ADDITIVE OPERATIONS BETWEEN CONNECTIVE K-THEORY AND CHOW THEORY

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ABSTRACT. We determine all additive operations, stable and unstable, between connective K-theory and Chow theory modulo a prime integer p. It is proved the module of all stable operations is free of rank $p-1$ over the reduced Steenrod algebra.

1. INTRODUCTION

Let F be a field of characteristic 0 and write $\mathbf{Sm}_F$ for the category of smooth quasi-projective varieties over F. An *oriented cohomology theory* A^* over F is a functor from $\mathbf{Sm}_F^{op}$ to the category of $\mathbb{Z}$-graded commutative rings equipped with a push-forward structure and satisfying certain axioms (see [8, Definition 2.1]), including the localisation axiom. We write

$$A^*(X) = \coprod_{n\in\mathbb{Z}} A^n(X)$$

for a variety X in $\mathbf{Sm}_F$ and let $A^*(F)$ denote the *coefficient ring* $A^*(\operatorname{Spec} F)$.

Every oriented cohomology theory A^* admits a theory of Chern classes c_n^A of vector bundles.

The algebraic cobordism of Levine-Morel Ω^* is the universal oriented cohomology theory (see [4]). A *free theory* is an oriented cohomology theory obtained from Ω^* by change of coefficients (see [4, Remark 2.4.14] or [8, §4]).

Let A^* be an oriented cohomology theory. There is a (unique) associated formal group law

$$\mathrm{FGL}_A(x,y) = x + y + \sum_{i,j\geqslant 1} a_{i,j}^A x^i y^j \in A^*(F)[[x,y]]$$

that computes the first Chern class of the tensor product of two line bundles L and L' (see, for example, [4, §1.1] or [8, §2.3]):

$$c_1^A(L\otimes L') = F_A(c_1^A(L), c_1^A(L')).$$

Example 1.1. (see [3]) The *Chow theory* CH^* takes a smooth variety X to the Chow ring $\mathrm{CH}^*(X)$ of classes of algebraic cycles on X. The coefficient ring $\mathrm{CH}^*(F) = \mathbb{Z}$ is concentrated in degree 0 and $\mathrm{FGL}_{\mathrm{CH}}(x,y) = x+y$ is the *additive* group law.

Example 1.2. (see [1] and [2]) The *connective K-theory* takes a smooth variety X to the ring $\mathrm{CK}^*(X)$ of X (see [1] and [2]). The group $\mathrm{CK}^n(X)$ is defined as the image of the natural homomorphism $K_0(\mathcal{M}^n(X)) \to K_0(\mathcal{M}^{n-1}(X))$, where $\mathcal{M}^n(X)$ is the abelian category of coherent $\mathcal{O}_X$-modules whose support is of codimension at least n.

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The coefficient ring $\mathrm{CK}^*(F) = \mathbb{Z}[t]$ is the polynomial ring in the *Bott element* $t \in \mathrm{CK}^{-1}(F)$ and $\mathrm{FGL}_{\mathrm{CK}}(x,y) = x + y - txy$ is a *multiplicative* group law.

The two theories CH^* and CK^* are related by an exact sequence

$$\mathrm{CK}^{n+1}(X) \xrightarrow{t} \mathrm{CK}^n(X) \xrightarrow{c} \mathrm{CH}^n(X) \to 0,$$

where the first map is multiplication by t and c is the *class map* taking the class in $\mathrm{CK}^n(X)$ of an $\mathcal{O}_X$-module M from $\mathcal{M}^n(X)$ to its cycle class in $\mathrm{CH}^n(X)$. Thus, we have a canonical graded ring isomorphism

$$\mathrm{CH}^*(X) \xrightarrow{\sim} \mathrm{CK}^*(X)/t\,\mathrm{CK}^{*+1}(X).$$

The inclusion functor $\mathcal{M}^n(X) \hookrightarrow \mathcal{M}^0(X)$ yields a homomorphism $\mathrm{CK}^n(X) \to K_0(X)$. Its image is the subgroup $K_0(X)^{(n)} \subset K_0(X)$ generated by the classes of coherent $\mathcal{O}_X$-modules with codimension of support at least n. This map is an isomorphism if $n \leqslant 0$, so we can identify $\mathrm{CK}^n(X)$ with $K_0(X)$ for all $n \leqslant 0$. Moreover,

$$\mathrm{CK}^*(X) \otimes_{\mathbb{Z}[t]} \mathbb{Z}[t,t^{-1}] = \mathrm{CK}^*(X)[t^{-1}] \simeq K_0(X) \otimes_{\mathbb{Z}} \mathbb{Z}[t,t^{-1}].$$

The natural homomorphism $\mathbb{Z}[t] = \mathrm{CK}^*(F) \to \mathrm{CK}^*(X)$ yields a morphism

$$\varphi : \operatorname{Spec} \mathrm{CK}^*(X) \to \operatorname{Spec} \mathbb{Z}[t] = \mathbb{A}^1_{\mathbb{Z}}.$$

The fiber of φ over 0 is $\operatorname{Spec} \mathrm{CH}^*(X)$ and the fiber of φ over the complement $\mathbb{G}_m = \mathbb{A}^1_{\mathbb{Z}} - \{0\}$ is $\operatorname{Spec} K_0(X) \times \mathbb{G}_m$. Thus, we can view $\operatorname{Spec} \mathrm{CK}^*(X)$ as a "deformation space" deforming $K_0(X)$ to $\mathrm{CH}^*(X)$.

All cohomology theories in Examples 1.1 and 1.2 are free theories.

Definition 1.3. Let A^* and B^* be two oriented cohomology theories and m, n integers. An *additive operation* $A^m \to B^n$ is a morphism between the functors A^m and B^n considered as contravariant functors from $\mathbf{Sm}_F$ to the category of abelian groups (see [8, Definition 3.3]). All additive operations $A^m \to B^n$ form an abelian group $\mathbf{OP}^{m,n}(A^*, B^*)$.

Example 1.4. Multiplication by t yields an operation $\mathrm{CK}^{n+1} \to \mathrm{CK}^n$ that is an isomorphism if $n < 0$. The operation $c : \mathrm{CK}^n \to \mathrm{CH}^n$ is an example of a surjective additive operation.

Let p be a prime integer. The group of operations $\mathbf{OP}^{m,n}(\mathrm{CH}^*, \mathrm{CH}^* \otimes \mathbb{F}_p)$ was determined in [8, Theorem 6.6] (see Section 3). The $\mathbb{F}_p$-space $\mathbf{OP}^{m,n}(\mathrm{CH}^*, \mathrm{CH}^* \otimes \mathbb{F}_p)$ has a canonical basis $\{P^R\}$ of *Steenrod operations* indexed by all sequences $R = (r_0, r_1, \dots)$ of non-negative integers, almost all zero, such that

$$||R|| := \sum_{i \geqslant 0} r_i p^i = n \quad \text{and} \quad |R| := \sum_{i \geqslant 0} r_i = m.$$

Note that the $\mathbb{F}_p$-space $\mathbf{OP}^{m,n}(\mathrm{CH}^*, \mathrm{CH}^* \otimes \mathbb{F}_p)$ is nonzero if and only if $n \geqslant m \geqslant s_p(n)$, where $s_p(n)$ is the sum of digits of n written in base p, and $n - m$ is divisible by $p - 1$.

The operations P^R cannot be defined integrally, they cannot be even lifted modulo p^2.

In the present paper we determine the structure of the groups $\mathbf{OP}^{m,n}(\mathrm{CK}^*, \mathrm{CH}^* \otimes \mathbb{F}_p)$. The composition $Q : \mathrm{CK}^m \xrightarrow{c} \mathrm{CH}^m \xrightarrow{P} \mathrm{CH}^n$ for an operation P is an operation in

$\mathbf{OP}^{m,n}(\mathrm{CK}^*, \mathrm{CH}^* \otimes \mathbb{F}_p)$. But P is nonzero for special values of (m,n) (see above). Nevertheless, we show that the group $\mathbf{OP}^{m,n}(\mathrm{CK}^*, \mathrm{CH}^* \otimes \mathbb{F}_p)$ is nontrivial for all positive integers $n \geqslant m$.

Precisely, let m' be the smallest integer such that $m' \geqslant m$, $m' \geqslant s_p(n)$ and $n - m'$ is divisible by $p-1$. There are two maps

$$\mathbf{OP}^{m,n}(\mathrm{CK}^*, \mathrm{CH}^* \otimes \mathbb{F}_p) \to \mathbf{OP}^{m',n}(\mathrm{CK}^*, \mathrm{CH}^* \otimes \mathbb{F}_p) \leftarrow \mathbf{OP}^{m',n}(\mathrm{CH}^*, \mathrm{CH}^* \otimes \mathbb{F}_p), \tag{1.5}$$

where the first map is given via multiplication by $t^{m'-m}$ and the second map is induced by $c : \mathrm{CK}^* \to \mathrm{CH}^*$. We prove (see Theorem 5.2 and Corollary 5.3) that both maps in (1.5) are isomorphisms.

In other words, for every operation $Q \in \mathbf{OP}^{m,n}(\mathrm{CK}^*, \mathrm{CH}^* \otimes \mathbb{F}_p)$ there is a unique operation $P \in \mathbf{OP}^{m',n}(\mathrm{CH}^*, \mathrm{CH}^* \otimes \mathbb{F}_p)$ (and for every P there is a unique operation Q) such that the diagram

$$\begin{tikzcd} \mathrm{CK}^{m'} \arrow[r, "t^{m'-m}"] \arrow[d, "c"'] & \mathrm{CK}^m \arrow[d, "Q"] \\ \mathrm{CH}^{m'} \arrow[r, "P"] & \mathrm{CH}^n \otimes \mathbb{F}_p \end{tikzcd}$$

is commutative.

In Section 5 we also explicitly construct an $\mathbb{F}_p$-basis for $\mathbf{OP}^{m,n}(\mathrm{CK}^*, \mathrm{CH}^* \otimes \mathbb{F}_p)$ that corresponds to the Steenrod basis in $\mathbf{OP}^{m',n}(\mathrm{CH}^*, \mathrm{CH}^* \otimes \mathbb{F}_p)$ under the isomorphisms (1.5).

In topology an operation is stable if it commutes with the suspension isomorphism. The algebraic version of stability was defined, for example, in [8, §3.1]. By Example 6.6, the algebra $\mathcal{A}$ of stable operations $\mathrm{CH}^* \otimes \mathbb{F}_p \to \mathrm{CH}^* \otimes \mathbb{F}_p$ is the *reduced Steenrod algebra*, the factor algebra of the Steenrod algebra modulo the Bockstein operation (see [6]).

In Section 6 we determine the group $\mathbf{OP}^*_{\mathrm{st}}(\mathrm{CK}^*, \mathrm{CH}^* \otimes \mathbb{F}_p)$ of stable operations $\mathrm{CK}^* \to \mathrm{CH}^* \otimes \mathbb{F}_p$. We prove that it is a free left $\mathcal{A}$-module of rank $p-1$ and give an explicit basis.

In Section 7 we determine the group of integral operations $\mathbf{OP}^{m,n}(\mathrm{CK}^*, \mathrm{CH}^*)$. It appeared to be an infinite cyclic group with a canonical generator. We also determine the image of the natural map $\mathbf{OP}^{m,n}(\mathrm{CK}^*, \mathrm{CH}^*) \to \mathbf{OP}(\mathrm{CK}^m, \mathrm{CH}^n \otimes \mathbb{F}_p)$, i.e., those operations in $\mathbf{OP}(\mathrm{CK}^m, \mathrm{CH}^n \otimes \mathbb{F}_p)$ that can be lifted to an operation $\mathrm{CK}^m \to \mathrm{CH}^n$ over $\mathbb{Z}$. Recall that the Steenrod operations P^R cannot be lifted to integral operations (see Example 3.3).

In order to make the exposition clearer we moved the proofs of some technical statements to Appendix.

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2. Vishik's theorem

We will be using the following fundamental theorem due to A. Vishik.

Theorem 2.1. [8, Theorem 6.2] *Let A^* be a free cohomology theory and let B^* be any oriented cohomology theory over a field F of characteristic zero. Then there is an isomorphism between the group $\mathbf{OP}^{m,n}(A, B)$ of additive operations $G : A^m \to B^n$ and the* *group consisting of the following data* $\{g_s\}_{s\geqslant 0}$*:*

$$g_s \in \operatorname{Hom}\big(A^{m-s}(F), B^*(F)[[x_1, x_2, \dots, x_s]]_{(n)}\big),$$

where $B^*(F)[[x_1, x_2, \dots, x_s]]_{(n)}$ *denotes the subgroup of all homogeneous degree* n *power series, satisfying*
1) $g_s(\alpha)$ *is a symmetric power series for all* s *and* $\alpha \in A^{m-s}(F)$*,*
2) $g_s(\alpha)$ *is divisible by* $x_1x_2\cdots x_s$ *for all* s *and* α*,*
3) $g_s(\alpha)(y +_B z, x_2, \dots, x_s) = g_s(\alpha)(y, x_2, \dots, x_s) + g_s(\alpha)(z, x_2, \dots, x_s)+$

$$\sum_{i,j\geqslant 1} g_{i+j+s-1}(\alpha \cdot a^A_{i,j})(y^{\times i}, z^{\times j}, x_2, \dots, x_s),$$

where $a^A_{i,j}$ *are the coefficients of the formal group law of* A^* *and the sum* $y +_B z$ *is taken with respect to the formal group law of* B^**.* □

Note that $g_0 : A^m(F) \to B^n(F)$ is the operation G on $\operatorname{Spec} F$.

The power series $g_s(\alpha)$ are determined by the rule

$$G\big(\alpha \cdot c^A_1(L_1) \cdot \ldots \cdot c^A_1(L_s)\big) = g_s(\alpha)\big(c^B_1(L_1), \dots, c^B_1(L_s)\big),$$

where $L_1, \dots, L_s$ are line bundles over a smooth variety.

Consider the following cohomology theories: $A^* = \mathrm{CK}^*$ and $B^* = \mathrm{CH}^* \otimes S$, where S is a commutative ring ($\mathbb{Z}$ or $\mathbb{F}_p$ in the sequel). The formal group laws of CK^* and $\mathrm{CH}^* \otimes S$ are the multiplicative group law $x+y-txy$ and the additive group law $x+y$, respectively.

We have $\mathrm{CK}^*(F) = \mathbb{Z}[t]$, where $t \in \mathrm{CK}^{-1}(F)$ is the Bott element and $\mathrm{CH}^*(F) \otimes S = S$. Let $G : \mathrm{CK}^m \to \mathrm{CH}^n \otimes S$ be an additive operation. Let us assume that $m > 0$. Note that

$$\mathrm{CK}^{m-s}(F) = \begin{cases} 0, & \text{if } s < m; \\ \mathbb{Z}t^{s-m}, & \text{if } s \geqslant m. \end{cases}$$

By Vishik's theorem, G is given by a collection of homogeneous polynomials

$$g_s := g_s(t^{s-m}) \in S[x_1, x_2, \dots, x_s] \quad \text{for all} \quad s \geqslant m$$

of degree n such that
(1) g_s is a symmetric polynomial for all $s \geqslant m$,
(2) g_s is divisible by $x_1x_2\cdots x_s$ for all $s \geqslant m$,
(3) $g_{s+1} = -\partial(g_s)$ for $s \geqslant m$, where for a polynomial h in s variables we define its *derivative*

$$\partial(h)(y, z, x_2, \dots, x_s) := h(y + z, x_2, \dots, x_s) - h(y, x_2, \dots, x_s) - h(z, x_2, \dots, x_s)$$

in $S[y, z, x_2, \dots, x_s]$.

All polynomials g_s with $s > m$ are uniquely determined by g_m in view of (3).

Notation 2.2. Let S be a commutative ring and $n \geqslant m$ positive integers. We write $V_{m,n}(S)$ for the group of homogeneous symmetric polynomials f of degree n in $S[x_1, x_2, \dots, x_m]$ that are divisible by $x_1x_2\cdots x_m$ and such that $\partial^i(f)$ are symmetric for all $i > 0$.

We have proved the following statement:

Proposition 2.3. *Let* m *and* n *be positive integers.*

(1) *The map*
$$\mathbf{OP}(\mathrm{CK}^m, \mathrm{CH}^n \otimes S) \to V_{m,n}(S)$$
taking an operation $G = \{g_s\}_{s\geqslant 0}$ *to the polynomial* $(-1)^{m+n} g_m$ *is an isomorphism.*

(2) *The diagram*
$$\begin{tikzcd} \mathbf{OP}(\mathrm{CK}^m, \mathrm{CH}^n \otimes S) \arrow[r, "\sim"] \arrow[d, "t"'] & V_{m,n}(S) \arrow[d, "\partial"] \\ \mathbf{OP}(\mathrm{CK}^{m+1}, \mathrm{CH}^n \otimes S) \arrow[r, "\sim"] & V_{m+1,n}(S) \end{tikzcd}$$
is commutative.

Example 2.4. Clearly, $V_{m,n}(S) = 0$ if $n < m$ and $V_{m,m}(S) = S \cdot x_1 x_2 \cdots x_m$. Therefore, $\mathbf{OP}(\mathrm{CK}^m, \mathrm{CH}^n \otimes S) = 0$ if $n < m$. The polynomial $x_1 x_2 \cdots x_m$ in $V_{m,m}(S)$ corresponds to the canonical operation $\mathrm{CK}^m \xrightarrow{c} \mathrm{CH}^m \to \mathrm{CH}^m \otimes S$.

3. Steenrod operations

Notation 3.1. Let S be a commutative ring and $n \geqslant m$ positive integers. We write $W_{m,n}(S)$ for the group of homogeneous symmetric polynomials f of degree n in $S[x_1, x_2, \ldots, x_m]$ that are divisible by $x_1 x_2 \cdots x_m$ and such that $\partial(f) = 0$.

Clearly, $W_{m,n}(S) \subset V_{m,n}(S)$.

Recall that the coefficient ring $\mathrm{CH}^*(F) = \mathbb{Z}$ is concentrated in degree 0. By Theorem 2.1, an additive operation $G : \mathrm{CH}^m \to \mathrm{CH}^n \otimes S$ is given by a collection of polynomials $\{g_s\}_{s\geqslant 0}$, where $g_s = 0$ for $s \neq m$ and $g_m \in W_{m,n}(S)$. It follows that the assignment $G \mapsto (-1)^{m+n} g_m$ yields an isomorphism in the top row of the commutative diagram

$$\begin{tikzcd} \mathbf{OP}(\mathrm{CH}^m, \mathrm{CH}^n \otimes S) \arrow[r, "\sim"] \arrow[d, "c"'] & W_{m,n}(S) \arrow[d, hook] \\ \mathbf{OP}(\mathrm{CK}^m, \mathrm{CH}^n \otimes S) \arrow[r, "\sim"] & V_{m,n}(S). \end{tikzcd} \tag{3.2}$$

Example 3.3. The group $W_{m,n}(\mathbb{Z})$ coincides with $\mathbb{Z} \cdot x_1 x_2 \cdots x_m$ if $m = n$ and zero otherwise. Therefore, the only integral operations $\mathrm{CH}^m \to \mathrm{CH}^n$ are multiples of the identity when $m = n$.

Let p be a prime integer. We simply write $W_{m,n}$ for $W_{m,n}(\mathbb{F}_p)$.

Lemma 3.4. (see [8, Theorem 6.6]) Let f be a nonzero symmetric homogeneous polynomial of degree n in $\mathbb{F}_p[x_1, x_2, \ldots, x_m]$ that is divisible by $x_1 x_2 \cdots x_m$. Then $f \in W_{m,n}$ if and only if every variable x_i enters each monomial of f in degree a power of p.

Proof. Clearly, if every variable x_i enters each monomial of f in degree a power of p, we have $\partial(f) = 0$. Conversely, let $c x_1^{a_1} \cdots x_m^{a_m}$ be a nonzero monomial of f such that $\partial(f) = 0$. It suffices to show that all a_i are p-powers. Suppose a_i is not a p-power for some i. Since f is symmetric there is a monomial of f of the form $c x_1^{a_i} \cdots$. Since a_i is not a p-power, the derivative of this monomial is not zero. As the derivatives of distinct monomials don't have common monomials, we have $\partial(f) \neq 0$, a contradiction. $\square$

We can determine a basis of $W_{m,n}$ as follows.

Notation 3.5. Write $\mathcal{R}$ for the set of all nonzero sequences $R = (r_0, r_1, r_2, \dots)$ of non-negative integers, almost all zero. Clearly, the component-wise sum of two sequences in $\mathcal{R}$ also belongs to $\mathcal{R}$. Set

$$||R|| := \sum_{i\geqslant 0} r_i p^i \quad \text{and} \quad |R| := \sum_{i\geqslant 0} r_i.$$

Note that $||R|| - |R|$ is divisible by $p-1$ and $||R|| \geqslant |R| \geqslant s_p(||R||)$, where $s_p(n)$ is the sum of digits of n written in base p (see Lemma 4.3 below).

Notation 3.6. For every $R \in \mathcal{R}$ denote by f^R the "smallest" symmetric homogenous polynomial in the variables $x_1, \dots, x_m$ with $m = |R|$ containing the monomial

$$(x_1 x_2 \cdots x_{r_0})(x_{r_0+1} \cdots x_{r_0+r_1})^p (x_{r_0+r_1+1} \cdots x_{r_0+r_1+r_2})^{p^2} \cdots$$

of degree $n = ||R||$. The polynomial f^R is divisible by $x_1 x_2 \cdots x_m$.

Clearly, all polynomials f^R in $\mathbb{F}_p[x_1, x_2, \dots, x_m]$ with $||R|| = n$ and $|R| = m$ belong to $W_{m,n}$ and by Lemma 3.4 they form a basis of the $\mathbb{F}_p$-space $W_{m,n}$. Note that by Lemma A.5, $W_{m,n} \neq 0$ if and only if $n - m$ is divisible by $p-1$ and $n \geqslant m \geqslant s_p(n)$.

Notation 3.7. Let $n \geqslant m$ be positive integers and let $R \in \mathcal{R}$ be such that $||R|| = n$ and $|R| = m$. Write P^R for the operation $\mathrm{CH}^m \to \mathrm{CH}^n \otimes \mathbb{F}_p$ corresponding to f^R under the isomorphism in (3.2). Thus, all operations P^R form a basis for the space $\mathbf{OP}(\mathrm{CH}^m, \mathrm{CH}^n \otimes \mathbb{F}_p)$. Note that the operation P^R shifts the codimension by $n - m = \sum_{i\geqslant 0} r_i(p^i - 1)$.

If $m \geqslant i > 0$ and $R = (m-i, i, 0, 0, \dots)$ then the corresponding operation $\mathrm{CH}^m \to \mathrm{CH}^{m+i(p-1)}$ (denoted P^i) is known as the *reduced power operation.* It is equal to the operation $x \mapsto x^p$ if $m = i$.

4. The spaces $V_{m,n}$

Let p be a prime integer.

Definition 4.1. We say that a sequence $R = (r_0, r_1, \dots) \in \mathcal{R}$ is *small* if $r_i < p$ for all i. Clearly, R is small if and only if R is the sequence of p-adic digits of a positive integer n. We write $R =: R(n)$ for such a sequence. Note that $||R|| = n$ and $|R(n)| = s_p(n)$.

Definition 4.2. For a sequence $R \in \mathcal{R}$ we define the *level* of R:

$$l(R) := \frac{|R| - s_p(||R||)}{p-1}.$$

Lemma 4.3. *Let $R \in \mathcal{R}$. Then*
(1) *$l(R)$ is a non-negative integer;*
(2) *$l(R) = 0$ if and only if R is small.*

Proof. Clearly, $s_p(||R||) \equiv ||R|| \equiv |R|$ modulo $p-1$, so $l(R)$ is an integer. We have

$$|R| = \sum r_i \geqslant \sum s_p(r_i) = \sum s_p(r_i\, p^i) \geqslant s_p(\sum r_i\, p^i) = s_p(||R||),$$

hence $l(R) \geqslant 0$. All the inequalities are equalities if and only if $r_i < p$ for all i, i.e., when R is small. $\square$

Definition 4.4. Let m be a positive integer. An *m-sequence* is sequence $a = (a_1, a_2, \dots, a_m)$ of positive integers. Write

$$\operatorname{type}(a) := \sum_{j=1}^{m} R(a_j) \in \mathcal{R}$$

and call this sequence the *type* of a. In other words, if $a_j = \sum_{i\geqslant 0} x_{ij} \cdot p^i$ with $0 \leqslant x_{ij} < p$, then $\operatorname{type}(a) = (r_0, r_1, \dots)$ where $r_i = \sum_{j=1}^{m} x_{ij}$.

We have

$$||\operatorname{type}(a)|| = \sum_{j=1}^{m} ||R(a_j)|| = \sum_{j=1}^{m} a_j =: ||a|| \quad \text{and} \quad |\operatorname{type}(a)| = \sum_{j=1}^{m} |R(a_j)| = \sum_{j=1}^{m} s_p(a_j).$$

For an m-sequence a consider the (generalized) binomial coefficient

$$\langle a\rangle := \frac{||a||!}{a_1! \cdot a_2! \cdot \ldots \cdot a_m!} \in \mathbb{Z}.$$

Lemma 4.5. *Let a be an m-sequence of type R. Then*

(1) *$|R| \geqslant m$ and $|R| = m$ if and only if every a_j is a p-power.*

(2)
$$v_p\langle a\rangle = \frac{\sum_j s_p(a_j) - s_p(||a||)}{p-1} = l(R),$$
where v_p is the p-adic valuation.

Proof. (1): We have

$$|R| = \sum_j s_p(a_j) \geqslant m$$

since $s_p(a_j) \geqslant 1$ for all j. The equality $|R| = m$ holds if and only if $s_p(a_j) = 1$ for all j, i.e., every a_j is a p-power.

(2) Follows from the equality $v_p(n!) = \frac{n - s_p(n)}{p-1}$ for every positive integer n. □

We set $x^a := x_1^{a_1} \cdots\ x_m^{a_m}$ for an m-sequence a and variables $x_1, \dots, x_m$.

Since $\partial^{m-1}(x^n)$ is the sum of all monomials in $(x_1 + x_2 + \cdots + x_m)^n$ that are divisible by $x_1 x_2 \cdots x_m$, the Binomial Theorem yields the equality

$$\partial^{m-1}(x^n) = \sum \langle a\rangle \cdot x^a \quad \text{in} \quad \mathbb{Z}[x_1, x_2, \dots, x_m], \tag{4.6}$$

where the sum is taken over all m-sequences a such that $||a|| = n$.

Lemma 4.7. *For an m-sequence a every monomial of $\partial\big(\langle a\rangle \cdot x^a\big)$ in $\mathbb{Z}[x_1, x_2, \dots, x_{m+1}]$ is of the form $\langle b\rangle \cdot x^b$ for an $(m+1)$-sequence b such that either $v_p\langle b\rangle > v_p\langle a\rangle$ or* $\operatorname{type}(b) = \operatorname{type}(a)$.

Proof. Every monomial of $\partial\big(\langle a\rangle \cdot x^a\big)$ is of the form $e \cdot x^b$ for an $(m+1)$-sequence b, where $b = (b_1, b_2, \dots, b_{m+1})$ so that $a = (b_1 + b_2, b_3 \dots, b_{m+1})$, and

$$e = \binom{b_1 + b_2}{b_1} \cdot \langle a\rangle = \langle b\rangle.$$

If $\binom{b_1+b_2}{b_1}$ is divisible by p, then $v_p\langle b\rangle > v_p\langle a\rangle$. Otherwise $s_p(b_1 + b_2) = s_p(b_1) + s_p(b_2)$, hence $R(b_1 + b_2) = R(b_1) + R(b_2)$ and therefore, $\operatorname{type}(b) = \operatorname{type}(a)$. □

Lemma 4.8. *Suppose that a nonzero multiple of $x_1^{a_1}x_2^{a_2}\cdots x_k^{a_k}\cdots$ is a monomial of $\partial^{k-1}(f)$ for a polynomial $f \in \mathbb{F}_p[x_1,x_2,\dots,x_m]$. Then $\langle a\rangle$ is not zero in $\mathbb{F}_p$, where $a=(a_1,a_2,\dots,a_k)$.*

Proof. The polynomial f must contain a nonzero multiple of a monomial $x_1^n\cdots$, where $n=||a||$. Since the monomial $x_1^{a_1}x_2^{a_2}\cdots x_k^{a_k}\cdots$ appears in $\partial^{k-1}(x_1^n\cdots)$ with coefficient $\langle a\rangle$, we have $\langle a\rangle\neq 0$ in $\mathbb{F}_p$. $\square$

Corollary 4.9. *Let $n\geqslant m\geqslant s_p(n)$. Then there is no polynomial $f\in\mathbb{F}_p[x_1,x_2,\dots,x_m]$ such that $\partial^{p-1}(f)$ is a nonzero polynomial in $W_{m+p-1,n}$.*

Proof. Suppose $\partial^{p-1}(f)$ is a nonzero polynomial in $W_{m+p-1,n}$, i.e., $\partial^{p-1}(f)$ is a nonzero linear combination of f^R for all sequences R with $||R||=n$ and $|R|=m+p-1$. Every two such f^R have no common monomials (up to a scalar multiple), and every such R is not small since $|R|>m\geqslant s_p(n)=s_p(||R||)$ (see Lemma 4.3). Choose any such R and an $i\geqslant 0$ with $r_i\geqslant p$. The polynomial f^R and hence $\partial^{p-1}(f)$ contains a monomial of the form $(x_1x_2\dots x_p)^{p^i}\cdots$. This contradicts Lemma 4.8 since $\langle a\rangle$ is divisible by p for $a=(p^i,p^i,\dots,p^i)$ (p times). $\square$

Notation 4.10. For a real number t write $[\![t]\!]$ for the smallest non-negative integer that is $\geqslant t$. For positive integers $n\geqslant m$ set

$$l_{m,n}=[\![\frac{m-s_p(n)}{p-1}]\!]\geqslant 0.$$

Note that $l_{m,n}=0$ if and only if $m\leqslant s_p(n)$ and for $m>s_p(n)$ we have $l_{m+1,n}=l_{m,n}+1$ if and only if $n-m$ is divisible by $p-1$ and $l_{m+1,n}=l_{m,n}$ otherwise.

By Proposition A.1, $l_{m,n}$ is the minimum of $v_p\langle a\rangle$ over all m-sequences a such that $||a||=n$.

Proof. (1): In view of Lemma 4.5, $v_p\langle a\rangle=l(R)$ and the sequence R is small if and only if $l(R)=0$.

(2): We have $v_p\langle a\rangle=l(R)=\frac{|R|-s_p(n)}{p-1}\geqslant\frac{m-s_p(n)}{p-1}$, hence $v_p\langle a\rangle\geqslant l_{m,n}$. The integer $v_p\langle a\rangle$ is equal to $l_{m,n}=[\![\frac{m-s_p(n)}{p-1}]\!]$ if and only if $0\leqslant|R|-m<p-1$. $\square$

Definition 4.11. For all positive integers $n\geqslant m$ define the subset $\mathcal{R}_{m,n}\subset\mathcal{R}$ of sequences as follows.

(1) If $m\leqslant s_p(n)$, then $\mathcal{R}_{m,n}$ is the singleton $\{R(n)\}$ consisting of the only small sequence R with $||R||=n$. Note that $|R|=s_p(n)$.
(2) If $m>s_p(n)$, then $\mathcal{R}_{m,n}$ is the (finite) set of all sequences R such that $||R||=n$ and $0\leqslant|R|-m<p-1$. Note that $|R|$ is the only integer in the interval $[m,m+p-1)$ that is congruent to n modulo $p-1$.

Equivalently, $R\in\mathcal{R}_{m,n}$ if and only if $l(R)=l_{m,n}$.

By Lemma A.5, the set $\mathcal{R}_{m,n}$ is not empty. Note that $\mathcal{R}_{m,n}=\mathcal{R}_{m',n}$, where m' is the smallest integer such that $m'\geq m$, $m'\geqslant s_p(n)$ and $n-m'$ is divisible by $p-1$. In particular, $\mathcal{R}_{m,n}=\mathcal{R}_{m+1,n}$ if either $m<s_p(n)$ or $m>s_p(n)$ and $n-m$ is not divisible by $p-1$.

Lemma 4.12. *Let $n \geqslant m$ be positive integers and a an m-sequence with $||a|| = n$. Let $R = \operatorname{type}(a)$.*

(1) *If $m \leqslant s_p(n)$, then $l_{m,n} = 0$. Moreover, $v_p\langle a\rangle = 0$ if and only if R is small, i.e., $R = R(n) \in \mathcal{R}_{m,n}$.*
(2) *If $m > s_p(n)$, then $v_p\langle a\rangle \geqslant l_{m,n}$ and $v_p\langle a\rangle = l_{m,n}$ if and only if $0 \leqslant |R| - m < p-1$, i.e., $R \in \mathcal{R}_{m,n}$. In the latter case $|R| - m$ is the remainder on dividing $n - m$ by $p - 1$.*

Notation 4.13. For a positive integer n, set

$$s^!_p(n) := \prod_{i\geqslant 0} c_i!,$$

where c_i are p-adic digits of n. Note that $s^!_p(n)$ is not divisible by p.

Definition 4.14. Let $n \geqslant m$ be positive integers and $R \in \mathcal{R}_{m,n}$. Define the following polynomial in $\mathbb{F}_p[x_1, x_2, \dots, x_m]$:

$$q^R_{m,n} := \frac{1}{s^!_p(n)} \cdot \sum_{\operatorname{type}(a)=R} (-p)^{-l_{m,n}} \langle a\rangle \cdot x^a \quad \text{modulo} \quad p,$$

where the sum is taken over all m-sequences a such that $\operatorname{type}(a) = R$.

Clearly, $q^R_{m,n}$ is a symmetric polynomial.

Let $n \geqslant m$ be positive integers. Recall that $V_{m,n} := V_{m,n}(\mathbb{F}_p)$ is the vector space over $\mathbb{F}_p$ of homogeneous symmetric polynomials f of degree n in m variables $x_1, x_2, \dots, x_m$ that are divisible by $x_1x_2 \cdots x_m$ and such that $\partial^i(f)$ are symmetric for all $i > 0$, where

$$\partial : V_{*,n} \to V_{*+1,n}$$

is the derivation.

In the following theorem we determine structure of the $\mathbb{F}_p$-spaces $V_{m,n}$.

Theorem 4.15. *Let $n \geqslant m$ be positive integers and p a prime integer.*

(1) *If $m \leqslant s_p(n)$, then $V_{m,n}$ is a 1-dimensional vector space over $\mathbb{F}_p$ spanned by $q^R_{m,n}$ for $R = R(n)$;*
(2) *If $m \geqslant s_p(n)$, then the polynomials $q^R_{m,n}$ for $R \in \mathcal{R}_{m,n}$ form a basis for $V_{m,n}$;*
(3) *$V_{m,n} = W_{m,n}$ if $m \geqslant s_p(n)$ and $n - m$ is divisible by $p - 1$;*
(4) *The map $\partial : V_{m,n} \to V_{m+1,n}$ is an isomorphism if $m < s_p(n)$ or if $m > s_p(n)$ and $n - m$ is not divisible by $p - 1$;*
(5) *The map $\partial : V_{m,n} \to V_{m+1,n}$ is zero if $m \geqslant s_p(n)$ and $n - m$ is divisible by $p - 1$;*
(6) *If $R \in \mathcal{R}_{m,n}$ is such that $|R| = m$, then $q^R_{m,n} = f^R$;*
(7) *The polynomial $\frac{1}{s^!_p(n)} \cdot (-p)^{-l_{m,n}} \cdot \partial^{m-1}(x^n)$ modulo p is equal to $\sum q^R_{m,n} \in \mathbb{F}_p[x_1, x_2, \dots, x_m]$, where the sum in taken over all sequences $R \in \mathcal{R}_{m,n}$;*
(8) *If $R \in \mathcal{R}_{m,n}$, then*

$$\partial(q^R_{m,n}) = \begin{cases} q^R_{m+1,n}, & \text{if } m < |R|; \\ 0, & \text{if } m = |R|. \end{cases}$$

Proof. (3): Let $g \in V_{m,n}$. In view of Notation 3.1, it suffices to show that $\partial(g) = 0$. Suppose $\partial(g) \neq 0$ and choose an $i > 0$ such that $\partial^i(g) \neq 0$ but $\partial^{i+1}(g) = 0$. Then $\partial^i(g)$ is a nonzero polynomial in $W_{m+i,n}$. It follows that $n - m - i$ and hence i are divisible by $p-1$, so $i \geqslant p-1$. This contradicts Corollary 4.9 applied to $f = \partial^{i-p+1}(g)$.

(5) Follows from (3) since $W_{m,n} \subset \operatorname{Ker}(\partial)$.

(6): By Lemma 4.5(1), if a is an m-sequence, $R = \operatorname{type}(a)$ and $|R| = m$, then every a_i is a p-power. It follows that $q^R_{m,n}$ is a multiple of f^R. The multiplicity is determined in Proposition A.3.

(7) Follows from the formula (4.6), Lemma 4.12 and Definition 4.14.

(8): If $m < |R|$ then either $m < s_p(n)$ or $m > s_p(n)$ and $n - m$ is not divisible by $p-1$. It follows that $l_{m,n} = l_{m+1,n}$ and $\mathcal{R}_{m,n} = \mathcal{R}_{m+1,n}$. By Lemma 4.7, the type of every monomial in $\partial(q^R_{m,n})$ is equal to R. Therefore, the equality $\partial(q^R_{m,n}) = q^R_{m+1,n}$ follows from (7). If $m = |R|$, the statement follows from (6).

(2) and (4): Note first that it follows from (8) that all higher derivatives of $q^R_{m,n}$ are symmetric, hence $q^R_{m,n} \in V_{m,n}$.

The kernel of $\partial : V_{*,n} \to V_{*+1,n}$ is spanned by f^R that are polynomials in m variables with $m \geqslant s_p(n)$ and $n - m$ divisible by $p-1$. Hence the maps $\partial : V_{m,n} \to V_{m+1,n}$ are injective if $m > s_p(n)$ and $n-m$ is not divisible by $p-1$. To prove surjectivity in (4), let m' be the smallest integer such that $m' > m$ and $n - m'$ is divisible by $p-1$. It suffices to show that the composition

$$V_{m,n} \xrightarrow{\partial} V_{m+1,n} \xrightarrow{\partial} \cdots \xrightarrow{\partial} V_{m',n}$$

is surjective. By (3), $V_{m',n} = W_{m',n}$ has basis $\{f^R\}$ with $||R|| = n$ and $|R| = m'$. In view of (6) and (8), the image of the polynomial $q^R_{m,n}$ from $V_{m,n}$ under the composition is equal to f^R. This proves surjectivity of the composition and (2).

(1): As in the proof of (2) we see that $q^R_{m,n} \in V_{m,n}$. The rest follows from (4), (6) and (8) since $V_{s_p(n),n} = W_{s_p(n),n}$ is a 1-dimensional vector space spanned by f^R with $R = R(n)$. □

Remark 4.16. According to Theorem 4.15, one can construct a basis for $V_{m,n}$ as follows. Let m' be the smallest integer such that $m' \geqslant m$, $m' \geqslant s_p(n)$ and $n - m'$ is divisible by $p-1$. Then the polynomials $q^R_{m,n}$ for all sequences R with $||R|| = n$ and $|R| = m'$ form a basis for $V_{m,n}$. By Lemma A.5, such sequences R exist, hence the space $V_{m,n}$ is nonzero for every positive integers $n \geqslant m$.

5. Operations $\mathrm{CK}^* \to \mathrm{CH}^* \otimes \mathbb{F}_p$

Let n and m be two integers and p a prime integer. In this Section we compute the group of additive operations

$$\mathbf{OP}^{m,n} := \mathbf{OP}(\mathrm{CK}^m, \mathrm{CH}^n \otimes \mathbb{F}_p).$$

Recall that $\mathbf{OP}^{m,n}$ is isomorphic to $V_{m,n}$ if m and n are positive in view of Proposition 2.3.

The functors CK^m are canonically isomorphic to K_0 for all $m \leqslant 0$, so $\mathbf{OP}^{m,n} = \mathbf{OP}^{0,n}$ if $m \leqslant 0$.

Let $A_n : K_0 \to \mathrm{CH}^n$ be the n-th *additive* Chern class. It is uniquely determined by the condition $A_n([L]) = c_1^{\mathrm{CH}}(L)^n$ for a line bundle L.

We can also view A_n as an operation in $\mathbf{OP}^{0,n}$ between $\mathrm{CK}^0 = K_0$ and $\mathrm{CH}^n \otimes \mathbb{F}_p$. According to [7, §2.2], $\mathbf{OP}^{m,n} = \mathbf{OP}^{0,n}$ is a cyclic group generated by A_n if $m \leqslant 0$.

Notation 5.1. Let $n \geqslant m$ be positive integers. For a sequence $R \in \mathcal{R}_{m,n}$, write $Q^R_{m,n}$ for the operation in $\mathbf{OP}^{m,n}$ corresponding to the polynomial $q^R_{m,n}$ by Proposition 2.3.

Theorem 5.2. *Let $n \geqslant m$ be positive integers and p a prime integer.*

1. *If $m \leqslant s_p(n)$, then $\mathbf{OP}^{m,n}$ is a 1-dimensional vector space over $\mathbb{F}_p$ spanned by $Q^R_{m,n}$ for $R = R(n)$;*
2. *If $m \geqslant s_p(n)$, then the operations $Q^R_{m,n}$ for $R \in \mathcal{R}_{m,n}$ form a basis for $\mathbf{OP}^{m,n}$;*
3. *For a sequence R with $||R|| = n$ and $|R| = m$ the operation $Q^R_{m,n}$ coincides with the composition*
$$\mathrm{CK}^m \xrightarrow{c} \mathrm{CH}^m \xrightarrow{P^R} \mathrm{CH}^n \otimes \mathbb{F}_p.$$
The canonical map $\mathbf{OP}(\mathrm{CH}^m, \mathrm{CH}^n \otimes \mathbb{F}_p) \xrightarrow{c} \mathbf{OP}^{m,n}$ is an isomorphism.
4. *If $R \in \mathcal{R}_{m,n}$, then*
$$[\mathrm{CK}^{m+1} \xrightarrow{t} \mathrm{CK}^m \xrightarrow{Q^R_{m,n}} \mathrm{CH}^n \otimes \mathbb{F}_p] = \begin{cases} Q^R_{m+1,n}, & \text{if } m < |R|; \\ 0, & \text{if } m = |R|. \end{cases}$$
5. *If R is small with $||R|| = n$ and $m \leqslant s_p(n)$, then the composition*
$$\mathrm{CK}^m \xrightarrow{t^m} K_0 \xrightarrow{A_n} \mathrm{CH}^n \to \mathrm{CH}^n \otimes \mathbb{F}_p$$
coincides with $s^!_p(n) \cdot Q^R_{m,n}$.

Proof. The statements (1)-(4) follow from Theorem 4.15.

(5): It follows from (4) that it suffices to consider the case $m = 1$. Let L be a line bundle over a smooth base X. Then the composition
$$\mathrm{CK}^1(X) \xrightarrow{t} K_0(X) \xrightarrow{A_n} \mathrm{CH}^n(X) \otimes \mathbb{F}_p$$
takes $c_1^{\mathrm{CK}}(L)$ to $A_n(1 - L^\vee) = -c_1^{\mathrm{CH}}(L^\vee)^n = (-1)^{n+1} c_1^{\mathrm{CH}}(L)^n$, i.e., the composition is given by the polynomial $(-1)^{n+1}x^n$ in view of Proposition 2.3. Recall that the operation $Q^R_{1,n}$ is given by the polynomial $\frac{1}{s^!_p(n)}(-1)^{n+1}x^n$. □

Corollary 5.3. *Let n be a positive integer.*

1. *The natural maps*
$$\mathbb{F}_p \cdot A_n = \mathbf{OP}^{0,n} \to \mathbf{OP}^{1,n} \to \mathbf{OP}^{2,n} \to \cdots \to \mathbf{OP}^{s_p(n),n} = \mathbb{F}_p \cdot Q^R_{s_p(n),n},$$
where $R = R(n)$, are isomorphisms of 1-dimension vector spaces. The image of A_n in $\mathbf{OP}^{s_p(n),n}$ is equal to $s^!_p(n) \cdot Q^R_{s_p(n),n} = s^!_p(n) \cdot (P^R \circ c)$.
2. *If m is an integer such that $n \geqslant m > s_p(n)$ and $m \equiv n$ modulo $p - 1$, then the natural maps*
$$\mathbf{OP}^{m-p+2,n} \to \mathbf{OP}^{m-p+3,n} \to \cdots \to \mathbf{OP}^{m,n}$$
are isomorphisms. The set $\{P^R \circ c\}_{R \in \mathcal{R}_{m,n}}$ is a basis for $\mathbf{OP}^{m,n}$.

(3) *Let $n \geqslant m$ be positive integers and let m' be the smallest integer such that $m' \geqslant m$, $m' \geqslant s_p(n)$ and $n - m'$ is divisible by $p-1$. Then the maps*

$$\mathbf{OP}^{m,n} \xrightarrow{t^{m'-m}} \mathbf{OP}^{m',n} \xleftarrow{c} \mathbf{OP}^{m',n}(\mathrm{CH}^*, \mathrm{CH}^* \otimes \mathbb{F}_p),$$

are isomorphisms. The operation $Q^R_{m,n}$ corresponds to the Steenrod operation $P^R_{m',n}$ under these isomorphisms. □

It follows from the corollary that the groups $\mathbf{OP}^{m,n}$ are nonzero for all positive integers $n \geqslant m$.

Remark 5.4. By Theorem 5.2, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{CK}^{m+1} \arrow[r] \arrow[d] & \mathrm{CK}^m \arrow[r, two heads] \arrow[d] & \mathrm{CH}^m \arrow[d, "s_p^!(n)\cdot P^R"] \\
K_0^{(m+1)} \arrow[r, hook] & K_0^{(m)} \arrow[r, "A_n"] & \mathrm{CH}^n \otimes \mathbb{F}_p,
\end{tikzcd}$$

where $R = R(n)$ and $m = |R| = s_p(n)$. As $s_p^!(n)$ is invertible modulo p, it follows that the Steenrod operation $Q^R = P^R \circ c$ factors through $K_0^{(m/m+1)}$.

6. Stable operations $\mathrm{CK}^* \to \mathrm{CH}^* \otimes \mathbb{F}_p$

For any polynomial $f \in \mathbb{F}_p[x_1, x_2, \dots, x_m]$ divisible by $x_1 x_2 \cdots x_m$ define

$$\varphi(f) := (x_m^{-1} \cdot f)|_{x_m=0} \in \mathbb{F}_p[x_1, x_2, \dots, x_{m-1}].$$

Example 6.1. Let $R = (r_0, r_1, \dots) \in \mathcal{R}$. We have

$$\varphi(f^R) = \begin{cases} 0, & \text{if } r_0 = 0; \\ f^{R-\mathbb{1}}, & \text{if } r_0 > 0. \end{cases}$$

where $\mathbb{1} := (1, 0, 0, \dots)$.

Note that if $m \geqslant 2$, then φ and ∂ commute: $\varphi \circ \partial = \partial \circ \varphi$. It follows that φ yields a homomorphism

$$\varphi : V_{m,n} \to V_{m-1,n-1}.$$

Proposition 6.2. *Let $n \geqslant m > 1$ be integers, $R \in \mathcal{R}_{m,n}$. Then*

$$\varphi(q^R_{m,n}) = \begin{cases} 0, & \text{if } r_0 = 0; \\ q^{R-\mathbb{1}}_{m-1,n-1}, & \text{if } r_0 > 0. \end{cases}$$

Proof. Let $r_0 = 0$ and let a be an m-sequence with $\mathrm{type}(a) = R$. We have $a_i \neq 1$ for all i, hence $\varphi(x^a) = 0$ and therefore, $\varphi(q^R_{m,n}) = 0$.

Now assume that $r_0 > 0$. Note that $R - \mathbb{1} \in \mathcal{R}_{m-1,n-1}$. Let m' be the smallest integer such that $m' \geqslant m$, $m' \geqslant s_p(n)$ and $m' - n$ is divisible by $p-1$. By Theorem 4.15 and Example 6.1,

$$\partial^i(\varphi(q^R_{m,n})) = \varphi(\partial^i(q^R_{m,n})) = \varphi(f^R) = f^{R-\mathbb{1}} = q^{R-\mathbb{1}}_{m'-1,n-1} = \partial^i(q^{R-\mathbb{1}}_{m-1,n-1}),$$

where $i = m' - m$. By Theorem 4.15 again, $\partial^i : V_{m,n} \to V_{m',n}$ is an isomorphism, hence $\varphi(q^R_{m,n}) = q^{R-\mathbb{1}}_{m-1,n-1}$. □

Definition 6.3. A *stable operation* $A^* \to B^{*+d}$ of (relative) degree d is a collection of operations $G^{(i)} : A^i \to B^{i+d}$ for $i \geqslant 0$ such that $G^{(i)} = \Sigma^{-1}(G^{(i+1)})$, where Σ^{-1} is the desuspension map (see [8, §3.1]). Write $\mathbf{OP}^d_{\mathrm{st}}(A^*, B^*)$ for the group of all stable operation $A^* \to B^{*+d}$ of degree d. In other words,

$$\mathbf{OP}^d_{\mathrm{st}}(A^*, B^*) = \lim_i \mathbf{OP}(A^i, B^{i+d}),$$

where the limit is taken with respect to Σ^{-1} as transition maps.

If $A^* = B^*$ the composition of stable operations makes $\mathbf{OP}^*_{\mathrm{st}}(A^*) := \mathbf{OP}^*_{\mathrm{st}}(A^*, A^*)$ a graded ring.

Proposition 6.4. *Let $G : A^m \to B^n$ be an additive operation given by $\{g_s\}_{s\geqslant 0}$ in view of Theorem 2.1. Let $\{(\Sigma^{-1}g)_s\}_{s\geqslant 0}$ be the data for $\Sigma^{-1}G$. Then*

$$(\Sigma^{-1}g)_s(\alpha) = \varphi(g_{s+1}(\alpha)) \quad \text{in} \quad B^*(F)[[x_1, x_2, \ldots, x_s]]_{(n)}$$

for every $s \geqslant 0$ and $\alpha \in A^{m-s}(F)$.

Proof. Note that if R is a commutative ring and $\varepsilon \in R$ satisfies $\varepsilon^2 = 0$, then for every polynomial $f \in R[x_1, x_2, \ldots, x_{m+1}]$ divisible by $x_1 x_2 \cdots x_{m+1}$ we have

$$f(x_1, \ldots, x_m, \varepsilon) = \varphi(f)(x_1, \ldots, x_m)\varepsilon \in R[x_1, x_2, \ldots, x_m].$$

We apply this to $\varepsilon = c_1^B(L_{can}) \in B^1(\mathbb{P}^1_F)$. Let $\alpha \in A^{m-s}(F)$ for some $s \geqslant 0$ and let $L_i \to X$ for $i = 1, \ldots s$ be line bundles over a smooth base X. It follows from the definition of Σ^{-1} (see [8, §3.1]) that the following equalities hold in $B^s(X \times \mathbb{P}^1_F)$:

$$\begin{aligned}(\Sigma^{-1}g)_s(\alpha)(c_1^A(L_1), \ldots, c_1^A(L_s))\varepsilon &= (\Sigma^{-1}G)(\alpha c_1^A(L_1) \cdots c_1^A(L_s))\varepsilon \\ &= G(\alpha c_1^A(L_1) \cdots c_1^A(L_s) c_1^A(L_{can})) \\ &= g_{s+1}(\alpha)(c_1^B(L_1), \ldots, c_1^B(L_s), \varepsilon) \\ &= \varphi(g_{s+1}(\alpha))(c_1^B(L_1), \ldots, c_1^B(L_s))\varepsilon. \end{aligned}$$ □

Notation 6.5. Let $\mathcal{S}$ be the set of sequences $(s_1, s_2, \ldots)$ of nonnegative integers, almost all zero. Write $\alpha : \mathcal{R} \to \mathcal{S}$ for the (forgetful) map taking $R = (r_0, r_1, r_2, \ldots)$ to $\alpha(R) := (r_1, r_2, \ldots)$. We also write

$$||S|| := \sum_{i\geqslant 1} s_i(p^i - 1).$$

Example 6.6. (see [8, Theorem 6.6]) For every $S \in \mathcal{S}$ let P^S be the stable operation of degree $||S||$ given by the sequence f^R over all $R \in \mathcal{R}$ such that $\alpha(R) = S$ (see Example 6.1). It follows from Lemma 3.4 and Example 6.1 that the stable operations P^S for all $S \in \mathcal{S}$ form a basis of the space $\mathcal{A} := \mathbf{OP}^*_{\mathrm{st}}(\mathrm{CH}^* \otimes \mathbb{F}_p)$. In fact, $\mathcal{A}$ is the *reduced Steenrod algebra*, the factor algebra of the Steenrod algebra modulo the Bockstein operation (see [6]).

For $j \geqslant p - 2$ we have $j \cdot \mathbb{1} \in \mathcal{R}_{j-p+2,j}$. It follows from Propositions 6.2 and 6.4 that the sequence of operations

$$Q^{j\cdot\mathbb{1}}_{j-p+2,j} : \mathrm{CK}^{j-p+2} \to \mathrm{CH}^j \otimes \mathbb{F}_p$$

yields a stable operation $L : \mathrm{CK}^* \to \mathrm{CH}^{*+p-2} \otimes \mathbb{F}_p$ of degree $p - 2$.

The collections of canonical operations $\mathrm{CK}^i \xrightarrow{c} \mathrm{CH}^i \to \mathrm{CH}^i \otimes \mathbb{F}_p$ and $\mathrm{CK}^i \xrightarrow{t} \mathrm{CK}^{i-1}$ yield stable operations $I : \mathrm{CK}^* \to \mathrm{CH}^* \to \mathrm{CH}^* \otimes \mathbb{F}_p$ and $J : \mathrm{CK}^* \to \mathrm{CK}^{*-1}$ of degree 0 and -1, respectively.

Note that the group $\mathbf{OP}^*_{\mathrm{st}}(\mathrm{CK}^*, \mathrm{CH}^* \otimes \mathbb{F}_p)$ has the structure of a left module over the reduced Steenrod algebra $\mathcal{A}$ (see Example 6.6).

For any positive integers $n \geqslant m$, every $R = (r_0, r_1, \ldots) \in \mathcal{R}_{m+i,n+i}$ satisfies $r_0 > 0$ for sufficiently large i. It follows that the map $\mathcal{R}_{m+i,n+i} \to \mathcal{R}_{m+i+1,n+i+1}$ taking R to $R + \mathbb{1}$ is a bijection for $i \gg 0$. By Propositions 6.2 and 6.4 together with Theorem 5.2(2), the map

$$\Sigma^{-1} : \mathbf{OP}^{m+i+1,n+i+1} \to \mathbf{OP}^{m+i,n+i}$$

is an isomorphism for $i \gg 0$. Therefore, the natural map

$$\mathbf{OP}^{n-m}_{\mathrm{st}}(\mathrm{CK}^*, \mathrm{CH}^* \otimes \mathbb{F}_p) \to \mathbf{OP}^{m+i,n+i}$$

is an isomorphism for $i \gg 0$. In other words, stable operations of relative degree $d = n-m$ can be viewed as unstable operations in $\mathbf{OP}^{m,n}$ for large n and m.

Theorem 6.7. *Let p be a prime integer.*

(1) *The identities $L \circ J^{p-2} = I$ and $L \circ J^{p-1} = 0$ hold in $\mathbf{OP}^*_{\mathrm{st}}(\mathrm{CK}^*, \mathrm{CH}^* \otimes \mathbb{F}_p)$.*
(2) *Stable operations $L \circ J^i$ for $i = 0, 1, \ldots, p-2$ form a basis of the free left $\mathcal{A}$-module $\mathbf{OP}^*_{\mathrm{st}}(\mathrm{CK}^*, \mathrm{CH}^* \otimes \mathbb{F}_p)$.*

Proof. (1): In view of Theorem 4.15, $\partial^{p-2}(q^{j\cdot\mathbb{1}}_{j-p+2,j}) = q^{j\cdot\mathbb{1}}_{j,j} = x_1 x_2 \cdots x_j$ for every $j > p-2$. By Example 2.4, the latter polynomial corresponds to the canonical operation $\mathrm{CK}^j \xrightarrow{c} \mathrm{CH}^j \to \mathrm{CH}^j \otimes \mathbb{F}_p$. It follows that $L \circ J^{p-2} = I$. Since $\partial(q^{j\cdot\mathbb{1}}_{j,j}) = 0$, we have $L \circ J^{p-1} = 0$.

(2): Let d be a non-negative integer and s is the remainder on dividing d by $p-1$. Consider the composition

$$\mathcal{A}^{d-s} \to \mathbf{OP}^d_{\mathrm{st}}(\mathrm{CK}^*, \mathrm{CH}^* \otimes \mathbb{F}_p) \to \mathbf{OP}^{d-s}_{\mathrm{st}}(\mathrm{CK}^*, \mathrm{CH}^* \otimes \mathbb{F}_p), \tag{6.8}$$

where the first map is the composition with $L \circ J^{p-2-s} : \mathrm{CK}^* \to \mathrm{CH}^{*+s} \otimes \mathbb{F}_p$ and the second map is the composition with J^s. The second map in (6.8) is an isomorphism by Corollary 5.3.

The composition in (6.8) is given by the composition with $L \circ J^{p-2-s} \circ J^s = L \circ J^{p-2} = I$ and hence is an isomorphism by Theorem 5.2(3). It follows that the first map in (6.8) is an isomorphism, whence the result. □

7. Integral operations $\mathrm{CK}^* \to \mathrm{CH}^*$

Notation 7.1. For a pair of integers $n \geqslant m$ set

$$c_{m,n} := \prod_p p^{\llbracket \frac{m - s_p(n)}{p-1} \rrbracket},$$

where p runs over all prime integers and if $n \geqslant m+1$ set

$$d_{m,n} := \frac{c_{m+1,n}}{c_{m,n}} \in \mathbb{Z}.$$

The integer $d_{m,n}$ is the product of all primes p such that $m \geqslant s_p(n)$ and $n-m$ is divisible by $p-1$.

According to Proposition A.1
$$\gcd\langle a\rangle = c_{m,n},$$
where the gcd is taken over all m-sequences a such that $||a|| = n$.

Consider the polynomials
$$g_{m,n} = \frac{1}{c_{m,n}} \cdot \sum \langle a\rangle \cdot x^a \in \mathbb{Z}[x_1, x_2, \dots, x_m],$$
where the sum is taken over all m-sequences a such that $||a|| = n$. The coefficients of $g_{m,n}$ are relatively prime.

Proposition 7.2. *The polynomial $g_{m,n}$ is a generator of the infinite cyclic group $V_{m,n}(\mathbb{Z})$.*

Proof. Let $f \in V_{m,n}(\mathbb{Z})$. By [5, Proposition 2.8], there is an $h \in \mathbb{Q}[x]$ such that $f = \partial^{m-1}(h)$. Clearly, h is a multiple of the monomial x^n. By the formula (4.6),
$$\partial^{m-1}(x^n) = c_{m,n} \cdot g_{m,n}.$$
Therefore, f is an integral multiple of $g_{m,n}$. □

Corollary 7.3. *The image of the generator $g_{m,n}$ under the map $\partial : V_{m,n}(\mathbb{Z}) \to V_{m+1,n}(\mathbb{Z})$ is equal to $d_{m,n} \cdot g_{m+1,n}$.*

Proof. We have
$$\partial(g_{m,n}) = c_{m,n}^{-1} \cdot \partial^m(x^n) = c_{m,n}^{-1} \cdot c_{m+1,n} \cdot g_{m+1,n} = d_{m,n} \cdot g_{m+1,n}$$ □

Let p be a prime integer. Reducing modulo p we get a homomorphism
$$V_{m,n}(\mathbb{Z}) \to V_{m,n}(\mathbb{F}_p) = V_{m,n}.$$
In view of Theorem 4.15(7) it follows from the definition of the polynomials $g_{m,n}$ and $q^R_{m,n}$ that the image of the generator $g_{m,n}$ is equal to a nonzero multiple of the sum $\sum q^R_{m,n}$ over all $R \in \mathcal{R}_{m,n}$. Therefore, $\sum q^R_{m,n}$ lifts to a polynomial in $V_{m,n}(\mathbb{Z})$. In particular, by Theorem 4.15(6), for every positive integers $n \geqslant m$ such that $m \geqslant s_p(n)$ and $n-m$ is divisible by $p-1$, the sum $\sum f^R$ over all R with $||R|| = n$ and $|R| = m$ lifts to an integral polynomial in $V_{m,n}(\mathbb{Z})$. We have proved the following theorem.

Theorem 7.4. *Let $n \geqslant m$ be positive integers and p a prime.*

1. *The group $\mathbf{OP}(\mathrm{CK}^m, \mathrm{CH}^n)$ is infinite cyclic with a canonical generator $G_{m,n}$. The image of $G_{m,n}$ under the map $\mathbf{OP}(\mathrm{CK}^m, \mathrm{CH}^n) \to \mathbf{OP}(\mathrm{CK}^{m+1}, \mathrm{CH}^n)$ is equal to $d_{m,n} \cdot G_{m+1,n}$.*
2. *An operation $Q \in \mathbf{OP}(\mathrm{CK}^m, \mathrm{CH}^n \otimes \mathbb{F}_p)$ lifts to an operation in $\mathbf{OP}(\mathrm{CK}^m, \mathrm{CH}^n)$ if and only if Q is a multiple of $\sum_{R \in \mathcal{R}_{m,n}} Q^R_{m,n}$.*
3. *If $m \geqslant s_p(n)$ and $n-m$ is divisible by $p-1$, an operation $P \in \mathbf{OP}(\mathrm{CH}^m, \mathrm{CH}^n \otimes \mathbb{F}_p)$ lifts to an operation in $\mathbf{OP}(\mathrm{CK}^m, \mathrm{CH}^n)$ if and only if P is a multiple of $\sum_{R \in \mathcal{R}_{m,n}} P^R_{m,n}$.*

It follows from Theorem 7.4 that the operation $Q^R_{m,n}$ in $\mathbf{OP}(\mathrm{CK}^m, \mathrm{CH}^n \otimes \mathbb{F}_p)$ lifts to an integral operation in $\mathbf{OP}(\mathrm{CK}^m, \mathrm{CH}^n)$ if and only if $\mathcal{R}_{m,n}$ is a singleton.

Definition 7.5. Let $R = (r_0, r_1, \dots) \in \mathcal{R}$. We say that R is *quasi-small* if at least one of the following holds:

(1) R is small;
(2) $r_i = 0$ for all $i \geqslant 2$ and $r_1 \leqslant p$;
(3) $r_0 < p$ and there is an $s \geqslant 1$ such that $r_s = p$, $r_{s+1} < p$ and $r_i = 0$ for all $i = 1, \dots, s-1$ and $i \geqslant s+2$.

Theorem 7.6. *The operation $Q^R_{m,n} : \mathrm{CK}^m \to \mathrm{CH}^n \otimes \mathbb{F}_p$, where $R \in \mathcal{R}_{m,n}$ lifts to an integral operation $\mathrm{CK}^m \to \mathrm{CH}^n$ if and only if R is quasi-small. In particular, the reduced power operation $P^i : \mathrm{CH}^{m+i} \to \mathrm{CH}^{m+ip} \otimes \mathbb{F}_p$ with $m \geqslant 0$ lifts to an integral operation $\mathrm{CK}^{m+i} \to \mathrm{CH}^{m+ip}$ if and only if $i \leqslant p$.*

Proof. The first statement follows from Theorem 7.4 and Proposition A.6. We have $P^i = P^R$ with $R = (m, i, 0, 0 \dots)$ and R is quasi-small if and only if $i \leqslant p$. $\square$

We show that the reduced power operation Q^i for large i cannot be even lifted modulo p^2.

Proposition 7.7. *The reduced power operation $Q^i = P^i \circ c : \mathrm{CK}^{m+i} \to \mathrm{CH}^{m+ip} \otimes \mathbb{F}_p$ with $m \geqslant 0$ cannot be lifted to an operation modulo p^2 if $i \geqslant p+1$.*

Proof. The polynomial corresponding to P^i is f^R, where $R = (m, i, 0, 0, \dots)$. Note that f^R is defined over $\mathbb{Z}$. Suppose there is an integer polynomial g in $m+i$ variables such that the polynomials $\partial^k(f^R + pg)$ are symmetric modulo p^2 for all $k \geqslant 0$. Consider the monomial $h = x_1^p x_2 x_3 \cdots x_{m+1} x_{m+2}^p \cdots x_{m+i}^p$ of the polynomial f^R. We have

$$\partial^{p-1}(h) = p! \cdot x_1 x_2 \cdots x_{m+p} x_{m+p+1}^p \cdots x_{m+i+p-1}^p.$$

By Lemma 4.8, a nonzero multiple of $x_1 x_2 \cdots x_{m+p} x_{m+p+1}^p \cdots x_{m+i+p-1}^p$ is not a monomial of $\partial^{p-1}(g)$ modulo p. Since $p! \equiv -p$ modulo p^2, $-p \cdot x_1 x_2 \cdots x_{m+p} x_{m+p+1}^p \cdots x_{m+i+p-1}^p$ is a monomial of the polynomial $\partial^{p-1}(f^R + pg)$ modulo p^2. As the polynomial $\partial^{p-1}(f^R + pg)$ is symmetric, it also contains the monomial

$$-p \cdot x_1^p x_2^p \cdots x_{i-1}^p x_i \cdots x_{m+i+p-1}.$$

Note that this monomial is not in $\partial^{p-1}(f^R)$, therefore $x_1^p x_2^p \cdots x_{i-1}^p x_i \cdots x_{m+i+p-1}$ is a monomial of $\partial^{p-1}(g)$ modulo p. But this contradicts Lemma 4.8 since $i - 1 \geqslant p$. $\square$

A. Appendix

For a real number a write $[\![a]\!]$ for the smallest non-negative integer that is not smaller than a.

Proposition A.1. *Let $n \geqslant m$ be positive integers. Then*

$$\min v_p\langle a\rangle = [\![\frac{m - s_p(n)}{p-1}]\!],$$

where the minimum is taken over all m-sequences a such that $||a|| = n$.

Proof. Choose a prime integer p. In view of Lemma 4.5, for an m-sequence a such that $||a|| = n$ we have

$$v_p\langle a\rangle = \frac{\sum_i s_p(a_i) - s_p(n)}{p-1} \geqslant \frac{m - s_p(n)}{p-1},$$

hence

$$v_p\langle a\rangle \geqslant [\![\frac{m - s_p(n)}{p-1}]\!].$$

We will find an m-sequence a such that $||a|| = n$ and

$$v_p\langle a\rangle = [\![\frac{m - s_p(n)}{p-1}]\!]. \tag{A.2}$$

Write $n = \sum_{i=1}^{s_p(n)} p^{t_i}$ as the sum of $s_p(n)$ powers of p.

Case 1: $m \leqslant s_p(n)$. Set $a_i = p^{t_i}$ for $i = 1, \ldots, m-1$ and a_m is defined by the condition $a_1 + a_2 + \cdots + a_m = n$. We have $\sum_{i=1}^m s_p(a_i) = s_p(n)$, hence the equality (A.2) holds by Lemma 4.5(2).

Case 2: $m > s_p(n)$. Set $a_i = p^{t_i}$ for $i = 1, \ldots, s_p(n)$, so the condition $a_1 + a_2 + \cdots + a_{s_p(n)} = n$ holds. But the number of a_i's is smaller than m. Choose one of the a_i's that is equal to p^s for $s \geqslant 1$. Then replace a_i with p numbers that are all equal to p^{s-1}. The integer $||a||$ does not change but the number of a_i's increases by $p-1$. Continue doing this we stop when the first time the number m' of a_i's becomes at least m, say $m' = m + j$, where $0 \leqslant j < p-1$. Note that $m' = s_p(n) + (p-1)b$, where $b = [\![\frac{m-s_p(n)}{p-1}]\!]$.

Since $m > s_p(n)$ there is t such that the equality $a_i = p^t$ holds for at least p values of i. Now choose $j+1$ numbers among the a_i's that are equal to p^t and replace all of them by one integer $(j+1)p^t$. The number of a_i's is equal to m now. We have $\sum_i s_p(a_i) = (m-1) + (j+1) = m'$ and hence again by Lemma 4.5(2),

$$v_p\langle a\rangle = \frac{(\sum_i s_p(a_i)) - s_p(n)}{p-1} = \frac{m' - s_p(n)}{p-1} = b = [\![\frac{m - s_p(n)}{p-1}]\!]. \qquad \square$$

Proposition A.3. *Let a be an m-sequence such that all a_i are powers of p and set $n = ||a||$. Then*

$$(-p)^{\frac{-m+s_p(n)}{p-1}} \cdot \langle a\rangle \equiv c_0!c_1! \cdots c_r! \quad \textit{modulo} \quad p,$$

where c_i are p-adic digits of n.

Proof. For a positive integer n set

$$\theta(n) := p^{-v_p(n)} \cdot n + p\mathbb{Z} \in \mathbb{F}_p^{\times}.$$

The function θ is multiplicative.

Lemma A.4. *The function θ satisfies the following properties:*

(1) $\theta((kp)!) = (-1)^k \theta(k!)$ *for every* $k \geqslant 0$.

(2) $\theta(n!) = (-1)^{\frac{n-s_p(n)}{p-1}} \cdot c_0!c_1! \cdots c_r!$ *modulo p, where c_i are p-adic digits of n.*

Proof. (1): For any $i \geqslant 0$ let
$$a_i := (1+ip)(2+ip)\cdots(p-1+ip).$$
We have $(kp)! = a_0a_1\cdots a_{k-1}\cdot p^k\cdot k!$ and $\theta(a_i)=-1$ by Wilson Theorem, whence the result.

(2): We prove the statement by induction on the number r of p-adic digits of n. The statement is clear if $r=1$. Write n in the form $n=c_0+kp$, where
$$k=c_1+c_2p+\cdots+c_rp^{r-1}.$$
Let $t=(1+kp)(2+kp)\cdots(c_0+kp)$, so $t\equiv c_0!$ modulo p. We have $n! = t\cdot(kp)!$ and by the first statement of the lemma,
$$\theta(n!)=\theta(t)\cdot\theta((kp)!)\equiv c_0!\cdot(-1)^k\cdot\theta(k!).$$
By induction, $\theta(k!)\equiv(-1)^{\frac{k-s_p(k)}{p-1}}\cdot c_1!c_2!\cdots c_r!$. It remains to notice that
$$\frac{k-s_p(k)}{p-1}+k=\frac{kp-s_p(kp)}{p-1}=\frac{n-s_p(n)}{p-1}. \qquad \square$$

It follows from Lemma A.4(2) that $\theta((p^k)!)=(-1)^{\frac{p^k-1}{p-1}}$. Since every a_i is a p-power, we have
$$\theta(a_1!a_2!\cdots a_m!)=(-1)^{\sum_{i=1}^m\frac{a_i-1}{p-1}}=(-1)^{\frac{n-m}{p-1}}.$$
By Lemma A.4, the residue of $p^{\frac{-m+s_p(n)}{p-1}}\cdot\langle a\rangle$ modulo p is equal to
$$\theta(\langle a\rangle)=\theta(n!)/\theta(a_1!a_2!\cdots a_m!)=(-1)^{\frac{n-s_p(n)}{p-1}}\cdot c_0!c_1!\cdots c_r!/(-1)^{\frac{n-m}{p-1}}=(-1)^{\frac{m-s_p(n)}{p-1}}\cdot c_0!c_1!\cdots c_r!$$
$\square$

Lemma A.5. *Let n and m be two positive integers such that $n\geqslant m\geqslant s_p(n)$ and $n-m$ is divisible by $p-1$. Then there is an R such that $||R||=n$ and $|R|=m$.*

Proof. We prove the statement by induction on m. If $m=s_p(n)$, then $R=R(n)$ is the sequence of p-adic digits of n.

($m\Rightarrow m+p-1$): Let R be so that $||R||=n$ and $|R|=m$. Write $R=(r_0,r_1,r_2,\ldots)$. As $m<n$, there is $i>0$ such that $r_i>0$. Then for
$$R'=(r_0,\ldots,r_{i-2},r_{i-1}+p,r_i-1,r_{i+1},\ldots),$$
we have $||R'||=n$ and $|R'|=m+p-1$. $\square$

Proposition A.6. *Let $R\in\mathcal{R}_{m,n}$. Then $\mathcal{R}_{m,n}=\{R\}$ is the singleton if and only if R is quasi-small (see Definition 7.5).*

Proof. Replacing m by the smallest integer m' such that $m'\geqslant m$, $m'\geqslant s_p(n)$ and $n-m'$ is divisible by $p-1$ we may assume that $|R|=m$ for every $R\in\mathcal{R}_{m,n}$.

Let $\mathcal{R}_{m,n}$ be a singleton. We will show that R is quasi-small. If $r_k>p$ for some $k\geqslant 1$ consider R' with
$$r'_i=\begin{cases} r_i+p, & \text{if } i=k-1;\\ r_i-p-1, & \text{if } i=k;\\ r_i+1, & \text{if } i=k+1;\\ r_i, & \text{otherwise.}\end{cases}$$

Then $R' \neq R$ and $||R'|| = n$, $|R'| = m$. Thus, $r_i \leqslant p$ for all $i \geqslant 1$.

If R is not small then there is $s \geqslant 0$ with $r_s \geqslant p$ (and hence $r_s = p$ if $s \geqslant 1$). If $s = 0$ we claim that $r_i = 0$ if $i \geqslant 2$. Let $r_k \geqslant 1$ for some $k \geqslant 2$. If $k = 2$ consider R' with

$$r'_i = \begin{cases} r_i - p, & \text{if } i = 0; \\ r_i + p + 1, & \text{if } i = 1; \\ r_i - 1, & \text{if } i = 2; \\ r_i, & \text{otherwise.} \end{cases}$$

Then $R' \neq R$ and $||R'|| = n$, $|R'| = m$. If $k \geqslant 3$ consider R' with

$$r'_i = \begin{cases} r_i - p, & \text{if } i = 0; \\ r_i + 1, & \text{if } i = 1; \\ r_i + p, & \text{if } i = k-1; \\ r_i - 1, & \text{if } i = k; \\ r_i, & \text{otherwise.} \end{cases}$$

Then $R' \neq R$ and $||R'|| = n$, $|R'| = m$. We proved the claim, so R is quasi-small.

Now consider the case $r_0 < p$ and $r_s = p$ for $s \geqslant 1$. We claim that $r_{s+1} < p$ and $r_i = 0$ for all $i = 1, \ldots, s-1$ and $i \geqslant s+2$, i.e., R is quasi-small. Suppose $r_k \geqslant 1$ for some $k = 1, \ldots, s-1$. Consider R' with

$$r'_i = \begin{cases} r_i + p, & \text{if } i = k-1; \\ r_i - 1, & \text{if } i = k; \\ r_i - p, & \text{if } i = s; \\ r_i + 1, & \text{if } i = s+1; \\ r_i, & \text{otherwise.} \end{cases}$$

Then $R' \neq R$ and $||R'|| = n$, $|R'| = m$, a contradiction. Note that this argument (with $k = s$ and s replaced by $s+1$) also shows that $r_{s+1} < p$.

Suppose $r_k \geqslant 1$ for some $k \geqslant s+2$. If $k = s+2$, consider R' with

$$r'_i = \begin{cases} r_i - p, & \text{if } i = s; \\ r_i + p + 1, & \text{if } i = s+1; \\ r_i - 1, & \text{if } i = k = s+2; \\ r_i, & \text{otherwise.} \end{cases}$$

Then $R' \neq R$ and $||R'|| = n$, $|R'| = m$, a contradiction. If $k > s+2$, consider R' with

$$r'_i = \begin{cases} r_i - p, & \text{if } i = s; \\ r_i + 1, & \text{if } i = s+1; \\ r_i + p, & \text{if } i = k-1; \\ r_i - 1, & \text{if } i = k; \\ r_i, & \text{otherwise.} \end{cases}$$

Then $R' \neq R$ and $||R'|| = n$, $|R'| = m$, a contradiction. The claim is proved.

Now assume that R is quasi-small and $R' \in \mathcal{R}_{m,n}$. We will show that $R' = R$.

If R is small, then since $l(R') = l(R) = 0$, the sequence R' is also small. It readily follows that $R' = R$.

Write $R' = (r'_0, r'_1, \dots, r'_{s'}, 0, 0, \dots)$ with $r'_{s'} \neq 0$. We have:

$$\sum_{i\geqslant 0} r_i = |R| = |R'| = \sum_{i\geqslant 0} r'_i \quad \sum_{i\geqslant 0} r_i p^i = ||R|| = ||R'|| = \sum_{i\geqslant 0} r'_i p^i.$$

It follows that

$$\sum_{i\geqslant 1} r_i q_i = \sum_{i\geqslant 1} r'_i q_i,$$

where $q_i = \frac{p^i-1}{p-1} = 1 + p + \dots + p^{i-1}$.

Suppose $r_i = 0$ if $i \geqslant 2$ and $r_1 \leqslant p$. Then $\sum_{i\geqslant 1} r_i q_i = r_1 \leqslant p$. Since $q_i > p$ if $i \geqslant 2$, we have $r'_i = 0$ if $i \geqslant 2$. It follows that $R' = R$.

Finally suppose that there is $s \geqslant 1$ such that $r_s = p$ and $r_0 < p$, $r_{s+1} < p$ and $r_i = 0$ for $i = 1, \dots, s-1$ and $i \geqslant s+2$. We have

$$\sum_{i\geqslant 1} r'_i q_i = \sum_{i\geqslant 1} r_i q_i = pq_s + r_{s+1}q_{s+1} \leqslant pq_s + (p-1)q_{s+1} < q_{s+2}.$$

It follows that $r'_i = 0$ if $i \geqslant s+2$.

We claim that $r_{s+1} \geqslant r'_{s+1}$. Indeed,

$$(r_{s+1}+1)q_{s+1} > pq_s + r_{s+1}q_{s+1} = \sum_{i=1}^{s+1} r'_i q_i \geqslant r'_{s+1}q_{s+1},$$

therefore, $r_{s+1} + 1 > r'_{s+1}$ whence the claim.

We claim that $r_0 > r'_0$ if $R \neq R'$. Indeed in this case either $r'_i > 0$ for some $i = 1, \dots, s-1$ or $r'_{s+1} \neq r_{s+1}$ (and hence $r_{s+1} > r'_{s+1}$), and therefore,

$$(r_{s+1} - r'_{s+1})(q_{s+1} - q_s) + \sum_{i=1}^{s} r'_i (q_s - q_i) > 0.$$

Equivalently,

$$r_{s+1}(q_{s+1} - q_s) + (\sum_{i=1}^{s+1} r'_i)q_s - \sum_{i=1}^{s+1} r'_i q_i > 0. \tag{A.7}$$

Recall that

$$\sum_{i=1}^{s+1} r'_i q_i = \sum_{i=1}^{s+1} r_i q_i = r_s q_s + r_{s+1}q_{s+1}$$

and

$$\sum_{i=0}^{s+1} r'_i = |R'| = |R| = \sum_{i=0}^{s+1} r_i = r_0 + r_s + r_{s+1}.$$

It follows that the left hand side of the inequality (A.7) is equal to $r_0 q_s - r'_0 q_s$, hence $r_0 > r'_0$. The claim is proved.

Since r_0 is congruent to r'_0 modulo p and $r'_0 \geqslant 0$, we deduce that $r_0 \geqslant p$, a contradiction. $\square$

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