CHOW FILTRATION ON REPRESENTATION RINGS OF ALGEBRAIC GROUPS

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ABSTRACT. We introduce and study a filtration on the representation ring $R(G)$ of an affine algebraic group G over a field. This filtration, which we call *Chow filtration*, is an analogue of the coniveau filtration on the Grothendieck ring of a smooth variety. We compare it with the other known filtrations on $R(G)$ and show that all three define on $R(G)$ the same topology. For any $n \geq 1$, we compute the Chow filtration on $R(G)$ for the special orthogonal group $G := O^+(2n+1)$. In particular, we show that the graded group associated with the filtration is torsion-free. On the other hand, the Chow ring of the classifying space of G over any field of characteristic $\neq 2$ is known to contain non-zero torsion elements. As a consequence, any sufficiently good approximation of the classifying space yields an example of a smooth quasi-projective variety X such that its Chow ring is generated by Chern classes and at the same time contains non-zero elements vanishing under the canonical homomorphism onto the graded ring associated with the coniveau filtration on the Grothendieck ring of X.

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1. Introduction

Let G be an affine group scheme of finite type over a field and let $R(G)$ be its representation ring (the Grothendieck ring of the category of finite-dimensional linear G-representations). In this paper we introduce and study a ring filtration on $R(G)$ which is an analogue of the coniveau filtration on the Grothendieck ring of a smooth variety. We call it *Chow filtration* because of a close relation with the Chow ring of the classifying space of G. In topology, for finite groups, a similar filtration has been considered by Atiyah in [1].

The ring $R(G)$ is an augmented λ-ring and therefore has two other important filtrations: the filtration by powers of the augmentation ideal and Grothendieck's γ-filtration. The latter can be defined via Chern classes and for this reason we call it Chern filtration in the paper. It has been shown by Totaro (see [21, proof of Theorem 3.1]) that the Chern filtration is equivalent to the augmentation filtration in the sense that they define on $R(G)$ the same topology. We show that the Chow filtration is also equivalent to them (see Corollary 4.8).

Our initial motivation came from following conjecture raised in [11]:

Conjecture 1.1. *Let G be a split semisimple algebraic group over a field, let P be a* special *parabolic subgroup of G (i.e., all P-torsors over all extension fields of the base field are trivial), let E be a standard generic G-torsor, and let X be the quotient variety E/P. Then the canonical homomorphism of the Chow ring* $\mathrm{CH}(X)$ *onto the graded ring associated with the coniveau filtration on the Grothendieck ring of X is an isomorphism.*

Using computations of Chow rings of classifying spaces of special parabolic subgroups, it has been shown in [12] that for X as above the ring $\mathrm{CH}(X)$ is generated by Chern classes (of vector bundles). It was not immediately clear to the authors of the present paper that this condition alone is insufficient for the conclusion of the conjecture to hold. Examples showing that it indeed is are produced here (see Theorem 5.5). Unexpectedly, they are also related to computations of Chow rings for classifying spaces of algebraic groups. In fact, analogous examples are first obtained with a classifying space in place of X. Then one takes for X a variety which is (in a certain specific sense) its sufficiently good approximation.

Before we can construct the example with a classifying space of an algebraic group G, we need to introduce the Chow filtration on the ring $R(G)$ (which can be viewed as the Grothendieck ring of the classifying space of G). This is done in §4, where we also study some basic properties of the filtration introduced and relate it to the other two filtrations on $R(G)$ in various ways.

Proving Theorem 5.5, we are using a computation of the Chow ring for the classifying space of the orthogonal group $O^+(2n+1)$ made first by B. Totaro over $\mathbb{C}$ and then by L. A. Molina Rojas and A. Vistoli over an arbitrary field of characteristic $\neq 2$. For the sake of completeness, we do the corresponding computation over a field of characteristic 2 in Appendix O. It turns out that the answer in characteristic 2 differs from the answer in other characteristics. In particular, examples of Theorem 5.5 do not extend to characteristic 2.

By a *variety* we mean an integral separated scheme of finite type over a field. An *algebraic group* is an affine group scheme of finite type over a field (not necessarily connected, not necessarily smooth).

2. Chern classes

Let R be λ-ring (see [6]). We say that R is an *augmented λ-ring* if there is given an (*augmentation*) homomorphism of λ-rings $\mathrm{rk} : R \to \mathbb{Z}$ (where $\mathbb{Z}$ is considered with its standard λ-ring structure), that is $\mathrm{rk}(\lambda^i(a)) = \binom{\mathrm{rk}(a)}{i}$. An augmented λ-ring R is an *enhanced λ-ring*, if an involution (*duality automorphism*) $R \to R$, $a \mapsto a^\vee$ is given such that $\mathrm{rk}(a^\vee) = \mathrm{rk}(a)$ for all $a \in R$.

Example 2.1. Let X be a smooth variety over a field F and $R = K(X)$, the Grothendieck ring $K_0(X)$ of classes of locally-free sheaves on X. Exterior powers of locally-free sheaves yield structure of a λ-ring on R. The rank map $R \to \mathbb{Z}$ is an augmentation and duality on R is given by dual sheaves. Thus, R is an enhanced λ-ring.

Example 2.2. Let G be an algebraic group over F and $R = R(G)$, the representation ring of G. Exterior powers of representations yield structure of a λ-ring on R. The dimension map $R \to \mathbb{Z}$ is an augmentation and duality on R is given by dual representations. Thus, R is an enhanced λ-ring.

Let R be a λ-ring. Recall that the *total λ-operation*

$$\lambda_t : R \to R[[t]],$$

where t is a variable, is defined by $\lambda_t(a) = \sum_{i\geq 0} \lambda^i(a)t^i$ and the *total γ-operation*

$$\gamma_t : R \to R[[t]]$$

satisfies $\gamma_t := \lambda_{t/(1-t)}$. We have $\gamma^0 = 1$, $\gamma^1 = \mathrm{id}$ and $\gamma^n = \sum_{i=1}^n \binom{n-1}{i-1}\lambda^i$ for all $n \geq 1$.

If R is an enhanced λ-ring, we define *Chern classes* $c_i^R : R \to R$ for all $i \geq 0$ by

$$c_i^R(a) := \gamma^i(\mathrm{rk}(a) - a^\vee).$$

The *total Chern class* c_t^R satisfies $c_t^R(a + b) = c_t^R(a)c_t^R(b)$. We have $c_0^R = 1$ and $c_1^R(a) = \mathrm{rk}(a) - a^\vee$.

We define *Chern filtration*

$$R = R^{[0]} \supset R^{[1]} \supset \dots$$

where $R^{[i]}$ is the subgroup generated by all products

$$c_{i_1}^R(a_1) \cdots c_{i_n}^R(a_n)$$

with $n \geq 0$, $a_1, \dots, a_n \in R$, and $i_1 + \dots + i_n \geq i$. (Note that since $c_i^R(a) = c_i^R(a - \mathrm{rk}(a))$ for every $a \in R(G)$, it suffices to take a_i with $\mathrm{rk}(a_i) = 0$.) In other words, Chern filtration is the smallest ring filtration with the property that $c_i^R(a) \in R^{[i]}$ for any $a \in R$ and any $i \geq 0$ (cf., [9, Definition 2.6]).

We write $\mathrm{Chern} R = \bigoplus_{i\geq 0} \mathrm{Chern}^i R$ for the graded ring associated with the Chern filtration.

Let $I = \mathrm{Ker}(\mathrm{rk}) \subset R$ be the *augmentation ideal*. For any $a \in I$, we have

$$a = \gamma^1(a) = -\gamma^1(\mathrm{rk}(a) - a) = -c_1^R(a^\vee) \in R^{[1]}.$$

It follows that $I \subset R^{[1]}$ and hence $I^i \subset R^{[i]}$ for all i.

Note that both filtrations on R we introduced are determined by augmented λ-ring structure of R and do not depend on the duality automorphism.

Example 2.3. If X is a smooth variety over a field F, the Chern classes on $R = K(X)$ coincide with the K-theoretic Chern classes as defined in [18, Example 3.6.1]. We write c_i^K for c_i^R. In particular, $c_1^K(x) = 1 - x^{-1}$, where x is the class of a line bundle on X.

3. The Grothendieck ring of a smooth variety

We recall that a *variety* is an integral separated scheme of finite type over a field. Let X be a smooth variety over a field F. We write $K(X)$ for the Grothendieck ring $K_0(X)$ of classes of locally-free sheaves on X. We introduce three filtrations on $K(X)$.

The augmentation ideal $I(X) \subset K(X)$ is the kernel of the (augmentation) ring homomorphism $K(X) \to \mathbb{Z}$ given by the rank of locally-free sheaves. The *augmentation filtration* on $K(X)$ is given by powers $I(X)^i$, $i \geq 0$ of the augmentation ideal.

By Example 2.1, the ring $K(X)$ is an enhanced λ-ring. In particular, it has the *Chern filtration* (the same as Grothendieck's γ-*filtration*, see [6])

$$K(X) = K(X)^{[0]} \supset K(X)^{[1]} \supset \dots$$

The class in $K(X)$ of any coherent sheaf on X is obtained by taking the alternating sum of the terms of any its finite locally free resolution. For any $i \geq 0$, let $K(X)^{(i)} \subset K(X)$ be the subgroup generated by the classes of coherent sheaves whose support has codimension at least i. The finite filtration

$$K(X) = K(X)^{(0)} \supset K(X)^{(1)} \supset \dots \supset K(X)^{(\dim X+1)} = 0$$

thus obtained is a ring filtration known under various names in the literature: the filtration by codimension of support, topological filtration, geometric filtration, coniveau filtration. We call it *Chow filtration* because of its close relation to the Chow ring $\mathrm{CH}(X)$ (see below).

By [6, Theorem 3.9 of Chapter V], $c_i^K(a) \in K(X)^{(i)}$ for all $a \in K(X)$ and all i. It follows that the three filtrations are related by the inclusions:

$$I(X)^i \subset K(X)^{[i]} \subset K(X)^{(i)} \text{ for any } i.$$

Remark 3.1. Some of these inclusions are equalities: $I(X) = K(X)^{[1]} = K(X)^{(1)}$ and $K(X)^{[2]} = K(X)^{(2)}$ for all X, [9, Proposition 2.14(2)]. The inclusion of the Chern filtration into Chow filtration yields a graded ring homomorphism $\mathrm{Chern}K(X) \to \mathrm{Chow}K(X)$ of the associated graded rings that is neither injective nor surjective in general, but becomes an isomorphism after tensoring with $\mathbb{Q}$, [6, Proposition 5.5 of Chapter VI]. In particular, the kernel and cokernel of $\mathrm{Chern}K(X) \to \mathrm{Chow}K(X)$ are torsion groups.

There is a well defined surjective graded ring homomorphism

$$\varphi : \mathrm{CH}(X) \to \mathrm{Chow}K(X)$$

taking the class of a closed subvariety $Z \subset X$ of codimension i to the class of its structure sheaf O_Z in $\mathrm{Chow}^i K(X)$. The kernel of $\varphi^i : \mathrm{CH}^i(X) \to \mathrm{Chow}^i K(X)$ is killed by multiplication by $(i-1)!$ (see [5, Example 15.3.6]). In particular, φ^i is an isomorphism for $i \leq 2$.

Proposition 3.2. *The homomorphism φ commutes with Chern classes, that is $\varphi(c_i(a)) = c_i^K(a)$ modulo $K(X)^{(i+1)}$ for every $a \in K(X)$.*

Proof. In view of the splitting principle (see [6, Lemma 3.8 of Chapter V]) it suffices to consider the case $i = 1$ and $a = [\mathcal{L}(Z)]$ for an irreducible divisor $j : Z \hookrightarrow X$, where $\mathcal{L}(Z)$ is the locally-free sheaf on X associated with Z. We have an exact sequence of sheaves on X:

$$0 \to \mathcal{L}(-Z) \to O_X \to j_* O_Z \to 0.$$

It follows that

$$\varphi(c_i(a)) = \varphi([Z]) = [j_* O_Z] = [O_X] - [\mathcal{L}(-Z)] = 1 - a^{-1} = c_1^K(a). \qquad \square$$

Proposition 3.3. *The following holds for any integer $i \geq 0$.*

1. *If $K(X)$ is generated by classes of line bundles, then $I(X)^i = K(X)^{[i]}$.*
2. *If $\mathrm{CH}(X)$ is generated by Chern classes, then $K(X)^{[i]} = K(X)^{(i)}$.*
3. *If the group $\mathrm{Chern}K(X)$ is torsion-free, then $K(X)^{[i]} = K(X)^{(i)}$.*

Proof. (1) If $l \in K(X)$ is the class of a line bundle, then $c_1^K(l) = 1 - l^{-1} \in I(X)$ and $c_i^K(-l) = (l^{-1} - 1)^i \in I(X)^i$ for any i. If $a, b \in K(X)$ are such that $c_i^K(a), c_i^K(b) \in I(X)^i$ for all i, then $c_i^K(a + b) = \sum_j c_j^K(a) c_{i-j}^K(b) \in I(X)^i$.

(2) By Proposition 3.2, for any i, the surjective ring homomorphism $\varphi : \mathrm{CH}(X) \to \mathrm{Chow}K(X)$ takes the Chow-theoretical Chern class $c_i(a) \in \mathrm{CH}^i(X)$ with $a \in K(X)$ to the class modulo $K(X)^{(i+1)}$ of the K-theoretical Chern class $c_i^K(a)$. It follows that the ring $\mathrm{Chow}K(X)$ is generated by Chern classes. By descending induction on i, we see that $K(X)^{[i]} = K(X)^{(i)}$.

(3) If $\mathrm{Chern}K(X)$ is torsion-free, the homomorphism $\mathrm{Chern}K(X) \to \mathrm{Chow}K(X)$ is injective. The equality $K(X)^{[i]} = K(X)^{(i)}$ follows by ascending induction on i. $\square$

Corollary 3.4. *If $\mathrm{CH}(X)$ is generated by $\mathrm{CH}^1(X)$, then $I(X)^i = K(X)^{[i]} = K(X)^{(i)}$.*

Proof. By descending induction on i, we see that the subring of $K(X)$ generated by line bundles contains $K(X)^{(i)}$. Therefore $K(X)$ is generated by line bundles. The statement under proof follows then from Proposition 3.3 (1) and (2). $\square$

Note that the graded ring $\mathrm{Chern}K(X)$, associated with the Chern filtration, is always generated by Chern classes. So, if the Chern and Chow filtrations on $K(X)$ coincide, the ring $\mathrm{Chow}K(X)$ is generated by Chern classes. This however does not imply that the Chow ring $\mathrm{CH}(X)$ is generated by Chern classes:

Example 3.5. Let L/F be a biquadratic field extension with $\operatorname{char} F \neq 2$ and let T be the corresponding (3-dimensional) torus of elements of norm 1. Denote by E_i, $i = 1, 2, 3$ all quadratic subextensions, by T_i the subtorus in T of norm 1 elements in the extension L/E_i and by ξ_i the class of the sheaf O_{T_i} in $K_0(T)$. Then by [15, Example 9.15(2)],

$$K(T)^{(i)} = \begin{cases} \coprod_{k=1}^{3} \mathbb{Z}/2\mathbb{Z} \cdot \xi_k, & \text{if } i = 1; \\ \mathbb{Z}/2\mathbb{Z} \cdot (\xi_1 + \xi_2 + \xi_3), & \text{if } i = 2; \\ 0, & \text{if } i = 3. \end{cases}$$

In particular, $K(T)$ is generated by the classes of line bundles, $K(T)^{[i]} = K(T)^{(i)}$ for all i, $\mathrm{CH}^1(T)$ is a group of order 4 generated by the three elements $\alpha_k := c_1(\xi_k)$ of order 2 (with the relation $\alpha_1 + \alpha_2 + \alpha_3 = 0$). The group $\mathrm{CH}^2(T)$ is cyclic of order 2 generated by $\alpha_1\alpha_2 = \alpha_1\alpha_3 = \alpha_2\alpha_3$ and $\alpha_k^2 = 0$ for all k. Therefore, all polynomials in Chern classes in $\mathrm{CH}^3(T)$ are trivial. On the other hand, by [13, Proposition 5.3], the order of the class of the identity in $\mathrm{CH}^3(T)$ is equal to 2, hence $\mathrm{CH}^3(T) \neq 0$ and Chow ring $\mathrm{CH}(T)$ is not generated by Chern classes.

The following statement will be used in the next section:

Lemma 3.6. *Let $f : X \to Y$ be a flat morphism of smooth varieties such that the pull-back homomorphisms $K(Y) \to K(X)$ and $\mathrm{CH}(Y) \to \mathrm{CH}(X)$ are isomorphisms. Then the induced monomorphisms $K(Y)^{(i)} \to K(X)^{(i)}$ and $K(Y)^{[i]} \to K(X)^{[i]}$ are isomorphisms.*

Proof. To prove the statement on the Chow filtration, we proceed by descending induction on i. Looking at the commutative diagram

$$\begin{tikzcd} \mathrm{CH}^i(Y) \arrow[r, "\sim"] \arrow[d] & \mathrm{CH}^i(X) \arrow[d] \\ \mathrm{Chow}^i K(Y) \arrow[r] & \mathrm{Chow}^i K(X) \end{tikzcd}$$

we see that the bottom map is surjective. By the induction hypothesis, the map

$$K(Y)^{(i+1)} \to K(X)^{(i+1)}$$

is an isomorphism, hence $K(Y)^{(i)} \to K(X)^{(i)}$ is surjective. This proves the statement on the Chow filtration.

The statement on the Chern filtration does not require the assumption on $\mathrm{CH}(Y) \to \mathrm{CH}(X)$. It follows from the fact that flat pull-backs respect enhanced λ-ring structures. $\square$

4. Representation rings of algebraic groups

We recall that by an *algebraic group* we mean an affine group scheme of finite type over a field. Let G be an algebraic group over a field F and let $R(G)$ be its *representation ring* – the Grothendieck ring of the category of finite-dimensional linear G-representations. The augmentaion ideal $I(G) \subset R(G)$ is the kernel of the (augmentation) ring homomorphism $R(G) \to \mathbb{Z}$ given by dimension of G-representations. The augmentation filtration on $R(G)$ is given by powers $I(G)^i$, $i \geq 0$ of the augmentation ideal.

The ring $R(G)$ is an enhanced λ-ring by Example 2.2. We simply write c_i^R for the Chern classes $c_i^{R(G)}$. Recall that we have the Chern filtration

$$R(G) = R(G)^{[0]} \supset R(G)^{[1]} \supset \dots$$

with the property that $c_i^R(x) \in R(G)^{[i]}$ for all $x \in R(G)$ and any $i \geq 0$. As usual, we write $\mathrm{Chern}R(G)$ for the associated graded ring.

Our next goal is to define the Chow filtration on $R(G)$. Let V be a generically free G-representation over F. By [2, Exposé V, Théorème 8.1], there is a nonempty G-invariant open subset $U \subset V$ and a G-torsor $U \to U/G$ for some variety U/G over F. We say that U/G is an *n-approximation of* $\mathrm{B}G$ if $\operatorname{codim}_V(V \setminus U) \geq n$.

Example 4.1 (cf., [21, Remark 1.4])**.** Embed $G \hookrightarrow \mathrm{GL}(m)$ with $m > 0$ and choose an integer $N \geq 0$. Let U be the open subset of all injective linear maps $F^m \to F^{m+N}$ in the vector space V of all linear maps $F^m \to F^{m+N}$. We have $\operatorname{codim}_V(V \setminus U) = N + 1$. The group $\mathrm{GL}(m + N)$ acts linearly on V and acts transitively on U with the stabilizer $\begin{pmatrix} 1 & * \\ 0 & \mathrm{GL}(N) \end{pmatrix}$ of the canonical inclusion $F^m \hookrightarrow F^m \oplus F^N = F^{m+N}$. The group G acts on U via $\mathrm{GL}(m)$ and

$$U/G = \mathrm{GL}(m + N)/ \begin{pmatrix} G & * \\ 0 & \mathrm{GL}(N) \end{pmatrix}.$$

Thus, U/G is an $(N+1)$-approximation of $\mathrm{B}G$. Note that $U/\operatorname{GL}(m)$ is naturally isomorphic to the Grassmannian variety $\mathrm{Gr}(m, m + N)$.

Let U/G be an n-approximation of $\mathrm{B}G$. In [21], Totaro defined graded Chow ring $\mathrm{CH}(\mathrm{B}G)$ by

$$\mathrm{CH}^i(\mathrm{B}G) = \mathrm{CH}^i(U/G)$$

for $i < n$. This is independent of the choice of approximation.

Note that $\mathrm{CH}(\mathrm{B}G)$ coincides with the G-equivariant Chow ring of $\operatorname{Spec} F$, [3].

Let $E \to X$ be a G-torsor, where X is a smooth variety. We have a canonical ring homomorphism

$$\alpha_E : R(G) \to K(X),$$

taking the class of a G-representation W to the class of the vector bundle

$$(W \times E)/G \to X.$$

Since α_E is a homomorphism of enhanced λ-rings, α_E commutes with Chern classes c_i^R and c_i^K respectively.

If U/G is an approximation of $\mathrm{B}G$, we have a ring homomorphism

$$\alpha_U : R(G) \to K(U/G)$$

given by the G-torsor $U \to U/G$. The map α_U is the composition of the homotopy invariance isomorphism

$$R(G) = K^G(\operatorname{Spec} F) \xrightarrow{\sim} K^G(V)$$

in equivariant K-theory and the surjective restriction homomorphism

$$K^G(V) \to K^G(U) = K(U/G)$$

(see [20, Theorems 2.7 and 4.1]). Thus, α_U is surjective.

Composing α_U with (classical) Chern classes on U/G yields Chern classes

$$c_i : R(G) \to \mathrm{CH}^i(\mathrm{B}G).$$

Lemma 4.2. *Let $E \to X$ be a G-torsor and U/G an n-approximation of* $\mathrm{B}G$. *Then*

$$\alpha_U^{-1}(K(U/G)^{(n)}) \subset \alpha_E^{-1}(K(X)^{(n)}).$$

Proof. In the commutative diagram

$$
\begin{tikzcd}
 & R(G) \arrow[dl, "\alpha_U"'] \arrow[d, "\alpha_{U\times E}"] \arrow[dr, "\alpha_{V\times E}"] \arrow[drr, bend left, "\alpha_E"] & & \\
K(U/G) \arrow[r, "\beta"] & K((U\times E)/G) & K((V\times E)/G) \arrow[l, two heads, "\varepsilon"'] & K(X) \arrow[l, "\delta", "\sim"']
\end{tikzcd}
\tag{4.3}
$$

the map δ is the pull-back with respect to the vector bundle $(V \times E)/G \to X$ so that δ is an isomorphism of groups with Chow filtrations by Lemma 3.6. In particular,

$$\alpha_E^{-1}(K(X)^{(n)}) = \alpha_{V\times E}^{-1}(K((V \times E)/G)^{(n)}).$$

The homomorphism ε is the restriction to the open subset $(U \times E)/G \subset (V \times E)/G$ with complement of codimension at least n. Therefore ε is surjective on the terms of Chow filtrations and the kernel of ε is contained in $K((V \times E)/G)^{(n)}$ by localization property in K-theory. It follows that

$$\alpha_{V\times E}^{-1}(K((V \times E)/G)^{(n)}) = \alpha_{U\times E}^{-1}(K((U \times E)/G)^{(n)}).$$

As β respects Chow filtrations, we have

$$\alpha_U^{-1}(K(U/G)^{(n)}) \subset \alpha_{U\times E}^{-1}(K((U \times E)/G)^{(n)}).$$

The result follows. □

It follows from Lemma 4.2 that the subgroup $(\alpha_U)^{-1}(K(U/G)^{(n)})$ of $R(G)$ does not depend on the choice of an n-approximation U/G. We set

$$R(G)^{(n)} := (\alpha_U)^{-1}(K(U/G)^{(n)})$$

for any n-approximation U/G of $\mathrm{B}G$. This way we get the *Chow filtration*

$$R(G) = R(G)^{(0)} \supset R(G)^{(1)} \supset \dots$$

on $R(G)$.

It also follows from Lemma 4.2 that for a G-torsor $E \to X$, the map $\alpha_E : R(G) \to K(X)$ takes $R(G)^{(n)}$ into $K(X)^{(n)}$, i.e., α_E respects Chow filtrations.

As in Section 3, we have

$$I(G)^n \subset R(G)^{[n]} \subset R(G)^{(n)}$$

for all n. (However none of the filtrations is finite in general.) The second inclusion induces a ring homomorphism $\mathrm{Chern}R(G) \to \mathrm{Chow}R(G)$ which is neither injective nor surjective in general.

Let U/G be an $(n+1)$-approximation of $\mathrm{B}G$. The composition

$$\mathrm{CH}^n(\mathrm{B}G) = \mathrm{CH}^n(U/G) \xrightarrow{\varphi^n} \mathrm{Chow}^n K(U/G) = \mathrm{Chow}^n R(G)$$

yields a surjective graded ring homomorphism

$$\varphi : \mathrm{CH}(\mathrm{B}G) \twoheadrightarrow \mathrm{Chow}R(G).$$

The kernel of $\varphi^i : \mathrm{CH}^i(\mathrm{B}G) \to \mathrm{Chow}^i R(G)$ is killed by multiplication by $(i-1)!$. In particular, the maps φ^i are isomorphisms for $i \leq 2$. By Proposition 3.2, $\varphi(c_i(a)) = c_i^R(a)$ modulo $R(G)^{(i+1)}$ for every $a \in R(G)$.

Remark 4.4. For any approximation X of $\mathrm{B}G$, the homomorphism $R(G) \to K(X)$ maps $I(G)^n$ onto $I(X)^n$, $R(G)^{[n]}$ onto $K(X)^{[n]}$, and $R(G)^{(n)}$ onto $K(X)^{(n)}$ for any n. The statement on the Chern filtration holds because the homomorphism $R(G) \to K(X)$ commutes with Chern classes. For the statement on $R(G)^{(n)}$ consider the diagram (4.3) in the proof of Lemma 4.2 with $X = E/G$ being an approximation and U/G a k-approximation of $\mathrm{B}G$ for some $k > \dim(X)$ and $\geq n$. Then the homomorphism $K(X) \to K((U \times E)/G)$ is an isomorphism of rings with Chow filtrations. The map $R(G) \to K(U/G)$ maps $R(G)^{(n)}$ onto $K(U/G)^{(n)}$ by the definition of $R(G)^{(n)}$ and $K(U/G)^{(n)}$ is mapped onto $K((U \times E)/G)^{(n)} = K(X)^{(n)}$.

Remark 4.5. By the very definition of Chow filtration on $R(G)$, for any n-approximation X of $\mathrm{B}G$, the kernel of $R(G) \to K(X)$ is contained in $R(G)^{(n)}$. This statement can be partially inverted: if X is any approximation of $\mathrm{B}G$ such that $\mathrm{Ker}(R(G) \to K(X)) \subset R(G)^{(n)}$ for some n, then $R(G)^{(n)}$ is the inverse image of $K(X)^{(n)}$. (This does not mean that X is an n-approximation but does mean that X – like a n-approximation – can be used for computation of the Chow filtration on $R(G)$ in codimensions up to n.) Indeed, by Remark 4.4, the inverse image of $K(X)^{(n)}$ is

$$R(G)^{(n)} + \mathrm{Ker}(R(G) \to K(X)).$$

Similarly, the inverse image of $K(X)^{[n]}$ is $R(G)^{[n]}$ if the kernel is contained in $R(G)^{[n]}$ and the inverse image of $I(X)^n$ is $I(G)^n$ if the kernel is contained in $I(G)^n$.

Lemma 4.6. *For any G-torsor $E \to X$ (with X a smooth variety), the kernel of α_E contains $R(G)^{(n)}$ for some n.*

Proof. Let $n = \dim(X) + 1$. Since α_E respects Chow filtration, we have

$$\alpha_E(R(G)^{(n)}) \subset K(X)^{(n)} = 0. \qquad \square$$

Lemma 4.7 (cf. the beginning of Remark 4.5)**.** *For any n, k and any G, there exists a k-approximation X of $\mathrm{B}G$ such that the kernel of the surjective homomorphism $R(G) \to K(X)$ is contained in $I(G)^n$.*

Proof. Let us fix an embedding $G \hookrightarrow \mathrm{GL}(m)$ for some m. For any N, consider an $(N+1)$-approximation $U/G = \mathrm{GL}(m+N)/H$ of $\mathrm{B}G$ as in Example 4.1, where $H = \begin{pmatrix} G & * \\ 0 & \mathrm{GL}(N) \end{pmatrix}$. Note that $R(H) = R(G \times \mathrm{GL}(N))$ since the unipotent radical of H acts trivially on all simple representations of H. Moreover, $R(G \times \mathrm{GL}(N)) = R(G) \otimes R(\mathrm{GL}(N))$ as follows from Propositions R.2 and R.5.

By [14, Theorem 41],

$$\begin{aligned} K(U/G) &= K(\mathrm{GL}(m+N)/H) \\ &= \mathbb{Z} \otimes_{R(\mathrm{GL}(m+N))} R(H) = \mathbb{Z} \otimes_{R(\mathrm{GL}(m+N))} [R(G) \otimes R(\mathrm{GL}(N))]. \end{aligned}$$

Under this identification, the homomorphism $\alpha_U : R(G) \to K(U/G)$ (which we denote below by α^G) coincides with the natural (surjective) homomorphism

$$R(G) \to \mathbb{Z} \otimes_{R(\mathrm{GL}(m+N))} [R(G) \otimes R(\mathrm{GL}(N))].$$

It follows that

$$\alpha^G = \alpha^{\mathrm{GL}(m)} \otimes_{R(\mathrm{GL}(m))} R(G)$$

and therefore, the natural homomorphism

$$\mathrm{Ker}(\alpha^{\mathrm{GL}(m)}) \otimes_{R(\mathrm{GL}(m))} R(G) \to \mathrm{Ker}(\alpha^G)$$

is surjective.

Since $U/\mathrm{GL}(m) = \mathrm{Gr}(m, m+N)$, as computed in Example G.2, the kernel of $\alpha^{\mathrm{GL}(m)}$ is generated by some polynomials (namely, by the polynomials d_i^R, $i \geq N+1$) of degree at least $N+1$ in the Chern classes $c_1^R, \dots, c_m^R \in R(\mathrm{GL}(m))$ of the standard representation of $\mathrm{GL}(m)$, where c_i^R is of degree i. Therefore, $\mathrm{Ker}(\alpha^G)$ is generated by polynomials in the images of $c_1^R, \dots, c_m^R$ (these images are the Chern classes of the G-representation given by the fixed embedding $G \hookrightarrow \mathrm{GL}(m)$) of degree $> N$ and will indeed contain $I(G)^n$ for sufficiently large N (say, for $N \geq mn$). □

Corollary 4.8. *For any group G and any n, we have $I(G)^n \supset R(G)^{(N)}$ for some N. In particular, the three filtrations define the same topology on $R(G)$.*

Proof. By Lemma 4.7, we find an approximation X of $\mathrm{B}G$ such that

$$I(G)^n \supset \mathrm{Ker}(R(G) \to K(X)).$$

By Lemma 4.6, the kernel contains $R(G)^{(N)}$ for some N. □

As follows from the proof, in Corollary 4.8 one can actually take

$$N = m^2(n+1) - \dim G + 1,$$

where m is such that G can be embedded in $\mathrm{GL}(m)$. This formula for N is linear in n.

Corollary 4.9. *We have $I(G) = R(G)^{[1]} = R(G)^{(1)}$ and $R(G)^{[2]} = R(G)^{(2)}$. The map* $\mathrm{Chern}R(G) \to \mathrm{Chow}R(G)$ *becomes an isomorphism after tensoring with $\mathbb{Q}$.*

Proof. Let X be an approximation of $\mathrm{B}G$ such that the kernel of $R(G) \to K(X)$ is contained in $I(G)^{n+1}$. By Remark 4.5, for $i \leq n+1$, $R(G)^{(i)}$ and $R(G)^{[i]}$ are inverse images of $K(X)^{(i)}$ and $K(X)^{[i]}$, respectively. As a consequence, $\mathrm{Chern}^i R(G) = \mathrm{Chern}^i K(X)$ and $\mathrm{Chow}^i R(G) = \mathrm{Chow}^i K(X)$ for $i \leq n$. The statements follow from Remark 3.1. □

Now we compute the groups $\mathrm{Chern}^i(G)$ for $i \leq 1$. Clearly, $\mathrm{Chern}^0(G) = R(G)/I(G) = \mathbb{Z}$.

Lemma 4.10. *We have*

$$\sum_{i=0}^{n} \gamma^i([V] - n) = \lambda^n([V]) \text{ in } R(G)$$

for every G-representation V of dimension n.

Proof. Since $\gamma_t(-1) = 1 - t$ and $\lambda^i([V]) = 0$ for $i > n$, we have

$$\gamma_t([V] - n) = \gamma_t([V])(1-t)^n = \sum_{i=0}^{\infty} \lambda^i([V]) \frac{t^i}{(1-t)^i}(1-t)^n = \sum_{i=0}^{n} \lambda^i([V]) t^i (1-t)^{n-i}.$$

It follows that $\gamma^i([V] - n) = 0$ for $i > n$. Finally, plug in $t = 1$. □

Corollary 4.11. *For every G-representation V of dimension n, we have*

$$\lambda^n([V]) - [V] + n - 1 \in R(G)^{[2]}.$$

Proof. Indeed, since $\gamma^0([V]-n) = 1$ and $\gamma^1([V]-n) = [V]-n$, we have

$$\lambda^n([V]) - [V] + n - 1 = \sum_{i=2}^{n} \gamma^i([V]-n) = \sum_{i=2}^{n} c_i^R(-[V^\vee]) \in R(G)^{[2]}. \qquad \square$$

Let $G^* = \mathrm{Hom}(G, \mathbb{G}_m)$ be the *character group* of G. Consider the homomorphism $\det : R(G) \to G^*$ taking a representation V of dimension n to the character of the 1-dimensional representation $\wedge^n(V)$.

Lemma 4.12. $\det(R(G)^{[2]}) = 1$.

Proof. Recall that $R(G)^{[2]}$ is generated by the products $c_{i_1}^R(a_1)\cdots c_{i_n}^R(a_n)$ with $i_1+\cdots+i_n \geq 2$ and $a_i \in I(G)$. In view of the product formula $\det(ab) = \det(a)^{\mathrm{rk}(b)} \det(b)^{\mathrm{rk}(a)}$, it suffices to show that $\det c_i^R(V) = 1$ for any representation $\rho : G \to \mathrm{GL}(V)$ and $i \geq 2$. By functoriality of the Chern classes with respect to ρ, it suffices to check the equality in the case $G = \mathrm{GL}(V)$ and V the standard representation of $\mathrm{GL}(V)$.

Let T be a split maximal torus of G. Since G^* restricts injectively to T^*, it suffices to prove the formula for T. But $R(T)$ is generated by the classes of 1-dimensional representations, hence $R(T)^{[2]}$ is generated by product of at least two (first) Chern classes. The statement follows again from the product formula. $\square$

By Lemma 4.12, the map det yields a homomorphism $\mathrm{Chern}^1(G) \to G^*$.

Proposition 4.13. *The homomorphism* $\mathrm{Chern}^1(G) \to G^*$ *is an isomorphism.*

Proof. We define a homomorphism $G^* \to \mathrm{Chern}^1(G)$ by $f \mapsto [L_f] - 1$ modulo $R(G)^{[2]}$, where L_f is a 1-dimensional representation of the character f. Both compositions of the maps between $\mathrm{Chern}^1(G)$ and G^* are the identities, one of them – in view of Corollary 4.11. $\square$

The next statement is an analogue of Proposition 3.3.

Proposition 4.14. *The following holds for any integer $i \geq 0$.*

1. *If $R(G)$ is generated by classes of 1-dimensional representations, then $I(G)^i = R(G)^{[i]}$.*
2. *If $\mathrm{CH}(\mathrm{B}G)$ is generated by Chern classes (of G-representaions), then $R(G)^{[i]} = R(G)^{(i)}$.*
3. *If the group $\mathrm{Chern}R(G)$ is torsion-free, then $R(G)^{[i]} = R(G)^{(i)}$.*

Proof. The proof of (1) is literally the same as in Proposition 3.3. If $l \in R(G)$ is the class of a 1-dimensional representation, then $c_1^R(l) = 1 - l^{-1} \in I(G)$ and $c_i^R(-l) = (l^{-1}-1)^i \in I(G)^i$ for any i. If $a, b \in R(G)$ are such that $c_i^R(a), c_i^R(b) \in I(G)^i$ for all i, then $c_i^R(a+b) = \sum_j c_j^R(a) c_{i-j}^R(b) \in I(G)^i$.

(2) For any i, the surjective ring homomorphism $\varphi : \mathrm{CH}(\mathrm{B}G) \to \mathrm{Chow}R(G)$ takes the Chow-theoretical Chern class $c_i(a) \in \mathrm{CH}^i(\mathrm{B}G)$ with $a \in R(G)$ to the class of $c_i^R(a) \in R(G)^{[i]} \subset R(G)^{(i)}$ modulo $R(G)^{(i+1)}$. It follows that the ring $\mathrm{Chow}R(G)$ is generated

by Chern classes. However, unlike the proof of Proposition 3.3(2), we are not able to show $R(G)^{[i]} = R(G)^{(i)}$ by descending induction on i as the filtrations in question can be infinite. We use Lemma 4.7 instead.

For X as in Lemma 4.7 (with any given n and arbitrary k), since $\mathrm{CH}(\mathrm{B}G)$ is generated by Chern classes and $\mathrm{CH}(\mathrm{B}G) \to \mathrm{CH}(X)$ is a surjective ring homomorphism mapping Chern classes to Chern classes, the ring $\mathrm{CH}(X)$ is also generated by Chern classes. It follows by Proposition 3.3(2) that $K(X)^{[n]} = K(X)^{(n)}$. Therefore $R(G)^{[n]} = R(G)^{(n)}$ by Remark 4.5.

(3) If $\mathrm{Chern}R(G)$ is torsion-free, the homomorphism $\mathrm{Chern}R(G) \to \mathrm{Chow}R(G)$ is injective. The equality $R(G)^{[i]} = R(G)^{(i)}$ follows by ascending induction on i. □

Corollary 4.15. *If* $\mathrm{CH}(\mathrm{B}G)$ *is generated by* $\mathrm{CH}^1(\mathrm{B}G)$*, then* $I(G)^i = R(G)^{[i]} = R(G)^{(i)}$.

Proof. If $\mathrm{CH}(\mathrm{B}G)$ is generated by $\mathrm{CH}^1(\mathrm{B}G)$, then $I(G)^i + R(G)^{(i+1)} = R(G)^{(i)}$ for any i so that $I(G)^i + R(G)^{(j)} = R(G)^{(i)}$ for any $j > i$. By Corollary 4.8, $R(G)^{(j)} \subset I(G)^i$ for some j, hence $R(G)^{(i)} \subset I(G)^i$. □

We finish this section by an example of G with the Chern filtration on $R(G)$ different from the Chow filtration:

Example 4.16. For $G := O^+(2n)$ with any $n \geq 3$ over the field of complex numbers, the Chern filtration on $R(G)$ differs from the Chow filtration. Indeed, according to [4, Corollary 2], the Chow ring $\mathrm{CH}(\mathrm{B}G)$ is not generated by Chern classes. By [4, Theorem 1] (see also [17]), the Chern subring of $\mathrm{CH}(\mathrm{B}G)$ (i.e., the subring of $\mathrm{CH}(\mathrm{B}G)$ generated by Chern classes) contains every element of finite order of the group $\mathrm{CH}(\mathrm{B}G)$. Since the kernel of the surjective ring homomorphism $\mathrm{CH}(\mathrm{B}G) \to \mathrm{Chow}R(G)$ consists of elements of finite order, the two above statements together imply that the ring $\mathrm{Chow}R(G)$ is not generated by Chern classes. Since the ring $\mathrm{Chern}R(G)$ is generated by Chern classes (for any G), the two filtrations (for $G = O^+(2n)$) are not the same.

A similar example can be given with $G = T$ a quasi-split torus and, in particular, a special algebraic group:

Example 4.17. Let K/F be a cyclic cubic field extension and let T be the Weil restriction $T = R_{K/F}(\mathbb{G}_\mathrm{m})$. The character lattice $L = T_K^*$ of T_K has a basis a, b, c, cyclically permuted by the Galois group $\Gamma = \mathbb{Z}/3\mathbb{Z}$ of K/F. The homomorphism of graded rings $\mathrm{CH}(\mathrm{B}T) \to \mathrm{CH}(\mathrm{B}T_K)$ is injective, its image is the subring of Γ-invariant elements, and the graded ring $\mathrm{CH}(\mathrm{B}T_K)$ with Γ-action is identified with the symmetric ring $S(L) = \mathbb{Z}[a, b, c]$, see, e.g., [10, §3]. The homomorphism $R(T) \to R(T_K)$ of the representation rings is also injective with the image the subring of Γ-invariant elements, the ring $R(T_K)$ with Γ-action is the group ring $\mathbb{Z}[L] = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$, where $x := \exp(a)$, $y := \exp(b)$, and $z := \exp(c)$, see, e.g., [16, Theorem 12.30]. The elements $a, b, c \in \mathrm{CH}^1(\mathrm{B}T_K)$ are the first Chern classes of $x, y, z \in R(T_K)$ respectively. We claim that the Γ-symmetric polynomial $a^2b + b^2c + c^2a \in \mathrm{CH}^3(\mathrm{B}T)$ is not in the Chern subring of $\mathrm{CH}(\mathrm{B}T)$. We prove the claim by showing that every element of the Chern subring lying in $\mathrm{CH}^3(\mathrm{B}T)$ is S-invariant modulo 2, where $S = S_3$ is the symmetric group permuting a, b, c. Note that $\mathrm{CH}^1(\mathrm{B}T)$ and $\mathrm{CH}^2(\mathrm{B}T)$ consist of S-invariant elements only. In remains to show that c_3 of the

Γ-symmetrizer of $x^i y^j z^k$ (for any integers i, j, k) is S-invariant modulo 2. This is done by a direct calculation.

5. Orthogonal groups

Let G be a split reductive group (over an arbitrary field), $T \subset G$ a split maximal torus, W the Weyl group. We have a natural homomorphism of graded rings

$$\mathrm{CH}(\mathrm{B}G) \to \mathrm{CH}(\mathrm{B}T)^W = S(T^*)^W,$$

where T^* is the character group of T and $S(T^*)$ is its symmetric ring. Similarly, we have a ring homomorphism

$$R(G) \to R(T)^W = \mathbb{Z}[T^*]^W.$$

Proposition 5.1 ([19, Théorème 4], see also [16, Theorem 22.38]). *The homomorphism $R(G) \to \mathbb{Z}[T^*]^W$ is an isomorphism.*

Recall that an algebraic group G is *special*, if all G-torsors over field extensions of the base field are trivial.

Proposition 5.2 ([3, Proposition 6]). *The homomorphism* $\mathrm{CH}(\mathrm{B}G) \to S(T^*)^W$ *is an isomorphism provided that G is special.*

Example 5.3 (cf. [21, §15], [17, §3]). For the special groups $G = \mathrm{GL}(n), \mathrm{SL}(n), \mathrm{Sp}(2n)$ (over an arbitrary field), let $c_i \in \mathrm{CH}^i(\mathrm{B}G)$ be the ith Chern class of the standard G-representation. Then the following Chow rings are polynomial rings in the listed algebraically independent elements:

$$\mathrm{CH}(\mathrm{BGL}(n)) = \mathbb{Z}[c_1, c_2, \dots, c_n],$$

$$\mathrm{CH}(\mathrm{BSL}(n)) = \mathbb{Z}[c_2, \dots, c_n], \quad c_1 = 0,$$

$$\mathrm{CH}(\mathrm{BSp}(2n)) = \mathbb{Z}[c_2, c_4, \dots, c_{2n}], \quad c_{odd} = 0.$$

Let us fix an integer $n \geq 1$ and consider the symplectic group $H := \mathrm{Sp}(2n)$ (over an arbitrary field). By Example 5.3, the group $\mathrm{CH}(\mathrm{B}H)$ is torsion-free. Since the kernel of the surjective ring homomorphism $\mathrm{CH}(\mathrm{B}H) \to \mathrm{Chow}R(H)$ consists of torsion elements only, it follows that this map is an isomorphism. In particular, the group $\mathrm{Chow}R(H)$ is torsion-free.

Since the ring $\mathrm{CH}(\mathrm{B}H)$ is generated by Chern classes, we conclude by Proposition 4.14(2) that the Chow filtration on $R(H)$ coincides with the Chern filtration. It follows that the group $\mathrm{Chern}R(H)$ is torsion-free.

The Weyl groups and character groups of maximal tori (as modules over the Weyl groups) of $\mathrm{Sp}(2n)$ and $O^+(2n+1)$ are isomorphic. Set $G = O^+(2n+1)$. By Proposition 5.1, there is an isomorphism of enhanced lambda rings $R(H) \simeq R(G)$. It induces an isomorphism $\mathrm{Chern}R(H) \simeq \mathrm{Chern}R(G)$. In particular, the group $\mathrm{Chern}R(G)$ turns out to be torsion-free. By Proposition 4.14(3), this implies that the Chern filtration on $R(G)$ coincides with the Chow filtration. We conclude that the group $\mathrm{Chow}R(G)$ is torsion-free.

The ring $\mathrm{CH}(\mathrm{B}G)$ has been computed for $G = O^+(2n+1)$ over the complex numbers in [21, §16]; it has been then computed for an arbitrary base field of characteristic $\neq 2$:

Proposition 5.4 ([17, Theorem 5.1]). *For $G = O^+(2n+1)$ (with any $n \geq 1$) over a field of characteristic $\neq 2$, the ring homomorphism $\mathbb{Z}[c_1, \ldots, c_{2n+1}] \to \mathrm{CH}(\mathrm{B}G)$ of the polynomial ring, mapping c_i to the ith Chern class of the standard G-representation, is surjective; its kernel is generated by c_1 and all $2c_i$ with odd i.*

It follows that for G as in Proposition 5.4, the kernel of the homomorphism $\mathrm{CH}^3(\mathrm{B}G) \to \mathrm{Chow}^3 R(G)$ is non-zero.

Theorem 5.5. *For any field F of $\operatorname{char} F \neq 2$, there exists a smooth quasi-projective variety X over F such that its Chow ring is generated by Chern classes (of vector bundles over X) and at the same time the kernel of the homomorphism $\mathrm{CH}(X) \to \mathrm{Chow} K(X)$ is non-zero. Specifically, let X be any 4-approximation of $O^+(2n+1)$ (for any $n \geq 1$). Then the ring $\mathrm{CH}(X)$ is generated by Chern classes and $\mathrm{Ker}(\mathrm{CH}^3(X) \twoheadrightarrow \mathrm{Chow}^3 K(X)) \neq 0$.*

Proof. We have a surjective ring homomorphism $\mathrm{CH}(\mathrm{B}G) \twoheadrightarrow \mathrm{CH}(X)$ with $G = O^+(2n+1)$, mapping Chern classes to Chern classes. Since $\mathrm{CH}(\mathrm{B}G)$ is generated by Chern classes (Proposition 5.4), the ring $\mathrm{CH}(X)$ is also generated by Chern classes.

Since X is a 4-approximation, the homomorphism $\mathrm{CH}^3(\mathrm{B}G) \to \mathrm{CH}^3(X)$ is an isomorphism so that $\mathrm{CH}^3(X)$ contains a non-trivial torsion by Proposition 5.4.

At the same time, we have a homomorphism $\mathrm{Chow} R(G) \twoheadrightarrow \mathrm{Chow} K(X)$ which is an isomorphism in codimensions < 4. Since $\mathrm{Chow} R(G)$ is torsion-free, the group $\mathrm{Chow}^3 K(X)$ is also torsion-free. □

Remark 5.6. If X is an r-approximation of $\mathrm{B}O^+(2n+1)$ with large enough n, then $\mathrm{CH}(X)$ is generated by Chern classes and $\mathrm{Ker}(\mathrm{CH}^i(X) \to \mathrm{Chow}^i K(X)) \neq 0$ for all $i \neq 4$ with $3 \leq i < r$.

Remark 5.7. We do not know if there exists a *projective* variety X with the properties as in Theorem 5.5.

Appendix G. General linear group

In this appendix, we provide some computations in the representation ring of a general linear group needed in the main part.

Example G.1. Let $G := \mathrm{GL}(n)$, $T \subset G$ the maximal torus of diagonal matrices. We have $R(T) = \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ a Laurent polynomial ring, where $x_i = \exp(t_i)$ and t_i are canonical generators of T^*. The Weyl group W is the nth symmetric group permuting the t_i's. It follows that

$$R(G) = R(T)^W = \mathbb{Z}[\sigma_1, \sigma_2, \ldots, \sigma_n, \sigma_n^{-1}],$$

where $\sigma_k := \sigma_k(x)$ are standard symmetric functions in the x_i's. Note that $\sigma_k = \lambda^k(x)$, the kth exterior power of x, where $x = x_1 + x_2 + \cdots + x_n$ is the class of the standard representation of G.

Since taking the dual $a \mapsto a^\vee$ of a representation yields an automorphism of the rings $R(T)$ and $R(G)$ taking x_i to x_i^{-1}, we have $\lambda^i(x^\vee) = \lambda^{n-i}(x)\lambda^n(x)^{-1}$ and $\lambda^n(x^\vee) = \lambda^n(x)^{-1}$. Therefore,

$$R(G) = \mathbb{Z}[\lambda^1(x^\vee), \lambda^2(x^\vee), \ldots, \lambda^n(x^\vee), \lambda^n(x^\vee)^{-1}].$$

The Chern class c_i^R is the ith standard elementary symmetric function in the first Chern classes $c_1^R(x_j) = 1 - x_j^{-1}$, $j = 1, \ldots, n$. In particular, c_i^R is the sum of $(-1)^i\lambda^i(x^\vee)$ and an integer linear combination of $\lambda^j(x^\vee)$ with $j < i$.

We also have

$$\sigma_n^{-1} = \prod[1 - (1 - x_i^{-1})],$$

hence

$$\sigma_n^{-1} = c_0^R - c_1^R + c_2^R - \cdots + (\pm 1)^n c_n^R.$$

It follows that

$$R(G) = \mathbb{Z}[c_1^R, c_2^R, \ldots, c_n^R, s_n^{-1}],$$

where c_i^R are algebraically independent and

$$s_n = \sum_{i=0}^{n} (-1)^i c_i^R.$$

Example G.2. Consider the homomorphism

$$\alpha^{\mathrm{GL}(m)} : R(\mathrm{GL}(m)) \to \mathbb{Z} \otimes_{R(\mathrm{GL}(m+N))} [R(\mathrm{GL}(m)) \otimes R(\mathrm{GL}(N))] = K(\mathrm{Gr}(m, m+N))$$

from the proof of Lemma 4.7. The target group is canonically isomorphic to

$$\big(R(\mathrm{GL}(m)) \otimes R(\mathrm{GL}(N))\big)/J,$$

where J is the ideal generated by the image of the augmentation ideal of $R(\mathrm{GL}(m+N))$ under $R(\mathrm{GL}(m+N)) \to R(\mathrm{GL}(m)) \otimes R(\mathrm{GL}(N))$. Recall (see Example G.1) that

$$R(\mathrm{GL}(m)) = \mathbb{Z}[c_1^R(x), c_2^R(x), \ldots, c_N^R(x), s_N^{-1}(x)],$$
$$R(\mathrm{GL}(N)) = \mathbb{Z}[c_1^R(y), c_2^R(y), \ldots, c_m^R(y), s_m^{-1}(y)],$$

where x and y are the classes of the standard representations of $\mathrm{GL}(m)$ and $\mathrm{GL}(N)$ respectively. Note that the class of the standard representation of $\mathrm{GL}(m+N)$ restricted to $\mathrm{GL}(m)\times\mathrm{GL}(N)$ is the sum of x and y. Moreover, the augmentation ideal of $R(\mathrm{GL}(m+N))$ is generated by $c_i^R(x+y)$ for $i = 1, 2, \ldots, m+N$. It follows that modding out J amounts to imposing the relation

$$c_t^R(x+y) = c_t^R(x) \cdot c_t^R(y) = 1,$$

where c_t^R is the total Chern class.

Invert the polynomial $c_t^R := c_t^R(x)$ formally:

$$(c_t^R)^{-1} =: \sum_{i \geq 0} d_i^R t^i,$$

where d_i^R is a homogeneous polynomial in $c_j^R(x)$ of degree i. Then

$$K(\mathrm{Gr}(m, m+N)) = \mathbb{Z}[c_1^R, c_2^R, \ldots, c_m^R]/(d_i^R, i \geq N+1).$$

(Note that the elements $s_m(x)$ and $s_N(y)$ are 1 plus nilpotents in $K(\mathrm{Gr}(m, m+N))$, hence, $s_m(x)$ and $s_N(y)$ are automatically invertible in $K(\mathrm{Gr}(m, m+N))$.)

Remark G.3. The ideal $(d_i^R, i \geq N+1)$ is generated by d_i^R for $i = N+1, \cdots, N+m$ only. Indeed, if $j > 0$, d_{N+m+j}^R is the negative of the linear combination $c_1^R d_{N+m+j-1}^R + \cdots + c_m^R d_{N+j}^R$ and hence the statement follows by induction on j.

It follows that the homomorphism

$$\alpha^{\mathrm{GL}(m)} : R(\mathrm{GL}(m)) \to K(\mathrm{Gr}(m, m+N))$$

is surjective, and its kernel is generated by polynomials in $c_1^R, c_2^R, \dots, c_m^R$ of degree at least $N+1$.

Appendix O. Orthogonal groups in characteristic 2

In this appendix, we determine $\mathrm{CH}(\mathrm{B}G)$ for $G = O(2n+1)$ and $G = O^+(2n+1)$ over a field F of characteristic 2. We start with some observations valid over a field of any characteristic.

For any $m \geq 1$, the factor variety $\mathrm{GL}(m)/O(m)$ is isomorphic to the variety of non-degenerate quadratic forms of dimension m that is an open subset in the affine space of all quadratic forms of dimension m (see, e.g., [8, §3]). Therefore, by [21, Proposition 14.2] (see also [8, Proposition 5.1]), $\mathrm{CH}(\mathrm{B}O(m))$ is generated by Chern classes $c_1, c_2, \dots, c_m$ of the standard representation of $O(m)$.

For any $n \geq 1$, we have $O(2n+1) = \mu_2 \times O^+(2n+1)$. Hence the restriction

$$\mathrm{CH}(\mathrm{B}O(2n+1)) \to \mathrm{CH}(\mathrm{B}O^+(2n+1))$$

is surjective. Therefore, the Chow ring $\mathrm{CH}(\mathrm{B}O^+(2n+1))$ of the classifying space of the group $O^+(2n+1)$ is also generated by the Chern classes of the standard representation.

From now on assume that $\mathrm{char}(F) = 2$. We have the following group homomorphisms:

$$O(2n) \to O^+(2n+1) \to \mathrm{Sp}(2n) \hookrightarrow \mathrm{GL}(2n).$$

The first map takes an automorphism α of the standard (non-degenerate) $2n$-dimensional hyperbolic quadratic form (V, q) to the automorphism $1 \oplus \alpha$ of (V', q'), where $V' = F \oplus V$ and $q'(a+v) = a^2 + q(v)$. Note that the subspace $F \subset V'$ coincides with the radical of the bilinear form of q'. The second map takes an automorphism β of (V', q') to the induced automorphism of $V = V'/F$ that preserves the associated nondegenerate alternating form on V. The first map and the composition $O(2n) \to \mathrm{Sp}(2n)$ are the embeddings. The second map is a (non-central) isogeny.

Let T be a split maximal torus in $O(2n)$. Its isomorphic images in $O^+(2n+1)$ and $\mathrm{Sp}(2n)$ are also maximal tori. Consider the composition

$$\text{(O.1)} \quad \mathrm{CH}(\mathrm{BGL}(2n)) \to \mathrm{CH}(\mathrm{BSp}(2n)) \to \mathrm{CH}(\mathrm{B}O^+(2n+1)) \to \mathrm{CH}(\mathrm{B}O(2n)) \to \mathrm{CH}(\mathrm{B}T).$$

For any i, the Chern class c_i in $\mathrm{CH}(\mathrm{BGL}(2n))$ clearly stays c_i in $\mathrm{CH}(\mathrm{BSp}(2n))$ as well as in $\mathrm{CH}(\mathrm{B}O(2n))$. We claim that the class c_i in $\mathrm{CH}(\mathrm{B}O^+(2n+1))$ (which is defined via the embedding of $O^+(2n+1)$ into $\mathrm{GL}(2n+1)$) stays c_i in $\mathrm{CH}(\mathrm{B}O(2n))$ (which is defined via the embedding of $O(2n)$ into $\mathrm{GL}(2n)$). In other words, the classes c_i in all groups correspond to each other.

To prove the claim, we use [21, Theorem 1.3], identifying the elements of $\mathrm{CH}(\mathrm{B}G)$ for reductive G with assignments to every G-torsor over a smooth quasi-projective variety X of an element in $\mathrm{CH}(X)$. So, let X be a smooth quasi-projective variety over F and let us consider a vector bundle $E \to X$ of rank $2n+1$ with a quadratic form representing an $O^+(2n+1)$-torsor. As $O^+(2n+1) \subset \mathrm{SL}(2n+1)$, we have $c_1(E) = 0$ (see Example

5.3). The radical $R \subset E$ of the associated bilinear form is a line sub-bundle of E. The factor bundle E/R carries a non-degenerate alternating form, thus representing a $\mathrm{Sp}(2n)$-torsor, hence $c_1(E/R) = 0$. It follows that $c_1(R) = 0$ and therefore, $c_i(E) = c_i(E/R) + c_1(R)c_{i-1}(E/R) = c_i(E/R)$. The claim is proved.

The composition $\mathrm{CH}(\mathrm{BSp}(2n)) \to \mathrm{CH}(\mathrm{B}T)$ of the homomorphisms in the sequence (O.1) is a monomorphism. The groups in the first line are generated by the Chern classes c_i, therefore, the maps in the first line are surjective. The odd Chern classes are trivial in $\mathrm{CH}(\mathrm{BSp}(2n))$ and hence in $\mathrm{CH}(\mathrm{B}O^+(2n+1))$ and in $\mathrm{CH}(\mathrm{B}O(2n))$. It follows that

$$\mathrm{CH}(\mathrm{BSp}(2n)) = \mathrm{CH}(\mathrm{B}O^+(2n+1)) = \mathrm{CH}(\mathrm{B}O(2n)) = \mathbb{Z}[c_2, c_4, \dots, c_{2n}].$$

Recall that $O(2n+1) = \mu_2 \times O^+(2n+1)$ and $\mathrm{CH}(\mathrm{B}\mu_2) = \mathbb{Z}[c_1]/(2c_1)$. It follows (see [21, §6]) that

$$\mathrm{CH}(\mathrm{B}O(2n+1)) = \mathbb{Z}[c_1, c_2, c_4, \dots, c_{2n}]/(2c_1).$$

Appendix R. Representation ring of a product

In this appendix, G and H are arbitrary algebraic groups over a field F. Recall that the abelian group $R(G)$ is free, the classes of all simple G-representations constitute a basis. The classes of two simple G-representations coincide if and only if the representations are isomorphic.

The ring homomorphisms

$$R(G), R(H) \to R(G \times H)$$

given by the projections $G \times H \to G, H$, yield a ring homomorphism

$$R(G) \otimes R(H) \to R(G \times H). \tag{R.1}$$

We say that an H-representation V is *pure*, if $\mathrm{End}_F(V) = F$.

The following statement has been used in the proof of Lemma 4.7:

Proposition R.2. *The homomorphism (R.1) is an isomorphism provided that every simple H-representation is pure.*

Proof. The classes of simple representations form a basis of the abelian group $R(G \times H)$. Tensor products of the classes of simple representations form a basis of the abelian group $R(G) \otimes R(H)$. It follows from the statements below that the homomorphism (O.1) yields a bijection of the bases. □

Lemma R.3. *Assume that we are given simple G-representations U, U' and simple H-representations V, V'. If the $(G \times H)$-representations $U \otimes U'$ and $V \otimes V'$ are isomorphic, then $U \simeq U'$ and $V \simeq V'$.*

Proof. The G-representation $U \otimes U'$ (resp., $V \otimes V'$) is a direct sum of $\dim U'$ (resp., $\dim V'$) copies of U (resp., V). It follows that there is a non-zero G-equivariant map $U \to V$ and consequently, $U \simeq V$. Similarly, $U' \simeq V'$. □

Proposition R.4 ([16, Proposition 4.21])**.**

(1) *For any simple G-representation U and any pure simple H-representation V, the $(G \times H)$-representation $U \otimes V$ is simple.*

(2) *If every simple H-representation is pure, then every simple $(G\times H)$-representation is of the form $U\otimes V$, where U and V are simple G- and H-representations respectively.*

Proposition R.5 ([7, Proposition 2.8 of Part II]). *For split reductive H, every simple H-representation is pure.*

Appendix S. Symplectic group

The statement of Lemma S.1 below has already been proved in §5 (and used in the proof of Theorem 5.5). In this appendix we provide a more direct proof.

Let us fix some $n \geq 1$ and consider the symplectic group $G := \mathrm{Sp}(2n)$ over an arbitrary field. The representation ring $R(T)$ of the standard split maximal torus $T = \mathbb{G}_{\mathrm{m}}^n$ is identified with the Laurent polynomial ring $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ in variables $x_1, \dots, x_n$. The Weyl group W of G with respect to T is a semidirect product of the symmetric group S_n with a direct product Π of n copies of $\mathbb{Z}/2\mathbb{Z}$. The action of W on $R(T)$ is described as follows: the subgroup $S_n \subset W$ acts by permutations of the variables and the ith copy of $\mathbb{Z}/2\mathbb{Z}$ acts by exchanging x_i and x_i^{-1}. It follows that the ring $R(T)^{\Pi}$ of the elements invariant under the action of Π (this ring can be viewed as the representation ring of the product of n copies of $\mathrm{SL}(2)$, containing T and contained in G) is the polynomial ring $\mathbb{Z}[y_1, \dots, y_n]$ with $y_i := x_i + x_i^{-1}$. We prefer to view it as the polynomial ring $\mathbb{Z}[z_1, \dots, z_n]$ with $z_i := 2 - y_i$. The ring $R(T)^W$ of W-invariant elements is the subring of symmetric polynomials in $\mathbb{Z}[z_1, \dots, z_n]$. So, $R(G) = R(T)^W = \mathbb{Z}[\sigma_1, \dots, \sigma_n]$, where σ_i is the ith elementary symmetric polynomial in $z_1, \dots, z_n$.

Let us consider the Chern classes $c_1^R, \dots, c_{2n}^R \in R(G)$ of the standard representation. We claim that the ring $R(G)$ is generated by the even Chern classes $c_2^R, c_4^R, \dots, c_{2n}^R$. Indeed, the class in $R(G)$ of the standard representation of G equals $y_1 + \dots + y_n$. The total Chern class of $y_i \in R(T)^{\Pi}$ equals

$$\big(1 + (1 - x_i^{-1})t\big)\big(1 + (1 - x_i)t\big) = 1 + z_i t + z_i t^2 = 1 + z_i(t + t^2) \in R(T)^{\Pi}[t]$$

so that the total Chern class of $y_1 + \dots + y_n$ equals

$$1 + \sigma_1(t + t^2) + \sigma_2(t + t^2)^2 + \dots + \sigma_n(t + t^2)^n \in R(G)[t].$$

It follows that $c_{2n}^R = \sigma_n$. More generally, c_{2i}^R for any i equals σ_i plus a polynomial in higher sigmas so that a descending induction on i gives the claim.

Let us consider the filtration on $R(G)$ given by the even Chern classes $c_2^R, \dots, c_{2n}^R$: its ith term is generated by monomials in these Chern classes of degree $\geq i$, where the degree of c_i^R is i. This filtration is contained in the Chern filtration on $R(G)$ and therefore also in the Chow filtration on $R(G)$. Let C be the associated graded ring of this filtration. We recall that $\mathrm{CH}(\mathrm{B}G)$ is a polynomial ring in the even Chow Chern classes $c_2, c_4, \dots, c_{2n} \in \mathrm{CH}(\mathrm{B}G)$. The homomorphism $\mathrm{CH}(\mathrm{B}G) \to C$, mapping c_{2i} to the class of c_{2i}^R in C^{2i}, is surjective. Composing it with the homomorphism $C \to \mathrm{Chow}R(G)$, induced by inclusion of filtrations, we get the standard homomorphism $\mathrm{CH}(\mathrm{B}G) \to \mathrm{Chow}R(G)$ which is an isomorphism because $\mathrm{CH}(\mathrm{B}G)$ is torsion-free. It follows that $C \to \mathrm{Chow}R(G)$ (as well as $\mathrm{CH}(\mathrm{B}G) \to C$) is an isomorphism so that the filtration on $R(G)$, given by $c_2^R, \dots, c_{2n}^R$, coincides with the Chow filtration as well as with the Chern filtration. In particular, we proved

Lemma S.1. *For $G = \mathrm{Sp}(2n)$, the group* $\mathrm{Chern}R(G)$ *is torsion-free.* □

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