatorname{Pic}(J_{\mathrm{sep}})^{\Gamma} \to \operatorname{Br}(F) \to \operatorname{Ker}\big(\operatorname{Br}(J) \to \operatorname{Br}(J_{\mathrm{sep}})\big) \to H^1(F, \operatorname{Pic}(J_{\mathrm{sep}})),$$

but the groups $\operatorname{Pic}(J_{\mathrm{sep}}) = \operatorname{Pic}(\widetilde{G}_{\mathrm{sep}})$ and $\operatorname{Br}(J_{\mathrm{sep}})' = \operatorname{Br}(\widetilde{G}_{\mathrm{sep}})'$ are trivial by Proposition 4.5. □

Let $f : J \to \operatorname{Spec}(F)$ be the structure morphism. Write $H^{i,2}(J/F)$ for the étale cohomology groups of the cone of the natural morphism $\mathbb{Z}_F(2) \to Rf_*\mathbb{Z}_J(2)$. We have an exact sequence

$$\cdots \to H^{i-1,2}(J/F) \to H^{i,2}(F) \to H^{i,2}(J) \to H^{i,2}(J/F) \to \ldots. \tag{4.8}$$

Since the groups $H^{i,2}(F_{\mathrm{sep}})$ are trivial for $i \geq 3$, we have

$$H^{i,2}(J_{\mathrm{sep}}/F_{\mathrm{sep}}) \simeq H^{i,2}(J_{\mathrm{sep}}) \quad \text{for} \quad i \geq 3. \tag{4.9}$$

The terms $E_2^{p,q}$ in the Hochschild-Serre spectral sequence of relative étale motivic cohomology groups

$$E_2^{p,q} = H^p(F, H^{q,2}(J_{\mathrm{sep}}/F_{\mathrm{sep}})) \Rightarrow H^{p+q,2}(J/F)$$

are trivial for $p \geq 1$ and $q \leq 2$ (see the proof of [8, Part 2, Theorem 8.9]). By Proposition [8, Part 2, Proposition 8.7],

$$H^{3,2}(J/F) \simeq A^1(J_{\mathrm{sep}}, K_2)^{\Gamma}. \tag{4.10}$$

In view of Proposition 4.5(7) and (4.9),

$$H^{4,2}(J_{\mathrm{sep}}/F_{\mathrm{sep}})' = H^{4,2}(J_{\mathrm{sep}})' = 0,$$

hence $E_2^{0,4} = 0$, and by (4.1) and (4.9),

$$E_2^{1,3} = H^1(F, H^{3,2}(J_{\mathrm{sep}})) = H^1(F, A^1(J_{\mathrm{sep}}, K_2)) = 0$$

since $A^1(J_{\mathrm{sep}}, K_2)$ is a permutation Galois module by Proposition 4.5(3). It follows that

$$H^{4,2}(J/F) = 0. \tag{4.11}$$

In view of (4.10) and (4.11), the exact sequence (4.8) reads as follows:

$$A^1(J_{\mathrm{sep}}, K_2)^{\Gamma} \to H^{4,2}(F) \to H^{4,2}(J) \to 0.$$

By (4.2) and Proposition 4.5(4), this sequence yields an exact sequence

$$A^1(\widetilde{G}_{\mathrm{sep}}, K_2)^{\Gamma} \xrightarrow{\alpha} H^3(F) \to A^0(J, H^3) \to 0. \tag{4.12}$$

Set $Q(\widetilde{G}) := A^1(\widetilde{G}, K_2)$. It follows from Proposition 4.5(3) that $Q(\widetilde{G})$ is a lattice with a canonical $\mathbb{Z}$-basis $\{q_i\}_{i \in \Omega}$, where Ω is the set of Γ_F-orbits in the set of irreducible components of the Dynkin diagram of G.

As in [8, Part 2, §9], write δ_J for the canonical composition

$$Q(\widetilde{G}) \xrightarrow{\sim} A^1(\widetilde{G}_{\mathrm{sep}}, K_2)^{\Gamma} \xrightarrow{\alpha} H^3(F).$$

We can then rewrite the exact sequence (4.12) as follows:

$$Q(\widetilde{G}) \xrightarrow{\delta_J} H^3(F) \to A^0(J, H^3) \to 0. \tag{4.13}$$

We have proved:

Proposition 4.14. *Let $\widetilde{G}$ be a simply connected group and J a $\widetilde{G}$-torsor over F. Then the sequence (4.13) is exact.*

Proposition 4.15. *Let $f : E \to Y$ be a $\widetilde{G}$-torsor over an integral variety Y, $J \to \operatorname{Spec} F(Y)$ its generic fiber. Then the image of the map $\delta_J : Q(\widetilde{G}) \to H^3(F(Y))$ is contained in $A^0(Y, H^3)$ and the sequence*

$$Q(\widetilde{G}) \xrightarrow{\delta_f} A^0(Y, H^3) \to A^0(E, H^3) \to 0,$$

where δ_f is a map such that δ_J is the composition of δ_f with the inclusion $A^0(Y, H^3) \hookrightarrow H^3(F(Y))$, is exact.

Proof. For a point $y \in Y$, let J_y be the fiber of f over y. Then J_y is a $\widetilde{G}$-torsor over $F(y)$. In the diagram

$$\begin{tikzcd}
 & & Q(\widetilde{G}) \arrow[d, "\delta_J"'] & \\
0 \arrow[r] & A^0(Y, H^3) \arrow[r] \arrow[d] & H^3(F(Y)) \arrow[r] \arrow[d] & \coprod_{y \in Y^{(1)}} H^2(F(y)) \arrow[d] \\
0 \arrow[r] & A^0(E, H^3) \arrow[r] & A^0(J, H^3) \arrow[r] \arrow[d] & \coprod_{y \in Y^{(1)}} A^0(J_y, H^2), \\
 & & 0 &
\end{tikzcd}$$

the top horizontal sequence is exact by the definition of $A^0(Y, H^3)$. The spectral sequence [28, Corollary 8.2]

$$\coprod_{y \in Y^{(p)}} A^q(J_y, H^{3-p}) \Rightarrow A^{p+q}(E, H^3)$$

for the torsor f and the cycle module H^* yields the exactness of the bottom sequence in the diagram. The vertical sequence is exact by Proposition 4.14. The right vertical map is an isomorphism by Lemma 4.7. The statement follows by a diagram chase. $\square$

Let G be a semisimple algebraic group over a field F, $\widetilde{G} \to G$ a simply connected cover and Z the kernel of $\widetilde{G} \to G$. Consider an approximation $V' \to V'/Z$ of the universal Z-torsor $\mathrm{E}Z \to \mathrm{B}Z$. Then $f : (\widetilde{G} \times V')/Z \to V'/Z = \mathrm{B}Z$ is a $\widetilde{G}$-torsor and the projection $(\widetilde{G} \times V')/Z \to \widetilde{G}/Z = G$ is an approximation of the vector bundle $(\widetilde{G} \times V)/Z \to G$. Let J be the generic fiber of the $\widetilde{G}$-torsor f. Applying Proposition 4.15 to f and using the Homotopy Invariance Property for vector bundles, we get the following statement.

Proposition 4.16. *The sequence*

$$Q(\widetilde{G}) \xrightarrow{\delta_f} A^0(\mathrm{B}Z, H^3) \to A^0(G, H^3) \to 0$$

is exact.

Remark 4.17. We have applied Proposition 4.15 to an approximation of the natural morphism of algebraic stacks $G \to \mathrm{B}Z$ that is a $\widetilde{G}$-torsor.

As explained in [8, Part 2, §9], there is a surjective homomorphism

$$r_{\widetilde{G}} : Q(\widetilde{G}) \to \mathrm{Inv}^3(\mathrm{B}\widetilde{G})$$

such that

$$r_{\widetilde{G}}(q)(J) = \delta_J(q) \tag{4.18}$$

for every $q \in Q(\widetilde{G})$ and every principal homogeneous $\widetilde{G}$-space J.

Proposition 4.19. *1. The map δ_f coincides with the composition*

$$Q(\widetilde{G}) \xrightarrow{r_{\widetilde{G}}} \mathrm{Inv}^3(\mathrm{B}\widetilde{G}) \xrightarrow{\mathrm{res}} \mathrm{Inv}^3(\mathrm{B}Z) \xrightarrow{\eta_Z} A^0(\mathrm{B}Z, H^3)$$

2. If G is quasi-split, we have $\delta_f = 0$ and $A^0(\mathrm{B}Z, H^3) \to A^0(G, H^3)$is an isomorphism.

Proof. 1. Let J be the generic fiber of the $\widetilde{G}$-torsor $(\widetilde{G} \times V')/Z \to V'/Z = \mathrm{B}Z$. The latter torsor is the push-forward of the Z-torsor $EZ = V' \to V'/Z = BZ$ with respect to $Z \to \widetilde{G}$. Therefore, J is the push-forward of the generic fiber I of the torsor $EZ \to BZ$. It follows from (4.18) that

$$(\eta \circ \mathrm{res} \circ r_{\widetilde{G}})(q) = (\mathrm{res} \circ r_{\widetilde{G}})(q)(I) = r_{\widetilde{G}}(q)(J) = \delta_J(q) = \delta_f(q).$$

2. If G is quasi-split, the map $Z \to \widetilde{G}$ factors through a quasi-trivial maximal torus T of G. Since T is special, the generic fiber of $(\widetilde{G} \times EG)/Z \to \mathrm{B}Z$ is trivial, hence the map δ_f is trivial. Thus, $A^0(BZ, H^3) \simeq A^0(G, H^3)$ if G is quasi-split. □

Let G be a semisimple group and let Z be the kernel of the simply connected cover $\widetilde{G} \to G$. We have a sequence of morphisms of functors

$$G \to \mathrm{B}Z \to \mathrm{B}\widetilde{G}.$$

The first morphism takes a point $g \in G(L)$ for a field extension L/F to the isomorphism class of the fiber of the Z-torsor $\widetilde{G} \to G$ over g.

By [22, Corollary 1.8] or [9, Lemma 7.1], the image of the induced restriction map $\mathrm{Inv}^3(\mathrm{B}\widetilde{G}) \to \mathrm{Inv}^3(\mathrm{B}Z)$ is contained in $\mathrm{Inv}^3_h(\mathrm{B}Z)$. Therefore, we have a complex

$$\mathrm{Inv}^3(\mathrm{B}\widetilde{G}) \to \mathrm{Inv}^3_h(\mathrm{B}Z) \to \mathrm{Inv}^3_h(G).$$

Theorem 4.20. *Let G be a semisimple group and let Z be the kernel of the simply connected cover $\widetilde{G} \to G$. Then the sequence*

$$\mathrm{Inv}^3(\mathrm{B}\widetilde{G}) \xrightarrow{\mathrm{res}} \mathrm{Inv}^3_h(\mathrm{B}Z) \to \mathrm{Inv}^3_h(G) \to 0$$

is exact.

Proof. The diagram

$$\begin{tikzcd}
\mathrm{Inv}^3(\mathrm{B}\widetilde{G}) \arrow[r, "\mathrm{res}"] & \mathrm{Inv}^3(\mathrm{B}Z) \arrow[r] \arrow[d, "\eta_Z"', "\simeq"] & \mathrm{Inv}^3(G) \arrow[d] & \\
Q(\widetilde{G}) \arrow[u, "r_{\widetilde{G}}"] \arrow[r, "\delta_J"] & \overline{A}^0(\mathrm{B}Z, H^3) \arrow[r] & \overline{A}^0(G, H^3) \arrow[r] & 0,
\end{tikzcd}$$

where the right vertical homomorphism is the evaluation at the generic point of G, is commutative by Proposition 4.19. The middle vertical homomorphism is an isomorphism by Example 3.1 and the low row is exact by Proposition 4.16. This proves exactness of the sequence in the middle term.

Now we prove surjectivity on the second map in the sequence. Let $\varphi \in \mathrm{Inv}^3_h(G)$. Since the restriction of the right vertical homomorphism in the diagram on $\mathrm{Inv}^3_h(G)$ is injective by Example 3.2, there is $\psi \in \mathrm{Inv}^3(\mathrm{B}Z)$ that maps to φ. It suffices to show that $\psi \in \mathrm{Inv}^3_h(\mathrm{B}Z)$. Consider the diagram

$$\begin{tikzcd}
 & \mathrm{Inv}^3(\mathrm{B}Z) \arrow[r] \arrow[d] & \mathrm{Inv}^3(G) \arrow[d] \\
\mathrm{Inv}^3(\mathrm{B}(\widetilde{G}^2)) \arrow[r, "\mathrm{res}"] & \mathrm{Inv}^3(\mathrm{B}(Z^2)) \arrow[r] & \mathrm{Inv}^3(G^2),
\end{tikzcd}$$

where the vertical homomorphisms are the alternating sums of the maps induced by the three maps $G^2 := G \times G \to G$, the two projections and the multiplication, and similarly for Z. We have already shown that the bottom sequence is exact.

The image Ψ of ψ in $\mathrm{Inv}^3(\mathrm{B}(Z^2))$ is an invariant of $\mathrm{B}(Z^2) = \mathrm{B}Z \times \mathrm{B}Z$ given by $\Psi(x, y) = \psi(x + y) - \psi(x) - \psi(y)$ for all x and y in $H^1(K, Z)$ for a field extension K/F. As φ is a homomorphism, the image of Ψ in $\mathrm{Inv}^3(G^2)$ is trivial, hence Ψ comes from $\mathrm{Inv}^3(\mathrm{B}(\widetilde{G}^2))$. By [8, Proposition 9.7], $\mathrm{Inv}^3(\mathrm{B}(\widetilde{G}^2)) \simeq \mathrm{Inv}^3(\mathrm{B}\widetilde{G}) \oplus \mathrm{Inv}^3(\mathrm{B}\widetilde{G})$, hence there are $\varepsilon, \mu \in \mathrm{Inv}^3(\mathrm{B}(\widetilde{G}))$ such that

$$\Psi(x, y) = \varepsilon(\bar{x}) + \mu(\bar{y}),$$

where $\bar{x}$ is the image of x in $H^1(K, \widetilde{G})$. Plugging in $y = 0$ (respectively, $x = 0$), we get $\varepsilon(\bar{x}) = 0$ (respectively, $\mu(\bar{y}) = 0$), hence $\Psi(x, y) = 0$ for all x and y. It follows that $\psi \in \mathrm{Inv}^3_h(\mathrm{B}Z)$. □

Remark 4.21. The composition $H^2(F, Z^\circ) \xrightarrow{\nu} \mathrm{Inv}^3_h(\mathrm{B}Z) \to \mathrm{Inv}^3_h(G) \xrightarrow{\lambda_G} \overline{H}^3(F(G))$ takes an element $\alpha \in H^2(F, Z^\circ)$ to the cup-product $\alpha_{F(G)} \cup \beta$, where $\beta \in H^1(F(G), Z)$ is the class of the generic fiber of the Z-torsor $\widetilde{G} \to G$.

5. A proof of Theorem B

Let G be an adjoint semisimple group and let Z be the kernel of a simply connected cover $\widetilde{G} \to G$. Recall that the group $Q(\widetilde{G})$ is a lattice with a canonical $\mathbb{Z}$-basis $\{q_i\}_{i\in\Omega}$, where Ω is the set of Galois orbits in the set of irreducible components of the Dynkin diagram of G.

Consider the homomorphism

$$\kappa : Q(\widetilde{G}) \to H^2(F, Z^\circ)$$

taking a basis element q_i to the ith component t_i° of the dual Tits class t_G°. The main result of [10] can be stated as follows.

Proposition 5.1. *The composition $Q(\widetilde{G}) \xrightarrow{r_{\widetilde{G}}} \mathrm{Inv}^3(\mathrm{B}\widetilde{G}) \xrightarrow{\mathrm{res}} \mathrm{Inv}^3(\mathrm{B}Z)$ coincides with with negative of the composition*

$$Q(\widetilde{G}) \xrightarrow{\kappa} H^2(F, Z^\circ) \xrightarrow{\nu} \mathrm{Inv}^3(\mathrm{B}Z).$$

Corollary 5.2. *The image of the restriction map* $\mathrm{Inv}^3(\mathrm{B}\widetilde{G}) \to \mathrm{Inv}^3(\mathrm{B}Z)$ *is generated by* $\nu(t_i^\circ), i \in \Omega$.

Write Φ_G for the image of κ, i.e., Φ_G is the subgroup of $H^2(F, Z^\circ)$ generated by all t_i°. If G is simple, Φ_G is a cyclic subgroup.

The composition

$$\varepsilon : H^2(F, Z^\circ) \xrightarrow{\nu} \mathrm{Inv}_h^3(\mathrm{B}Z) \to \mathrm{Inv}_h^3(G)$$

takes an element $\alpha \in H^2(F, Z^\circ)$ to the invariant of G given by

$$g \in G(K) \mapsto \alpha_K \cup \theta_K(g) \in H^3(K)$$

for a field extension K/F, where $\theta_K : G(K) \to H^1(K, Z)$ is the connecting homomorphism. Now we can prove Theorem B.

Theorem 5.3. *Let G be an adjoint semisimple group and let Z be the kernel of the simply connected cover $\widetilde{G} \to G$. The homomorphism ε yield an isomorphism*

$$H^2(F, Z^\circ)/\Phi_G \xrightarrow{\sim} \mathrm{Inv}_h^3(G),$$

where Φ_G is the subgroup of $H^2(F, Z^\circ)$ generated by the components t_i° of the dual Tits class t_G°.

Proof. The statement is a combination of Theorem 4.20. Proposition 3.4 and Corollary 5.2. □

Remark 5.4. The group G and a quasi-split inner twisted form G^{qs} of G have the same group $H^2(F, Z^\circ)$. But $\Phi_{G^{\mathrm{qs}}} = 0$ since the Tits class of G^{qs} is trivial, while Φ_G is not trivial in general.

6. Unramified cohomology

Let K/F be a field extension and v a discrete valuation of K trivial on F. Then the residue field $\kappa(v)$ is a field extension of F and there is the *residue homomorphism*

$$\partial_v : H^d(K) \to H^{d-1}(\kappa(v)).$$

An element $x \in H^d(K)$ is called *unramified with respect to* v if $\partial_v(x) = 0$. We write $H^d(K)_{\mathrm{nr}}$ for the subgroup of all elements that are unramified with respect to all discrete valuations of K/F, and write $\overline{H}^d(K)_{\mathrm{nr}}$ for the cokernel of the natural homomorphism $H^d(F) \to H^d(K)_{\mathrm{nr}}$.

An invariant I in $\mathrm{Inv}^d(\mathcal{A})$ is called *unramified* if for every field extension K/F and every $a \in \mathcal{A}(K)$ we have $I(a) \in H^d(K)_{\mathrm{nr}}$.

Let G be an algebraic group over F. The isomorphism η_G (see Example 3.1) identifies the group $\mathrm{Inv}^d(\mathrm{B}G)$ with the subgroup of $\overline{H}^d\big(F(\mathrm{B}G)\big)$ and the group $\mathrm{Inv}^d(\mathrm{B}G)_{\mathrm{nr}}$ of all unramified invariants in $\mathrm{Inv}^d(\mathrm{B}G)$ with the subgroup $\overline{H}^d\big(F(\mathrm{B}G)\big)_{\mathrm{nr}}$ of all unramified elements in $\overline{H}^d\big(F(\mathrm{B}G)\big)$ (see [18, Proposition 6.4]). Thus, we have the following sequence of subgroups of $\overline{H}^d\big(F(\mathrm{B}G)\big)$:

$$\overline{H}^d\big(F(\mathrm{B}G)\big)_{\mathrm{nr}} = \mathrm{Inv}^d(\mathrm{B}G)_{\mathrm{nr}} \subset \mathrm{Inv}^d(\mathrm{B}G) = \overline{A}^0(\mathrm{B}G, H^d) \subset \overline{H}^d\big(F(\mathrm{B}G)\big).$$

A homomorphic invariant I in $\operatorname{Inv}_h^d(G)$ is called *unramified* if for every field extension K/F and every $g\in G(K)$ we have $I(g)\in H^d(K)_{\mathrm{nr}}$.

Lemma 6.1. *A homomorphic invariant $I\in\operatorname{Inv}_h^d(G)$ is unramified if and only if the value of I in $H^d(F(G))$ at the generic point of G is unramified.*

Proof. Suppose that the generic value $u\in A^0(G,H^d)_{\mathrm{mult}}$ of the invariant I is unramified. Let K/F be a field extension and v a discrete valuation on K that is trivial on F. We show that for every $g\in G(K)$, the value $I(g)$ in $H^d(K)$ is unramified with respect to v. We may assume that K is finitely generated over F.

We can view g as a morphism $\operatorname{Spec}(K)\to G$ over F. By the proof of [24, Theorem 2.3], the value $I(g)$ is the image of u under the pull-back map

$$A^0(G,H^d)\xrightarrow{g^*}A^0(\operatorname{Spec}(K),H^d)=H^d(K).$$

Let X be an integral variety over F such that $F(X)\simeq K$. Then g yields a rational morphism $g':X\dashrightarrow G$. By [25, §2.5], the map g' yield a map

$$A^0(G,H^d)_{\mathrm{nr}}\to A^0(X,H^d)_{\mathrm{nr}}=H^d(K)_{\mathrm{nr}}$$

which is the restriction of g^*. Therefore, $I(g)\in H^d(K)_{\mathrm{nr}}$. □

Write $\operatorname{Inv}_h^d(G)_{\mathrm{nr}}$ for the subgroup of homomorphic unramified invariants in $\operatorname{Inv}_h^d(G)$. If G is an adjoint semisimple group, by Lemma 6.1 and Theorem A, we have the following sequence of subgroups of $\overline{H}^d\big(F(G)\big)$:

$$\overline{H}^d\big(F(G)\big)_{\mathrm{nr}}=\operatorname{Inv}_h^d(G)_{\mathrm{nr}}\subset\operatorname{Inv}_h^d(G)=A^0(G,H^d)_{\mathrm{mult}}\subset\overline{H}^d\big(F(G)\big).$$

Let L be an étale algebra over F. Then $L=L_1\times L_2\times\cdots\times L_k$, where L_i are finite separable field extensions. Let G be a semisimple group over L, i.e., G is a tuple $(G_1,G_2,\dots,G_k)$ of semisimple groups over fields $L_1,L_2,\dots,L_k$, respectively. We write $L(G)$ for the product of fields $L_i(G_i)$ and $H^d(L(G))_{\mathrm{nr}}$ for the direct sum of $H^d(L_i(G_i))_{\mathrm{nr}}$.

Proposition 6.2. *Let L/F be an étale F-algebra, G an adjoint semisimple group over L and $H=R_{L/F}(G)$. Then there is a natural isomorphism*

$$\overline{H}^3(F(H))_{\mathrm{nr}}\simeq\overline{H}^3(L(G))_{\mathrm{nr}}.$$

Proof. If Z is the kernel of the simply connected cover of G, then $C:=R_{L/F}(Z)$ is the center of the simply connected cover of H. Moreover,

$$H^2(F,C^\circ)\simeq H^2(L,Z^\circ). \tag{6.3}$$

It follows from [10, Lemmas 7.1 and 7.2] that this isomorphisms restricts to an isomorphism between subgroups Φ_H and Φ_G. In view of Theorem 5.3, there is an isomorphism

$$\operatorname{Inv}_h^3(H)\simeq\operatorname{Inv}_h^3(G). \tag{6.4}$$

Let $u\in H^2(F,C^\circ)$ and $v\in H^2(L,Z^\circ)$ correspond to each other under the isomorphism (6.3) and let $I\in\operatorname{Inv}_h^3(H)$ and $J\in\operatorname{Inv}_h^3(G)$ be the corresponding invariants. For a field extension K/F, choose an element $h\in H(K)$ and the corresponding element $g\in G(KL)$, where $KL=K\otimes_F L$. By [10, Lemma 2.12],

$$I(h)=\theta_H(h)\cup v_K=N_{KL/K}(\theta_G(g)\cup u_{KL})=N_{KL/K}(J(g))\in H^3(K). \tag{6.5}$$

It follows that if $J(g)$ is unramified in $H^3(KL)$, then $I(h)$ is also unramified in $H^3(K)$. Hence if the invariant J is unramified, then so is I.

Let E be a field that is an L-algebra. We have $EL = E \otimes_F L \simeq E \times E'$ for some L-algebra E'. Let $g \in G(E)$ and $g' = (g, 1) \in G(E) \times G(E') = G(EL)$ and let $h \in H(E) = G(EL)$ be the corresponding element. The formula (6.5) reads as follows:

$$I(h) = N_{EL/E}(J(g')) = J(g),$$

i.e., every value of J is a value of I. Hence, if I is unramified, then so is J. Thus, the unramified invariants correspond to each other under the isomorphism (6.4), whence the result. □

Every adjoint semisimple group H over F is isomorphic to the corestriction $R_{L/F}(G)$, where G is an adjoint group over an étale F-algebra L such that all G_i are adjoint absolutely simple ([30, §3.1.2]). By Proposition 6.2, it suffices to consider adjoint absolutely simple groups. We will consider such groups case-by-case following the classification in [14].

We start with some preliminary remarks.

Lemma 6.6. *If G is a quasi-split adjoint semisimple group over F. Then the variety of G is rational. In particular, $\overline{H}^3(F(G))_{\mathrm{nr}} = 0$.*

Proof. Let B a Borel subgroup of G over F. As B is a special group, the B-torsor $G \to G/B$ is split generically. Therefore, G is birational to $B \times G/B$. The group B is a semi-direct product of the unipotent radical and a quasi-trivial torus, hence B is a rational variety. The variety G/B is projective homogeneous with a rational point, hence G/B is also rational. □

Lemma 6.7. *Let G be an adjoint semisimple group such that the Dynkin diagram of G is simply laced and Z^* is a cyclic group of order n with trivial Galois action. Let A be a Tits algebra of a generator of Z^*. Then*

1. $\mathrm{Inv}^3_h(G) \simeq {}_n\mathrm{Br}(F)/\operatorname{span}([A])$;
2. *The group $\overline{H}^3(F(G))_{\mathrm{nr}}$ is trivial.*

Proof. By assumption, $Z \simeq \boldsymbol{\mu}_n \simeq Z^\circ$. Therefore, $H^2(F, Z^\circ) \simeq {}_n\mathrm{Br}(F)$. Since the Dynkin diagram of G is simply laced, $\hat{\rho} : Z \to Z^\circ$ is the identity, hence $\Phi_G = \operatorname{span}([A])$. Then the first statement follows from Theorem 5.3.

Let $I \in \overline{H}^3(F(G))_{\mathrm{nr}} = \mathrm{Inv}^3_h(G)_{\mathrm{nr}}$ and let $a \in {}_n\mathrm{Br}(F)$ be an element corresponding to I. Consider the variety X of Borel subgroups of G. Over the field $F(X)$ the group G is quasi-split, hence the invariant $I_{F(X)}$ is zero by Lemma 6.6. It follows that $a_{F(X)} = 0$. By Theorem 2.1, we have

$$\operatorname{Ker}\big(\mathrm{Br}(F) \to \mathrm{Br}\,F(X)\big) = \operatorname{span}([A]) = \Phi_G,$$

therefore, $I = 0$. □

Below G is an adjoint absolutely simple semisimple group over a field F and Z is the kernel of a simply connected cover $\widetilde{G} \to G$. Recall that we assume that $\mathrm{char}(F)$ is relatively prime to the order of Z. We compute $\mathrm{Inv}^3_h(G)$ and $\overline{H}^3(F(G))_{\mathrm{nr}}$.

6a. **Adjoint groups of type ${}^1\!A_{n-1}$.** We have $G = \mathbf{PGL}_1(A)$, where A is a central simple algebra of degree n over F. Then $Z = \boldsymbol{\mu}_n = Z^\circ$. By Lemma 6.7, $\Phi_G = \operatorname{span}([A])$ and by Theorem 5.3,

$$\operatorname{Inv}^3_h(G) \simeq {}_n\operatorname{Br}(F)/\operatorname{span}([A]).$$

If $a \in {}_n\operatorname{Br}(F)$, the corresponding invariant is defined by

$$gF^\times \mapsto a \cup \operatorname{Nrd}_A(g),$$

for $g \in G(F) = A^\times/F^\times$, where $\operatorname{Nrd}_A : A^\times \to F^\times$ is the reduced norm map for A.

The variety of the group G is rational, hence $\overline{H}^3(F(G))_{\mathrm{nr}} = 0$. This also follows from Lemma 6.7.

6b. **Type ${}^2\!A_{n-1}$, n odd.** We have $G = \mathbf{PGU}(B, \tau)$, where B is a central simple K-algebra of degree n with an involution τ of the second kind, where K/F be a separable quadratic field extension with $\Gamma = \operatorname{Gal}(K/F)$. We have

$$G(F) = \{b \in B^\times \quad \text{such that} \quad \tau(b) \cdot b \in F^\times\}/F^\times,$$

$$Z = Z^\circ = \boldsymbol{\mu}_{n,[K]} := \operatorname{Ker}\big(R_{K/F}(\boldsymbol{\mu}_{n,K}) \xrightarrow{N_{K/F}} \boldsymbol{\mu}_n\big)$$

and $Z^* = \mathbb{Z}/n\mathbb{Z}$ with τ acting by -1. Then $Z_* = \mathbb{Z}/n\mathbb{Z}$ and the map $\hat\rho : Z \to Z^\circ$ is the identity.

Since n is odd, the exact sequence $1 \to Z \to R_{K/F}(\boldsymbol{\mu}_{n,K}) \xrightarrow{N_{K/F}} \boldsymbol{\mu}_n \to 1$ is split. It follows that

$$H^1(F, Z) = \operatorname{Ker}\big(K^\times/K^{\times n} \xrightarrow{N_{K/F}} F^\times/F^{\times n}\big),$$

$$H^2(F, Z^\circ) = \operatorname{Ker}\big({}_n\operatorname{Br}(K) \xrightarrow{N_{K/F}} {}_n\operatorname{Br}(F)\big).$$

The connecting map $\theta_G : G(F) \to H^1(F, Z)$ takes a similitude g to $\operatorname{Nrd}_B(b)$. The invariant corresponding to a class $a \in H^2(F, Z^\circ)$ is defined by

$$g \mapsto N_{K/F}\big(a \cup \operatorname{Nrd}_B(b)\big).$$

The subgroup Φ_G is spanned by $[B]$ in $H^2(F, Z^\circ)$. Thus, by Theorem 5.3,

$$\operatorname{Inv}^3_h(G) \simeq \operatorname{Ker}\big({}_n\operatorname{Br}(K) \xrightarrow{N_{K/F}} {}_n\operatorname{Br}(F)\big)/\operatorname{span}([B]).$$

The variety of the group G is rational by [33, Theorem 8], hence $\overline{H}^3(F(G))_{\mathrm{nr}} = 0$.

6c. **Type ${}^2\!A_{n-1}$, n even.** In the notation of the previous section assume that $n = 2m$ is even.

We have $R = A_{n-1}$. The weight lattice is $\Lambda_w = \mathbb{Z}^n/\mathbb{Z}s$, where $s = \sum e_i$. Write $\bar e_i$ for the image of e_i in Λ_w. The system of simple roots is

$$\Pi = \{\alpha_1 = \bar e_1 - \bar e_2,\ \alpha_2 = \bar e_2 - \bar e_3, \dots,\ \alpha_{n-1} = \bar e_{n-1} - \bar e_n\}.$$

The fundamental weights are $f_i = \bar e_1 + \bar e_2 + \cdots + \bar e_i$ for $i = 1, \dots, n-1$. The $*$-action permutes α_i and α_{n-i}, and leaves stable α_m. We have $Z = \boldsymbol{\mu}_{n,[K]}$ and $Z^* = \mathbb{Z}/n\mathbb{Z}$ with τ acting by -1. The restriction of f_m to Z is a character over F, and the corresponding Tits algebra is the *discriminant algebra* $D(B, \tau)$ (see [14, §27.B]). The restriction of f_i

to Z, $i \neq m$, is a character over K, and the corresponding Tits algebra is the ith tensor power of B over K.

We have (see [14, Proposition 30.13]),

$$H^1(F,Z) = \{(x,y) \in F^\times \oplus K^\times \mid x^n = N_{K/F}(y)\}/\{(N_{K/F}(z), z^n) \mid z \in K^\times\},$$
$$H^2(F,Z^\circ) = \{(d,b) \in {}_2\mathrm{Br}(F) \oplus {}_n\mathrm{Br}(K) \mid d_K = mb \quad \text{and} \quad N_{K/F}(b) = 0\}.$$

The connecting map $\theta_G : G(F) \to H^1(F,Z)$ takes a similitude g to $(\mu(g), \mathrm{Nrd}_B(g))$, where $\mu(g) = \tau(g)g$ is the *multiplier* of g. The invariant I corresponding to a pair $(d,b) \in H^2(F,Z^\circ)$ is defined as follows. Take $g \in G(F)$ and let $x = \mu(g)$ and $y = \mathrm{Nrd}_B(g)$. Then $(x,y) \in H^1(F,Z)$. Since $N_{K/F}(y/x^{-m}) = 1$, by Hilbert Theorem 90, there is a $z \in K^\times$ such that $y/x^{-m} = z/\tau(z)$. Then $I(g) = d \cup (x) + N_{K/F}(b \cup (z))$ (see [22, Corollary 1.6]).

The subgroup Φ_G in $H^2(F,Z^\circ)$ is spanned by the pair $([D],[B])$, where $D := D(B,\tau)$ is the discriminant algebra. Thus, by Theorem 5.3,

$$\mathrm{Inv}^3_h(G) \simeq \{(d,b) \in \mathrm{Br}(F) \oplus \mathrm{Br}(K) \mid mb = d_K \quad \text{and} \quad N_{K/F}(b) = 0\}/\operatorname{span}([D],[B]).$$

Now we compute the group $\overline{H}^3(F(G))_{\mathrm{nr}} = \mathrm{Inv}^3_h(G)_{\mathrm{nr}}$. Take an invariant $I \in \mathrm{Inv}^3(G)_{\mathrm{nr}}$ and let a pair $(d,b) \in H^2(F,C^\circ)$ correspond to I via the isomorphism above. We claim that I is unramified if and only if $d \in \operatorname{span}([D])$ and $b \in \operatorname{span}([B])$. Suppose that I is unramified. Let X be the variety of Borel subgroups in G. Over the field $F(X)$, the group G is quasi-split, hence the invariant is zero by Lemma 6.6. It follows that $d_{F(X)} = 0$ and $b_{K(X)} = 0$. By Theorem 2.1, we have

$$\mathrm{Ker}\big(\mathrm{Br}(F) \to \mathrm{Br}\,F(X)\big) = \operatorname{span}([D]),$$
$$\mathrm{Ker}\big(\mathrm{Br}(K) \to \mathrm{Br}\,K(X)\big) = \operatorname{span}([B]).$$

It follows that $d \in \operatorname{span}([D])$ and $b \in \operatorname{span}([B])$.

Conversely, suppose $d \in \operatorname{span}([D])$ and $b \in \operatorname{span}([B])$. We show that the invariant I is unramified. Note that $2[D] = 0$. Modifying (d,b) by a multiple of $([D],[B])$, we may assume that $b = 0$ and $d = [D]$. In particular, $m[B] = [D_K] = d_K = mb = 0$. Thus, under the condition $m[B] = 0$, we have $([D],0) \in H^2(F,C^\circ)$ and I is given by

$$I(g) = [D] \cup \mu(g).$$

This invariant I has been considered by Berhuy-Monsurro-Tignol in [1, Proposition 11].

We prove that the invariant I is unramified. Let L/F be a field extension with a discrete valuation v over F and $g \in G(L)$. If the integer $v(\mu(g))$ is even, we have $\partial_v(I(g)) = 0$ since $2[D] = 0$. If $v(\mu(g))$ is odd, since $\mu(g)h \simeq h$ for the underlined hermitian form h, we have h is hyperbolic (see [27]), hence $I(g) = 0$ by [1, Lemma 10].

The pair $([D],0)$ represents a non-trivial invariant, i.e., it is different from a multiple of $([D],[B])$ if and only if $\exp(B)$ is even and $[D] \neq 0$. We have proved the following statement.

Theorem 6.8. *Let K/F be a quadratic field extension and let $G = \mathbf{PGU}(B,\tau)$ be the projective unitary group, where B is a central simple K-algebra of even degree with an involution τ of the second kind for the extension K/F. Assume that* $\mathrm{char}(F)$ *does not divide* $\deg(B)$*. Then $\overline{H}^3(F(G))_{\mathrm{nr}} = 0$ unless the exponent of B is even, the discriminant algebra*

D is not split but D_K is split. In the latter case, the group $\overline{H}^3(F(G))_{\mathrm{nr}} = \mathrm{Inv}^3_h(G)_{\mathrm{nr}}$ is cyclic of order 2 generated by the invariant $I(g) = \mu(g) \cup [D]$, where D is the discriminant algebra.

Note that if the exponent of B is even, the integer m is even as $m[B] = [D_K] = 0$. Therefore, n is divisible by 4.

Corollary 6.9. *Let K/F be a quadratic field extension and let B be a central simple K-algebra of degree n divisible by 4 with an involution τ of the second kind for the extension K/F. If the exponent of B is even and the discriminant algebra D is not split, then the variety of the projective unitary group $G = \mathbf{PGU}(B, \tau)$ is not rational.*

Proof. Let $X = R_{K/F}(\mathrm{SB}(D_K))$, where $\mathrm{SB}(D_K)$ is the Severi-Brauer variety of D_K, i.e., $X = X_\Sigma$ for $\Sigma = \Pi \setminus \{\alpha_1\}$.

. By Theorem 2.1, the map $\mathrm{Br}(F) \to \mathrm{Br}(F(X))$ is injective and

$$\mathrm{Ker}(\mathrm{Br}(K) \to \mathrm{Br}(K(X_K)) = \mathrm{span}([D_K]).$$

It follows that the algebra D_K is split over $F(X)$ and D is not split by $F(X)$. Let k is the exponent of B over $K(X_K)$. Then $k[B] \in \mathrm{span}([D_K])$. If $k[B] = 0$ in $\mathrm{Br}(K)$, then k is even. If $k[B] = [D_K]$, then $k[B] = m[B]$, hence $(k - m)[B] = 0$. It follows that $k - m$ and therefore k is even.

By Theorem 6.8, $\overline{H}^3(F(G))_{\mathrm{nr}} \neq 0$. It follows that G is not rational. □

The corollary was proved by a different method in [21, Theorem 5.14].

6d. **Type B_n.** We have $G = \mathbf{O}^+(q)$ the special orthogonal group, where q is a non-degenerate quadratic form of dimension $2n + 1$ over F. Then $Z = \boldsymbol{\mu}_2 = Z^\circ$ and $Z_* = \mathbb{Z}/2\mathbb{Z} = Z^*$. A point of G over a field extension L/F is an automorphism of q_L of determinant 1.

The map $\hat{\rho} : Z \to Z^\circ$ is trivial, hence $\Phi_G = 0$, and by Theorem 5.3,

$$\mathrm{Inv}^3_h(G) \simeq {}_2\mathrm{Br}(F).$$

Choose any $a \in {}_2\mathrm{Br}(F)$ and consider the corresponding invariant

$$g \mapsto a \cup \mathrm{sn}(g),$$

where $\mathrm{sn}(g) \in F^\times/F^{\times 2}$ is the *spinor norm* of g.

The variety of the group $\mathbf{O}^+(q)$ is rational (see [34]), hence $\overline{H}^3(F(G))_{\mathrm{nr}} = 0$.

6e. **Type C_n.** We have $G = \mathbf{PGSp}(A, \sigma)$, where A is a central simple algebra of degree $2n$ over F with a symplectic involution σ. We have

$$G(F) = \{a \in A^\times \quad \text{such that} \quad \sigma(a) \cdot a \in F^\times\}/F^\times,$$

$Z = \boldsymbol{\mu}_2 = Z^\circ$ and $Z^* = \mathbb{Z}/2\mathbb{Z} = Z_*$.

If $a \in {}_2\,\mathrm{Br}(F)$, we have the invariant given by

$$g \mapsto a \cup \mu(g),$$

where $\mu(g) := \sigma(g) \cdot g \in F^\times/F^{\times 2}$ is the *multiplier* of g.

Suppose first that n is even. The map $\hat{\rho} : Z \to Z^\circ$ is trivial, hence $\Phi_G = 0$ and

$$\mathrm{Inv}^3_h(G) \simeq {}_2\,\mathrm{Br}(F).$$

The weight lattice of the root system $R = \mathsf{C}_n$ is $\Lambda_w = \mathbb{Z}^n$. The system of simple roots is

$$\Pi = \{\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3, \dots,\ \alpha_{n-1} = e_{n-1} - e_n,\ \alpha_n = 2e_n\}.$$

The fundamental weights are $f_i = e_1 + e_2 + \cdots + e_i$ for $i = 1, \dots, n$.

Consider the projective homogeneous variety X corresponding to the subset $\Pi \setminus \{\alpha_n\}$ in Π. This is the variety of maximal isotropic ideals in A. The restriction of the fundamental weight f_n to Z is trivial since n is even. Therefore, the Tits algebra of the character f_n is trivial. By Theorem 2.1, the homomorphism

$$\mathrm{Br}(F) \to \mathrm{Br}\, F(X)$$

is injective.

Let I be an unramified invariant given by an element $a \in {}_2\mathrm{Br}(F)$. Since $\sigma_{F(X)}$ is hyperbolic, the multiplier map is surjective over any field containing $F(X)$. In particular, $a \cup t$ is the value of the invariant over the rational function field $F(X)(t)$. The residue of $a \cup t$ at $t = 0$ is equal to $a_{F(X)}$. As the invariant is unramified, we must have $a_{F(X)} = 0$. Hence $a = 0$, i.e., the invariant I is trivial. We proved that $\overline{H}^3(F(G))_{\mathrm{nr}} = 0$.

Now assume that n is odd. The map $\psi : Z \to Z^\circ$ is the identity and $\Phi_G = \mathrm{span}([A])$. Therefore,

$$\mathrm{Inv}^3_h(G) = {}_2\,\mathrm{Br}(F)/\,\mathrm{span}([A]).$$

Let I be an unramified invariant given by $a \in {}_2\,\mathrm{Br}(F)$ and let X be the projective variety of Borel subgroups. Over the field $F(X)$, the group G is quasi-split, hence the invariant I is zero by Lemma 6.6. It follows that $a_{F(X)} = 0$. By Theorem 2.1, we have

$$\mathrm{Ker}\big(\mathrm{Br}(F) \to \mathrm{Br}\, F(X)\big) = \mathrm{span}([A]).$$

It follows that $I = 0$. We proved that $\overline{H}^3(F(G))_{\mathrm{nr}} = 0$ in this case.

6f. **Type ${}^1\mathsf{D}_n$, n even.** We have $G = \mathbf{PGO}^+(A, \sigma)$, where A is a central simple F-algebra of degree $2n = 4m$ with an orthogonal involution σ of trivial discriminant and

$$G(F) = \{a \in A^\times \text{ such that } \mu(a) := \sigma(a) \cdot a \in F^\times \text{ and } \mathrm{Nrd}(a) = \mu(a)^n\}/F^\times.$$

The system of simple roots of the root system $R = \mathsf{D}_n$ is

$$\Pi = \{\alpha_1 = e_1 - e_2, \dots,\ \alpha_{n-1} = e_{n-1} - e_n,\ \alpha_n = e_{n-1} + e_n\}$$

in $\mathbb{Z}^n$. We write α_- and α_+ for α_{n-1} and α_n respectively. Let $f_1, \dots, f_n$ be the fundamental weights of R. We have $f_{n-1} = (e_1 + \cdots + e_{n-1} - e_n)/2$ and $f_n = (e_1 + \cdots + e_{n-1} + e_n)/2$. We write f_- and f_+ for f_{n-1} and f_n respectively and identify the dual root system $R^\vee$ with R.

We identify Z with $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ so that the first projection $Z \to \boldsymbol{\mu}_2$ is the restriction of f_+ on Z and the second projection is the restriction of f_-. The Clifford algebra of (A, σ) is the product $C^+ \times C^-$ of simple components and $[C^+] + [C^-] = [A]$, $2[C^\pm] = 0$ in $\mathrm{Br}(F)$.

We have $Z^* = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} = Z_*$. The pairing between Z^* and Z_* is given by

$$\langle f_+, f_+^\vee \rangle = \langle f_-, f_-^\vee \rangle = \frac{n}{4} \pmod{\mathbb{Z}}, \qquad \langle f_-, f_+^\vee \rangle = \frac{n-2}{4} \pmod{\mathbb{Z}}.$$

It follows that the map

$$\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 = Z \xrightarrow{\hat{\rho}} Z^\circ = \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$$

is the identity if $n \equiv 2$ modulo 4 and the exchange map $(u, v) \mapsto (v, u)$ if $n \equiv 0$ modulo 4. Therefore, the subgroup Φ_G in $H^2(F, Z) = {}_2\mathrm{Br}(F) \oplus {}_2\mathrm{Br}(F)$ is spanned by (C^+, C^-) if $n \equiv 2$ modulo 4 and by (C^-, C^+) if $n \equiv 0$ modulo 4.

By Theorem 5.3,

$$\mathrm{Inv}^3_h(G) \simeq \begin{cases} {}_2\mathrm{Br}(F) \oplus {}_2\mathrm{Br}(F)/\operatorname{span}([C^+], [C^-]), & \text{if } n \equiv 2 \text{ modulo } 4; \\ {}_2\mathrm{Br}(F) \oplus {}_2\mathrm{Br}(F)/\operatorname{span}([C^-], [C^+]), & \text{if } n \equiv 0 \text{ modulo } 4. \end{cases}$$

Since $H^1(F, Z) = F^\times/F^{\times 2} \oplus F^\times/F^{\times 2}$, we have the connecting homomorphism

$$\theta_G = (\varepsilon^+, \varepsilon^-) : G(F) \to F^\times/F^{\times 2} \oplus F^\times/F^{\times 2}.$$

The invariant of G corresponding to a pair $(b^+, b^-) \in {}_2\mathrm{Br}(F) \oplus {}_2\mathrm{Br}(F)$ takes an element $g \in G(F)$ to $b^+ \cup \varepsilon^+(g) + b^- \cup \varepsilon^-(g)$ in $H^3(F)$.

We claim that if the invariant given by a pair (b^+, b^-) is unramified, then it is trivial. Consider the components X^+ and X^- of the variety of maximal (of reduced rank n) isotropic subspaces. Thus, $X^\pm$ are projective homogeneous varieties corresponding to the subsets $\Pi \setminus \{\alpha_\pm\}$ in Π. The Tits algebra of the character $f_\pm$ is $C^\pm$. By Theorem 2.1,

$$\operatorname{Ker}\big(\mathrm{Br}(F) \to \mathrm{Br}\, F(X^\pm)\big) = \operatorname{span}([C^\pm]). \tag{6.10}$$

We pass to the field $F(X^+)$. Since the algebra A has an isotropic ideal of reduced rank n over $F(X^+)$, the involution σ is hyperbolic over $F(X^+)$ by [14, Proposition 6.7]. As C^+ is split over $F(X^+)$, by [22, Lemma 4.2], the pair (t^m, t^{m+1}) belongs to the image of θ_G for every field L containing $F(X^+)$ and every $t \in L^\times$. Taking for t a variable over $F(X^+)$ and computing the residue of $t^m \cup b^+ + t^{m+1} \cup b^-$ at $t = 0$, we get $b^+ = 0$ over $F(X^+)$ if $n \equiv 2$ modulo 4 and $b^- = 0$ if $n \equiv 0$ modulo 4. It follows from (6.10) that $b^+ \in \operatorname{span}([C^+])$ if $n \equiv 2$ modulo 4 and $b^- \in \operatorname{span}([C^+])$ if $n \equiv 0$ modulo 4. Similarly, $b^- \in \operatorname{span}([C^-])$ if $n \equiv 2$ modulo 4 and $b^+ \in \operatorname{span}([C^-])$ if $n \equiv 0$ modulo 4.

Consider the case $n \equiv 2$ modulo 4. Modifying the pair (b^+, b^-) by a multiple of $([C^+], [C^-])$, we may assume that $b^+ = [C^+] \neq 0$ and $b^- = 0$. Let Y be the involution variety of isotropic subspaces of reduced rank 1. Thus, Y is the projective homogeneous variety corresponding to the subset $\Pi \setminus \{\alpha_1\}$ in Π. The Tits algebra of the character f_1 is the algebra A. By Theorem 2.1,

$$\operatorname{Ker}\big(\mathrm{Br}(F) \to \mathrm{Br}\, F(Y)\big) = \operatorname{span}([A]). \tag{6.11}$$

Now we pass to the function field of the involution variety Y. Since A is split and the involution σ is isotropic over $F(Y)$, the spinor norm homomorphism sn is surjective. The commutativity of the diagram

$$\begin{tikzcd} \mathbf{O}^+(A,\sigma)(L) \arrow[r] \arrow[d, "\mathrm{sn}"'] & G(L) \arrow[d, "\theta_G"] \\ L^\times/L^{\times 2} \arrow[r, "d"] & L^\times/L^{\times 2} \oplus L^\times/L^{\times 2}, \end{tikzcd}$$

where d is the diagonal map, implies that all pairs (t, t) are in the image of θ_G for every field L containing $F(Y)$ and for every $t \in L^\times$. Taking for t a variable over $F(Y)$ and computing the residue of $b^+ \cup t + b^- \cup t = [C^+] \cup t$ at $t = 0$, we get $b^+ = [C^+] = 0$ over $F(Y)$. By (6.11), $[C^+] \in \operatorname{span}([A])$. As $[C^+] \neq 0$, we have $[C^+] = [A]$. It follows that $[C^-] = [A] - [C^+] = 0$. Therefore, $(b^+, b^-) = ([C^+], 0) = ([C^+], [C^-]) \in \Phi_G$, i.e., the invariant is trivial.

The case $n \equiv 0$ modulo 4 is considered similarly. We proved that $\overline{H}^3(F(G))_{\rm nr} = 0$.

6g. **Type ${}^2\mathsf{D}_n$, n even.** We have $G = \mathbf{PGO}^+(A, \sigma)$, where A is a central simple F-algebra of degree $2n$ with an orthogonal involution σ such that the discriminant quadratic algebra K/F is a field. The Clifford algebra $C = C(A, \sigma)$ is a central simple algebra over K.

We have $Z = R_{K/F}(\boldsymbol{\mu}_2) = Z^\circ$ and $H^2(F, Z^\circ) = {}_2\operatorname{Br}(K)$. Moreover, $\Phi_G = \operatorname{span}([C])$ if m is odd and $\Phi_G = \operatorname{span}([{}^\iota C])$ if m is even (here ι is the nontrivial automorphism of K/F). By Theorem 5.3,

$$\operatorname{Inv}^3_h(G) \simeq \begin{cases} {}_2\operatorname{Br}(K)/\operatorname{span}([C]), & \text{if } m \text{ is odd;} \\ {}_2\operatorname{Br}(K)/\operatorname{span}([{}^\iota C]), & \text{if } m \text{ is even.} \end{cases}$$

We have $C_K \simeq C \times {}^\iota C$. The computation of the invariants show that the natural homomorphism $\operatorname{Inv}^3_h(G) \to \operatorname{Inv}^3_h(G_K)$ is injective. Since $\overline{H}^3(K(G_K))_{\rm nr} = 0$ in the case ${}^1\mathsf{D}_n$, n even, we have $\overline{H}^3(F(G))_{\rm nr} = 0$.

6h. **Type ${}^2\mathsf{D}_n$, n odd.** We have $G = \mathbf{PGO}^+(A, \sigma)$, where A is a central simple F-algebra of degree $2n$ with an orthogonal involution σ such that the discriminant algebra K/F is a field. We have $Z = \boldsymbol{\mu}_{4,[K]}$ and $Z^* = \mathbb{Z}/4\mathbb{Z}$ with $\Gamma := \operatorname{Gal}(K/F) = \{1, \iota\}$ acting by -1.

As in the case ${}^2\mathsf{A}_3$, we have

$$\begin{aligned} H^1(F, Z) &= \{(x, y) \in F^\times \times K^\times \mid x^4 = N_{K/F}(y)\}/\{(N_{K/F}(z), z^4) \mid z \in K^\times\}, \\ H^2(F, Z^\circ) &= \{(a, c) \in {}_2\operatorname{Br}(F) \oplus {}_4\operatorname{Br}(K) \mid 2c = a_K \quad \text{and} \quad N_{K/F}(c) = 0\}. \end{aligned}$$

Consider the *extended Clifford group* $\mathbf{\Omega}(A, \sigma)$. There is an exact sequence

$$1 \to R_{K/F}(\mathbf{G}_{\rm m}) \to \mathbf{\Omega}(A, \sigma) \to G \to 1$$

(see [14, §13.35]). The group $\mathbf{\Omega}(A, \sigma)$ is a subgroup of the multiplicative group of the Clifford algebra $C := C(A, \sigma)$.

The connecting map $\theta_G : G(F) \to H^1(F, Z)$ takes a similitude g to (x, y) with $x = \underline{\mu}(g) := \bar{g} \cdot \underline{\sigma}(\bar{g})$, where $\underline{\sigma}$ is the canonical involution on the Clifford algebra C and $\bar{g}$ is a lift of g to $\mathbf{\Omega}(A, \sigma)(F)$ (see [14, page 189]). The invariant I corresponding to a pair $(a, c) \in H^2(F, Z^\circ)$ is defined as follows. For an element $g \in G(F)$, let $(x, y) = \theta_G(g) \in H^1(F, Z)$. Since $N_{K/F}(y/x^{-2}) = 1$, by Hilbert Theorem 90, there is a $z \in K^\times$ such that $y/x^{-2} = z/\iota(z)$. Then $I(g) = a \cup (x) + N_{K/F}(c \cup (z))$ (see [22, Corollary 1.6]).

The subgroup Φ_G in $H^2(F, Z^\circ)$ is spanned by the pair $([A], [C])$. By Theorem 5.3,

$$\operatorname{Inv}^3_h(G) \simeq \{(a, c) \in \operatorname{Br}(F) \oplus \operatorname{Br}(K) \mid 2c = a_K \quad \text{and} \quad N_{K/F}(c) = 0\}/\operatorname{span}([A], [C]).$$

Let $I \in \operatorname{Inv}^3_h(G)$ be an invariant corresponding to a pair $(a, c) \in H^2(F, Z^\circ)$. We claim that the invariant I is unramified if and only if $a \in \operatorname{span}([A])$ and $c \in \operatorname{span}([C])$. Suppose the invariant is unramified. Let X be the variety of Borel subgroups of G. Over the field

$F(X)$ the group G is quasi-split, hence the invariant I is zero by Lemma 6.6. It follows that $a_{F(X)} = 0$ and $c_{K(X)} = 0$. By Theorem 2.1, we have

$$\operatorname{Ker}\big(\operatorname{Br}(F) \to \operatorname{Br} F(X)\big) = \operatorname{span}([A]),$$
$$\operatorname{Ker}\big(\operatorname{Br}(K) \to \operatorname{Br} K(X)\big) = \operatorname{span}([C]).$$

It follows that $a \in \operatorname{span}([A])$ and $c \in \operatorname{span}([C])$.

Conversely, suppose $a \in \operatorname{span}([A])$ and $c \in \operatorname{span}([C])$. We show that the invariant I is unramified. Recall that $2[A] = 0$. Modifying (a, c) by a multiple of $([A], [C])$, we may assume that $c = 0$ and $a = [A] \neq 0$. In particular, $2[C] = [A_K] = a_K = 2c = 0$. Then $([A], 0) \in H^2(F, Z^\circ)$ and I is given by

$$I(g) = \underline{\mu}(g) \cup [A].$$

Let L/F be a field extension with a discrete valuation v over F. We show that for every $g \in G(L)$ the element $I(g)$ is unramified in $H^3(L)$. We have $\partial_v(I(g)) = v(\underline{\mu}(g))[A_{\kappa(v)}]$, where $\kappa(v)$ is the residue field of v. This is trivial if $v(\underline{\mu}(g))$ is even.

Suppose that $v(\underline{\mu}(g))$ is odd, hence $\partial_v(I(g)) = [A_{\kappa(v)}]$. If K is isomorphic to a subfield of $\kappa(v)$, we have $[A_{\kappa(v)}] = 0$ as $[A_K] = 0$.

Assume that K and $\kappa(v)$ are linearly disjoint over F. Let T be the sub-torus in $\mathbf{G}_{\mathrm{m}} \times R_{K/F}(\mathbf{G}_{\mathrm{m}})$ consisting of all pairs (x, y) such that $x^4 = N_{K/F}(y)$. There is a surjective homomorphism (see [14, page 189])

$$\mu_* : \mathbf{\Omega}(A, \sigma) \to T.$$

The kernel of μ_* is the spinor group $\mathbf{Spin}(A, \sigma)$.

The center of $\mathbf{\Omega}(A, \sigma)$ is the torus $R_{K/F}(\mathbf{G}_{\mathrm{m}})$, so we get an exact sequence

$$1 \to Z \to R_{K/F}(\mathbf{G}_{\mathrm{m}}) \xrightarrow{\varphi} T \to 1,$$

where the second map takes z to the pair $(N_{K/F}(z), z^4)$. Therefore, we have an exact sequence of the co-character Γ-modules

$$0 \to \mathbb{Z}[\Gamma] \to T_* \to Z_* \to 0.$$

The group T_* is identified with the subgroup in $\mathbb{Z} \oplus \mathbb{Z}[\Gamma]$ consisting of all pairs (α, β) such that $4\alpha = \varepsilon(\beta)$, where $\varepsilon : \mathbb{Z}[\Gamma] \to \mathbb{Z}$ is the augmentation. The first map in the sequence takes γ to the pair $(\varepsilon(\gamma), 4\gamma)$. Taking Γ-invariant elements we get an exact sequence

$$0 \to \mathbb{Z} \to T_*(\kappa(v)) \to Z_*(\kappa(v)) \to 0$$

with the cyclic group $T_*(\kappa(v))$ spanned by $\varphi := (1, 2 + 2\iota)$. The first map takes 1 to 2φ. Thus, $Z_*(\kappa(v))$ is a cyclic group of order 2 spanned by the class χ of φ. Under the identification of Z^* with $\mathbb{Z}/4\mathbb{Z}$, the character ξ is equal to $2 + 4\mathbb{Z}$. Then $A_{\kappa(v)}$ is a Tits algebra of χ over $\kappa(v)$.

As $v(\underline{\mu}(g))$ is odd, an odd multiple of the co-character φ in $T_*(\kappa(v))$ is in the image of the composition

$$\mathbf{\Omega}(A, \sigma)(L) \xrightarrow{\mu_*} T(L) \xrightarrow{\partial} T_*(\kappa(v)),$$

i.e., an odd multiple of the co-character φ is *special* over $\kappa(v)$ (see [17, §3]). In the notation of [17, §5] we have $X(\varphi) = \{\chi\}$. By [17, Proposition 5.8], the class of the Tits algebra A

of χ is split over $\kappa(v)$, i.e., $[A_{\kappa(v)}] = 0$ in $\mathrm{Br}(\kappa(v))$, hence $\partial_v(I(g)) = 0$. We have proved that the invariant I is unramified.

The pair $([A], 0)$ represents a non-trivial invariant, i.e., it is different from a multiple of $([D], [C])$ if and only if $[A] \neq 0$, $[C] \neq 0$ and $[A_K] = 0$. We have proved:

Theorem 6.12. *Let $G = \mathbf{PGO}^+(A, \sigma)$, where A is a central simple F-algebra of degree $2n \geq 6$, n odd, with an orthogonal involution σ of nontrivial discriminant K/F,* $\mathrm{char}(F) \neq 2$. *Then $\overline{H}^3(F(G))_{\mathrm{nr}} = 0$ unless both algebras A and the Clifford algebra $C(A, \sigma)$ are not split but A_K is split. In the latter case, the group $\overline{H}^3(F(G))_{\mathrm{nr}} = \mathrm{Inv}_h^3(G)_{\mathrm{nr}}$ is cyclic of order 2 generated by the invariant $I(g) = \underline{\mu}(g) \cup [A]$.*

Corollary 6.13. *Let $G = \mathbf{PGO}^+(A, \sigma)$, where A is a central simple F-algebra of degree $2n \geq 6$, n odd, with an orthogonal involution σ of nontrivial discriminant K/F. Suppose both algebras A and $C(A, \sigma)$ are not split. Then the variety of G is not rational.*

Proof. Let $X = R_{K/F}(\mathrm{SB}(A_K))$. By Theorem 2.1, the map $\mathrm{Br}(F) \to \mathrm{Br}(F(X))$ is injective and

$$\mathrm{Ker}(\mathrm{Br}(K) \to \mathrm{Br}(K(X_K)) = \mathrm{span}([A_K]).$$

It follows that the algebra A_K is split over $F(X)$ and A is not split by $F(X)$. If C is split over $K(X_K)$, then $[C] = [A_K] = 2[C]$, i.e., C is split, a contradiction. By Theorem 6.12, $\overline{H}^3(F(G))_{\mathrm{nr}} \neq 0$. It follows that G is not rational. $\square$

Remark 6.14. It is proved in [20, Theorem 2] that if n even, the discriminant of (A, σ) is nontrivial and $\mathrm{ind}(C) \geq 4$, then G is not rational.

6i. **Type $^1\mathsf{E}_6$.** Let G be an adjoint group of type $^1\mathsf{E}_6$, A be the Tits algebra corresponding to the generator of Z^*. By Lemma 6.7,

$$\mathrm{Inv}_h^3(G) \simeq {}_3\mathrm{Br}(F)/\,\mathrm{span}([A])$$

and the group $\overline{H}^3(F(G))_{\mathrm{nr}}$ is trivial.

6j. **Type $^2\mathsf{E}_6$.** Let K/F be a quadratic field extension such that G_K is of inner type. Then $Z = \boldsymbol{\mu}_{3,[K]} \simeq Z^\circ$ and $H^2(F, Z^\circ) = \mathrm{Ker}\big({}_3\mathrm{Br}(K) \xrightarrow{N_{K/F}} {}_3\mathrm{Br}(F)\big)$. The Tits algebra B is a central simple algebra over K with $N_{K/F}([B]) = 0$. We have

$$\mathrm{Inv}_h^3(G) = \mathrm{Ker}\big({}_3\mathrm{Br}(K) \xrightarrow{N_{K/F}} {}_3\mathrm{Br}(F)\big)/\,\mathrm{span}([B]).$$

Since the map $\mathrm{Inv}_h^3(G) \to \mathrm{Inv}_h^3(G_K)$ is injective, we have $\overline{H}^3(F(G))_{\mathrm{nr}} = 0$.

6k. **Type E_7.** By Lemma 6.7,

$$\mathrm{Inv}_h^3(G) \simeq {}_2\mathrm{Br}(F)/\,\mathrm{span}([A]),$$

where A is the Tits algebra, and the group $\overline{H}^3(F(G))_{\mathrm{nr}}$ is trivial.

6l. **Types G_2, F_4 and E_8.** Adjoint groups are simply connected, hence $\mathrm{Inv}_h^3(G) = 0 = \overline{H}^3(F(G))_{\mathrm{nr}}$.

7. K-cohomology of $\mathrm{B}Z$

Let n be an integer prime to $\mathrm{char}(F)$. A model for the classifying space $\mathrm{B}\boldsymbol{\mu}_n$ is the *lens space* defined as follows (see [3, §6]). Let $\mathbb{P}^\infty$ denote the projective space $\mathbb{P}^r$ for $r >> 0$ and let $L \to \mathbb{P}^\infty$ be a line bundle with the sheaf of sections $O(n)$. Then $\mathrm{B}\boldsymbol{\mu}_n$ is the complement of the zero section in L. The localization exact sequence together with the Homotopy Invariance Property gives the following exact sequences

$$0 \to A^0(X \times \mathbb{P}^\infty, K_1) \to A^0(X \times \mathrm{B}\boldsymbol{\mu}_n, K_1) \to A^0(X \times \mathbb{P}^\infty, K_0) \xrightarrow{\partial_0} A^1(X \times \mathbb{P}^\infty, K_1) \to A^1(X \times \mathrm{B}\boldsymbol{\mu}_n, K_1) \to 0,$$

$$0 \to A^0(X \times \mathbb{P}^\infty, K_2) \to A^0(X \times \mathrm{B}\boldsymbol{\mu}_n, K_2) \to A^0(X \times \mathbb{P}^\infty, K_1) \xrightarrow{\partial_0} A^1(X \times \mathbb{P}^\infty, K_2) \to A^1(X \times \mathrm{B}\boldsymbol{\mu}_n, K_2) \to A^1(X \times \mathbb{P}^\infty, K_1) \xrightarrow{\partial_1} A^2(X \times \mathbb{P}^\infty, K_2) \to A^2(X \times \mathrm{B}\boldsymbol{\mu}_n, K_2) \to 0$$

for every smooth variety X. By the Projective Bundle Theorem, we have

$$A^i(X \times \mathbb{P}^\infty, K_1) \simeq A^i(X, K_1) \oplus A^{i-1}(X, K_0),$$
$$A^i(X \times \mathbb{P}^\infty, K_2) \simeq A^i(X, K_2) \oplus A^{i-1}(X, K_1) \oplus A^{i-2}(X, K_0).$$

With respect to these direct sum decompositions, the connecting maps ∂_i are multiplication by n on the component of the same codimension and zero between the other components. Recall that $A^0(X, K_1) = F[X]^\times$, $A^1(X, K_1) = \mathrm{CH}^1(X)$ and $A^2(X, K_2) = \mathrm{CH}^2(X)$.

Assume that $F[X]^\times = F^\times$ and $F^\times$ is separably closed. It follows that the group $A^0(X, K_1) = F^\times$ is divisible by every integer prime to $\mathrm{char}\, F$. Note also that the natural homomorphism $A^i(X, K_j) \to A^i(X \times \mathrm{B}\boldsymbol{\mu}_n, K_j)$ is canonically split by the pull-back with respect to the morphism $X \to X \times \mathrm{B}\boldsymbol{\mu}_n$ given by a rational point $p \in \mathrm{B}\boldsymbol{\mu}_n$ such that the fiber of $\mathrm{E}\boldsymbol{\mu}_n \to \mathrm{B}\boldsymbol{\mu}_n$ over p is a trivial $\boldsymbol{\mu}_n$-torsor. Overall, the exact sequences yield the following statement.

Proposition 7.1. *Let F be a separably closed field, n an integer prime to $\mathrm{char}(F)$ and X a smooth variety over F such that $F[X]^\times = F^\times$. Then*

(1) $F[X \times \mathrm{B}\boldsymbol{\mu}_n]^\times = F[X]^\times$,
(2) $\mathrm{CH}^1(X \times \mathrm{B}\boldsymbol{\mu}_n) = \mathrm{CH}^1(X) \oplus \mathbb{Z}/n\mathbb{Z}$,
(3) $A^0(X \times \mathrm{B}\boldsymbol{\mu}_n, K_2) = A^0(X, K_2) \oplus \mu_n(F)$,
(4) $A^1(X \times \mathrm{B}\boldsymbol{\mu}_n, K_2) = A^1(X, K_2) \oplus {}_n\mathrm{CH}^1(X)$,
(5) $\mathrm{CH}^2(X \times \mathrm{B}\boldsymbol{\mu}_n) = \mathrm{CH}^2(X) \oplus \mathrm{CH}^1(X)/n \oplus \mathbb{Z}/n\mathbb{Z}$.

Corollary 7.2. *In the conditions of the proposition we have*

(1) $F[\mathrm{B}\boldsymbol{\mu}_n]^\times = F^\times$,
(2) $\mathrm{CH}^1(\mathrm{B}\boldsymbol{\mu}_n) = \mathbb{Z}/n\mathbb{Z}$,
(3) $A^0(\mathrm{B}\boldsymbol{\mu}_n, K_2) = K_2(F) \oplus \mu_n(F)$,
(4) $A^1(\mathrm{B}\boldsymbol{\mu}_n, K_2) = 0$,
(5) $\mathrm{CH}^2(\mathrm{B}\boldsymbol{\mu}_n) = \mathbb{Z}/n\mathbb{Z}$.

Let Z be a finite diagonalizable group of order prime to char F and let n be a multiple of the order of Z. A character of Z can be viewed as a homomorphism $x : Z \to \boldsymbol{\mu}_n$ for some n. Then x yields a morphism $\mathrm{B}Z \to \mathrm{B}\boldsymbol{\mu}_n$ and therefore, a homomorphism

$$x^* : \mu_n(F) \hookrightarrow A^0(\mathrm{B}\boldsymbol{\mu}_n, K_2) \to A^0(\mathrm{B}Z, K_2).$$

Therefore, we have a pairing

$$Z^\circ(F) = Z^*(F) \otimes \mu_n(F) \to A^0(\mathrm{B}Z, K_2)$$

taking $x \otimes \xi$ to $x^*(\xi)$. Proposition 7.1(3) gives inductively the following statement:

Proposition 7.3. *Let F be a separably closed field and let Z be a diagonalizable group of order prime to* $\mathrm{char}(F)$. *Then the homomorphism*

$$K_2(F) \oplus Z^\circ(F) \to A^0(\mathrm{B}Z, K_2)$$

is an isomorphism.

By Proposition 7.1(4), we have

$$A^1(\mathrm{B}Z \times \mathrm{B}\boldsymbol{\mu}_n, K_2) = A^1(\mathrm{B}Z, K_2) \oplus {}_nZ^*(F).$$

It follows that

$$A^1(\mathrm{B}\boldsymbol{\mu}_n \times \mathrm{B}\boldsymbol{\mu}_n, K_2) = {}_n(\boldsymbol{\mu}_n)^*(F) = \mathbb{Z}/n\mathbb{Z}$$

is the cyclic group of order n with a canonical generator.

Let A and B be two abelian groups. By [7, §62], the group $\mathrm{Tor}(A, B)$ is generated by certain elements $[a, n, b]$, where $a \in A$ and $b \in B$ satisfy $na = 0$ and $nb = 0$. Let $\tau : \mathrm{Tor}(A, A) \to \mathrm{Tor}(A, A)$ be the exchange map taking $[x, n, y]$ to $[y, n, x]$. Write $\Upsilon^2(A)$ for the factor group of $\mathrm{Tor}(A, A)$ by $\mathrm{Ker}(1 - \tau)$. If A is a cyclic group, we have $\tau = 1$ and $\Upsilon^2(A) = 0$. Moreover, $\Upsilon^2(A \oplus B) \simeq \Upsilon^2(A) \oplus \mathrm{Tor}(A, B) \oplus \Upsilon^2(B)$.

We have a map

$$\mathrm{Tor}\big(Z^*(F), Z^*(F)\big) \to A^1(\mathrm{B}Z, K_2)$$

taking $[x, n, y]$, where $x, y : Z \to \boldsymbol{\mu}_n$ are two characters, to the image of the canonical element under the map

$$A^1(\mathrm{B}\boldsymbol{\mu}_n \times \mathrm{B}\boldsymbol{\mu}_n, K_2) \to A^1(\mathrm{B}Z, K_2)$$

induced by $(x, y) : Z \to \boldsymbol{\mu}_n \times \boldsymbol{\mu}_n$. This map factor through a unique homomorphism

$$\Upsilon^2(Z^*(F)) \to A^1(\mathrm{B}Z, K_2).$$

Then Proposition 7.1(4) yields inductively the following statement.

Proposition 7.4. *Let F be a separably closed field and Z be a diagonalizable group of order prime to* $\mathrm{char}(F)$. *Then the homomorphism*

$$\Upsilon^2(Z^*(F)) \to A^1(\mathrm{B}Z, K_2)$$

is an isomorphism.

Proposition 7.1(5) also yields inductively the following statement.

Proposition 7.5. *Let F be a separably closed field and Z be a diagonalizable group of order prime to* $\mathrm{char}(F)$. *Then*

$$\mathrm{CH}^2(\mathrm{B}Z) = S^2(\mathrm{CH}^1(\mathrm{B}Z)) \simeq S^2(Z^*(F)).$$

Consider the Hochschild-Serre spectral sequence for the relative cohomology $\overline{H}^{n,2}(\mathrm{B}Z) := H^{n,2}(\mathrm{B}Z)/H^{n,2}(F)$:

$$E_2^{p,q} = H^p(F, \overline{H}^{q,2}(\mathrm{B}Z_{\mathrm{sep}})) \Rightarrow \overline{H}^{p+q,2}(\mathrm{B}Z).$$

Note that $E_2^{p,q} = 0$ when $p \geq 1$ and $q \leq 1$ (see [2, Appendix B.10]). Therefore, in view of Propositions 7.3 and 7.4, we have an exact sequence

$$H^2(F, Z^\circ) \xrightarrow{j} \overline{H}_a^{4,2}(\mathrm{B}Z) \to H^1(F, \Upsilon^2(Z^*(F))), \tag{7.6}$$

where

$$\overline{H}_a^{4,2}(\mathrm{B}Z) := \mathrm{Ker}\big(\overline{H}^{4,2}(\mathrm{B}Z) \to \overline{H}^{4,2}(\mathrm{B}Z_{\mathrm{sep}})\big).$$

Lemma 7.7. *Let $1 \to Z \to T \to S \to 1$ be an exact sequence where T and S are split tori and Z is a finite group. Then the diagram*

$$\begin{tikzcd} Z^\circ(F) \arrow[r] \arrow[d] & S^\circ(F) \arrow[d] \\ A^0(\mathrm{B}G, K_2) \arrow[r] & A^0(S, K_2), \end{tikzcd}$$

where the right vertical homomorphism takes $x \otimes a \in S^* \otimes F^\times = S^\circ(F)$ to the symbol $\{x, a\} \in A^0(S, K_2)$, is commutative.

Proof. It suffices to consider the exact sequence $1 \to \boldsymbol{\mu}_n \to \mathbf{G}_{\mathrm{m}} \xrightarrow{n} \mathbf{G}_{\mathrm{m}} \to 1$. We view $\mathrm{B}\boldsymbol{\mu}_n$ as an open subscheme of the line bundle $L = O(n)$ over $\mathbb{P} := \mathbb{P}^r$, $r >> 0$ with complement the zero section isomorphic to $\mathbb{P}$.

Let $U := \mathbb{A}^{r+1} \setminus \{0\}$ and consider the base change with respect to the natural flat morphism $U \to \mathbb{P}$. By [5, Proposition 49.36], the diagram

$$\begin{tikzcd} 0 \arrow[r] & \overline{A}^0(\mathrm{B}\boldsymbol{\mu}_n, K_2) \arrow[r] \arrow[d] & A^0(L, K_1) \arrow[r] \arrow[d] & A^1(\mathbb{P}, K_2) \arrow[d] \\ 0 \arrow[r] & \overline{A}^0(\mathrm{B}\boldsymbol{\mu}_n \times_{\mathbb{P}} U, K_2) \arrow[r] & A^0(L \times_{\mathbb{P}} U, K_1) \arrow[r] & A^1(U, K_2) \end{tikzcd}$$

with the exact rows is commutative. Since the pull-back of the line bundle L with respect to π is trivial, we have $L \times_{\mathbb{P}} U \simeq \mathbb{A}^1 \times U$ and $\mathrm{B}\boldsymbol{\mu}_n \times_{\mathbb{P}} U \simeq \mathbf{G}_{\mathrm{m}} \times U$. Moreover, the projection $\mathrm{B}\boldsymbol{\mu}_n \times_{\mathbb{P}} U \to \mathrm{B}\boldsymbol{\mu}_n$ is a model for the morphism $\mathbf{G}_{\mathrm{m}} \to \mathrm{B}\boldsymbol{\mu}_n$. The right square of the diagram is equal to

$$\begin{tikzcd} F^\times \arrow[r, "n"] \arrow[d, "1"'] & F^\times \arrow[d] \\ F^\times \arrow[r] & 1 \end{tikzcd}$$

The result follows. $\square$

Lemma 7.8. *Let Z be the finite kernel of a surjective homomorphism $T \to S$ of quasi-trivial tori. Then the diagram*

$$\begin{tikzcd} H^2(F, Z^\circ) \arrow[r, "j"] \arrow[d, "\nu"'] & \overline{H}_a^{4,2}(\mathrm{B}Z) \arrow[d, "l"] \\ \mathrm{Inv}^3(\mathrm{B}Z) \arrow[r, "\eta", "\sim"'] & \overline{A}^0(\mathrm{B}Z, H^3), \end{tikzcd}$$

where the map l is defined in (4.3), is commutative.

Proof. We have a T-torsor $S \to \mathrm{B}Z$. The left square of the diagram

$$\begin{tikzcd} H^2(F, Z^\circ) \arrow[r] \arrow[d, "\psi"'] & H^2(F, A^0(\mathrm{B}Z, K_2)) \arrow[r] \arrow[d] & \overline{H}_a^{4,2}(\mathrm{B}Z) \arrow[r] \arrow[d] & \overline{A}^0(\mathrm{B}Z, H^3) \arrow[d, "\rho"] \\ H^2(F, S^\circ) \arrow[r] & H^2(F, A^0(S, K_2)) \arrow[r] & \overline{H}_a^{4,2}(S) \arrow[r] & \overline{A}^0(S, H^3) \end{tikzcd}$$

is commutative by Lemma 7.7. The functoriality of the Hochschild-Serre spectral sequence for the morphism $S \to \mathrm{B}Z$ yields commutativity of the other two squares of the diagram. The left vertical homomorphism is induced by the first homomorphism in the dual exact sequence $1 \to Z^\circ \to S^\circ \to T^\circ \to 1$. By [19, Proposition B4], the composition in the lower row is the cup-product map with the generic point $s_{\mathrm{gen}} \in S(F(S))$.

The image of the generic point s_{gen} of S under the connecting homomorphism

$$\varphi : S(F(S)) \to H^1(F(S), Z)$$

is the class of the generic fiber E of the Z-torsor $T \to S$. By [10, Proposition 2.10],

$$\psi(a) \cup s_{\mathrm{gen}} = a_{F(S)} \cup \varphi(s_{\mathrm{gen}}) = a_{F(S)} \cup [E]$$

in $H^3(F(S))$ for every $a \in H^2(F, Z^\circ)$.

The homomorphism ρ in the diagram is injective since the T-torsor $S \to \mathrm{B}Z$ has a rational splitting as T is a quasi-trivial torus. It follows that the composition in the upper row in the diagram is the cup-product with the class of E, i.e., it coincides with the composition $\eta \circ \nu$. ☐

8. A proof of Theorem A

Let G be an adjoint semisimple group over F and let X be the variety of Borel subgroups in G. If $\mathcal{B} : \mathit{Fields}(F) \to \mathit{Groups}$ is a functor, we write $\mathcal{B}_a(F)$ for the kernel of the map $\mathcal{B}(F) \to \mathcal{B}(F_{\mathrm{sep}})$ and $\mathcal{B}_v(F)$ for $\mathrm{Ker}\big(\mathcal{B}(F) \to \mathcal{B}(F(X))\big)$. We will combine indices:

$$\mathcal{B}_{av}(F) := \mathcal{B}_a(F) \cap \mathcal{B}_v(F).$$

Let Z be the kernel of a simply connected cover $\widetilde{G} \to G$. Recall that there is an exact sequence $1 \to Z \to T \to S \to 1$ with quasi-trivial tori $T = R_{L/F}(\mathbf{G}_{\mathrm{m}})$ and $S = R_{E/F}(\mathbf{G}_{\mathrm{m}})$ for some étale F-algebras L and E. Therefore, there is an exact sequence $1 \to Z^\circ \to S^\circ \to T^\circ \to 1$ with $S^\circ \simeq S$ and $T^\circ \simeq T$.

We have an exact sequence

$$0 \to H^2(F, Z^\circ) \to \mathrm{Br}(E) \to \mathrm{Br}(L)$$

and a similar one over the function field of X. Hence there is an exact sequence

$$0 \to H^2_v(F, Z^\circ) \to \operatorname{Ker}(i_E) \to \operatorname{Ker}(i_L), \tag{8.1}$$

where $i_L : \operatorname{Br}(L) \to \operatorname{Br}(L(X))$ and $i_E : \operatorname{Br}(E) \to \operatorname{Br}(E(X))$ are natural homomorphisms.

For any field extension K/F, the kernel of $i_K : \operatorname{Br}(K) \to \operatorname{Br}(K(X))$ coincides with the image of the Tits map $\beta_K : Z^*(K) \to \operatorname{Br}(K)$ by Theorem 2.1. Let Y be a geometrically irreducible smooth variety over F with a rational point. In the diagram

$$\begin{tikzcd}
Z^*(K) \arrow[r, "\beta_K"] \arrow[d, "\simeq"] & \operatorname{Br}(K) \arrow[r, "i_K"] \arrow[d, hook] & \operatorname{Br}(K(X)) \arrow[d, hook] \\
Z^*(K(Y)) \arrow[r, "\beta_{K(Y)}"] & \operatorname{Br}(K(Y)) \arrow[r, "i_{K(Y)}"] & \operatorname{Br}(K(X \times Y))
\end{tikzcd}$$

with exact rows the left vertical map is an isomorphism since the restriction homomorphism $\Gamma_{K(Y)} \to \Gamma_K$ of the absolute Galois groups is surjective and the other vertical homomorphisms are injective since Y has a rational point over K. A simple diagram chase shows that the natural homomorphism $\operatorname{Ker}(i_K) \to \operatorname{Ker}(i_{K(Y)})$ is an isomorphism. The exact sequence (8.1) then yields the following statement.

Lemma 8.2. *Let Y be a geometrically irreducible smooth variety over F with a rational point. Then the natural homomorphism*

$$H^2_v(F, Z^\circ) \to H^2_v(F(Y), Z^\circ)$$

is an isomorphism.

For an invariant $I \in \operatorname{Inv}^d(\mathrm{B}Z)$, we construct an invariant $\Delta(I) \in \operatorname{Inv}^d(\mathrm{B}Z^2)$ defined by

$$\Delta(I)(x, y) = I(x + y) - I(x) - I(y).$$

Clearly, $\Delta(I) = 0$ if and only if $I \in \operatorname{Inv}^d_h(\mathrm{B}Z)$. We define an invariant $\Delta^2(I) \in \operatorname{Inv}^d(\mathrm{B}Z^3)$ by the formula:

$$\Delta^2(I)(x, y, z) = I(x + y + z) - I(x + y) - I(x + z) - I(y + z) + I(x) + I(y) + I(z).$$

Lemma 8.3. *Let $I \in \operatorname{Inv}^3(\mathrm{B}Z)$ be such that $\Delta^2(I) \in \operatorname{Inv}^3_h(\mathrm{B}Z^3)$. Then*

(1) *$\Delta^2(I) = 0$.*
(2) *If $I \in \operatorname{Inv}^3_v(\mathrm{B}Z)$, then $I \in \operatorname{Inv}^3_h(\mathrm{B}Z)$.*

Proof. (1): By Proposition 3.4 applied to $\Delta^2(I)$, there are $a, b, c \in H^2(F, Z^\circ)$ such that

$$\Delta^2(I)(x, y, z) = a_K \cup x + b_K \cup y + c_K \cup z$$

for all $x, y, z \in H^1(K, Z)$ for all field extensions K/F. Since the invariant $x \mapsto \Delta^2(x, 0, 0)$ of $\mathrm{B}G$ is trivial, we must have $a = 0$ by Proposition 3.4 again. Similarly, $b = c = 0$. Therefore, $\Delta^2(I) = 0$.

(2): For every field extension K/F and every $y \in H^1(K, Z_K)$ consider an invariant $J_y \in \operatorname{Inv}^3(\mathrm{B}Z_K)$ defined by

$$J_y(x) := I(x + y) - I(x) - I(y).$$

Since by the first part of the lemma, $\Delta^2(I) = 0$, we have $J_y \in \operatorname{Inv}^3_h(\mathrm{B}Z_K)$. By Proposition 3.4, there is a unique $a_y \in H^2(K, Z^\circ)$ such that $J_y(x) = a_y \cup x$ for every x. Since

$\Delta^2(I)=0$, we have $J_{y+z}=J_y+J_z$ for every $y,z\in \mathrm{B}Z(K)$, hence $a_{y+z}=a_y+a_z$. Thus, we have an invariant $L_y\in \mathrm{Inv}^2_h(\mathrm{B}Z,M)$ taking y to a_y, where M is the cycle module over K defined by $M_n(E)=H^n(E,Z^*(n-1))$ for every field extension E/K. By Rost's theorem [8, Part I, Appendix C], the invariant L_y is uniquely determined by its value a_{gen} at the generic point in $H^2(K(\mathrm{B}Z),Z^\circ)$.

Since $I\in \mathrm{Inv}^3_v(\mathrm{B}Z)$ by assumption, the invariants J_y and hence L_y are split by the function field of the variety X of Borel subgroups of G. It follows that $a_{gen}\in H^2_v(K(\mathrm{B}Z),Z^\circ)$. In view of Lemma 8.2 applied to $\mathrm{B}Z$ (or rather to an approximation of $\mathrm{B}Z$) over K, we see that a_{gen} comes from $H^2_v(K,Z^\circ)$, i.e., the invariant L_y is constant, hence zero. It follows that $J_y=0$ which means the $I\in \mathrm{Inv}^3_h(\mathrm{B}Z)$. □

The following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{CH}^2(\mathrm{B}Z)' \arrow[r]\arrow[d] & H^{4,2}(\mathrm{B}Z)' \arrow[r]\arrow[d] & \mathrm{Inv}^3(\mathrm{B}Z) \arrow[r]\arrow[d] & 0\\
0 \arrow[r] & \mathrm{CH}^2(\mathrm{B}Z_{\mathrm{sep}})' \arrow[r] & H^{4,2}(\mathrm{B}Z_{\mathrm{sep}})' \arrow[r] & \mathrm{Inv}^3(\mathrm{B}Z_{\mathrm{sep}}) \arrow[r] & 0
\end{tikzcd}$$

with the exact rows (see (4.4)) gives an exact sequence

$$0\to \mathrm{CH}^2_a(\mathrm{B}Z)'\to H^{4,2}_a(\mathrm{B}Z)'\to \mathrm{Inv}^3_a(\mathrm{B}Z)\xrightarrow{\partial_Z} S^2(Z^*(F_{\mathrm{sep}}))'/\,\mathrm{CH}^2(\mathrm{B}Z)' \tag{8.4}$$

in view of Proposition 7.5. Then the exact sequence (7.6) yields an exact sequence

$$H^2(F,Z^\circ)\to \mathrm{Ker}(\partial_Z)\to H^1(F,\Upsilon^2(Z^*(F)))'/\,\mathrm{CH}^2_a(\mathrm{B}Z)'. \tag{8.5}$$

Note that by Lemma 7.8, the composition

$$H^2(F,Z^\circ)\to \mathrm{Ker}(\partial_Z)\hookrightarrow \mathrm{Inv}^3(\mathrm{B}Z)$$

coincides with ν. In particular, its image is equal to $\mathrm{Inv}^3_h(\mathrm{B}Z)$.

Lemma 8.6. *We have* $\mathrm{Inv}^3_v(\mathrm{B}Z)\cap \mathrm{Ker}(\partial_Z)\subset \mathrm{Inv}^3_h(\mathrm{B}Z)$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
H^2(F,Z^\circ) \arrow[r]\arrow[d, "\Delta^2"'] & \mathrm{Ker}(\partial_Z) \arrow[r]\arrow[d, "\Delta^2"'] & H^1(F,\Upsilon^2(Z^*(F)))'/\,\mathrm{CH}^2_a(\mathrm{B}Z)' \arrow[d, "\Delta^2"']\\
H^2(F,(Z^\circ)^3) \arrow[r] & \mathrm{Ker}(\partial_{Z^3}) \arrow[r] & H^1(F,\Upsilon^2((Z^*)^3(F)))'/\,\mathrm{CH}^2_a(\mathrm{B}Z^3)',
\end{tikzcd}$$

where the vertical homomorphisms are given by the alternating sum in the definition of Δ^2. The group $\Upsilon^2(A)$ for an abelian group A is "quadratic", hence the right vertical map is trivial. That is, the map $\Delta^2:\Upsilon^2(A)\to\Upsilon^2(A^3)$ takes $x*y:=[x,n,y]$ to

$$\begin{aligned}(x,x,x)*(y,y,y)-(x,x,0)*(y,y,0)-(x,0,x)*(y,0,y)-(0,x,x)*(0,y,y)+\\ (x,0,0)*(y,0,0)+(0,x,0)*(0,y,0)+(0,0,x)*(0,0,y)\end{aligned}$$

which is zero in $\Upsilon^2(A^3)$.

Let $I\in \mathrm{Inv}^3_v(\mathrm{B}Z)\cap\mathrm{Ker}(\partial_Z)$. From the diagram, $\Delta^2(I)$ is in the image of $H^2(F,(Z^\circ)^3)$, i.e, $\Delta^2(I)\in \mathrm{Inv}^3_h(\mathrm{B}Z^3)$. By Lemma 8.3, $I\in\mathrm{Inv}^3_h(\mathrm{B}Z)$. □

Lemma 8.7. *We have* $\mathrm{Inv}^3_{av}(\mathrm{B}Z) \subset \mathrm{Inv}^3_h(\mathrm{B}Z)$.

Proof. Consider the commutative diagram arising from (8.4):

$$\begin{tikzcd}
\mathrm{Inv}^3_a(\mathrm{B}Z) \arrow[r, "\partial_Z"] \arrow[d, "\Delta^2"'] & S^2(Z^*(F_{\mathrm{sep}}))' / \mathrm{CH}^2(\mathrm{B}Z)' \arrow[d, "\Delta^2"'] \\
\mathrm{Inv}^3_a(\mathrm{B}Z^3) \arrow[r, "\partial_{Z^3}"] & S^2((Z^*)^3(F_{\mathrm{sep}}))' / \mathrm{CH}^2(\mathrm{B}Z^3)'.
\end{tikzcd}$$

As in the proof of Lemma 8.6, we see that the right vertical map is zero. Therefore, $\Delta^2(I) \in \mathrm{Ker}(\partial_{Z^3})$ for every $I \in \mathrm{Inv}^3_{av}(\mathrm{B}Z)$. By Lemma 8.6, applied to $\Delta^2(I)$, we have $\Delta^2(I) \in \mathrm{Inv}^3_h(\mathrm{B}Z^3)$. It follows from Lemma 8.3 that $I \in \mathrm{Inv}^3_h(\mathrm{B}Z)$. $\square$

Now we can prove Theorem A. Let K/F be a field extension such that G_K is quasi-split. By Propositions 4.15 and 4.19(2), we have a commutative diagram

$$\begin{tikzcd}
Q(\widetilde{G}) \arrow[r] & A^0(\mathrm{B}Z, H^3) \arrow[r] \arrow[d] & A^0(G, H^3) \arrow[r] \arrow[d] & 0 \\
& A^0(\mathrm{B}Z_K, H^3) \arrow[r, "\sim"] & A^0(G_K, H^3) &
\end{tikzcd}$$

with the top exact row and the bottom isomorphism.

Plugging in $K = F_{\mathrm{sep}}$ and $K = F(X)$, we get an exact sequence

$$Q(\widetilde{G}) \to A^0_{av}(\mathrm{B}Z, H^3) \to A^0_{av}(G_K, H^3) \to 0. \tag{8.8}$$

Now let $u \in H^3(F(G))$ be an unramified element. In particular, $u \in A^0(G, H^3)$. Since over fields F_{sep} and $F(X)$ the group G is quasi-split, the element u vanishes over these fields by Lemma 6.6. Therefore, $u \in A^0_{av}(G, H^3)$. By (8.8), u lifts to an element in $A^0_{av}(\mathrm{B}Z, H^3)$ and then to an invariant $I \in \mathrm{Inv}^3_{av}(\mathrm{B}Z)$ in view of Example 3.1. By Lemma 8.7, $I \in \mathrm{Inv}^3_h(\mathrm{B}Z)$. It follows that the image of I in $\mathrm{Inv}^3(G)$ is contained in $\mathrm{Inv}^3_h(G)$. Therefore, the image of u in $\overline{H}^3(F(G))$ is contained in the subgroup $\mathrm{Inv}^3_h(G)$ of $\overline{H}^3(F(G))$.

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Department of Mathematics, University of California, Los Angeles, CA, 90095-1555, USA

E-mail address: `merkurev` *at* `math.ucla.edu`