# MATH 216B PROBLEMS

1. Prove that a left invertible element of a finite dimensional algebra is invertible.

2. Let $A$ be a finite dimensional division algebra and $B \subset A$ a subalgebra. Prove that $B$ is a division algebra. Prove that the center of a (finite dimensional) simple algebra is a field.

3. Let $B$ be an $F$-algebra and $A = \mathrm{End}_F(B)$, where $B$ is viewed as a vector space over $F$. Consider the following two subalgebras of $A$: the subalgebra $B_1$ of left multiplication endomorphisms by all elements in $B$ and the subalgebra $B_2$ of right multiplication endomorphisms by all elements in $B$. Prove that $B_1 \simeq B$, $B_2 \simeq B^{op}$, $B_2 = C_A(B_1)$ and $B_1 = C_A(B_2)$.

4. Let $A$ be a central simple $F$-algebra, $B \subset A$ a simple subalgebra and $L = Z(B)$. Prove that $C_A(B)$ is a central simple $L$-algebra and $[B]+[C_A(B)] = [A_L]$ in $\mathrm{Br}(L)$.

5. Let $L/F$ be a finite separable field extension. Prove that there exists a unique idempotent $e \in L \otimes_F L$ such that $e(x \otimes 1) = (1 \otimes x)e$ for all $x \in L$ and the image of $e$ under the product map $L \otimes_F L \to L$ is equal to 1. Show that $e(L \otimes_F L) \simeq L$.

6. Let $A$ be a central simple algebra over $F$ and $e \in A$ a nonzero idempotent. Prove that $eAe$ is a central simple algebra over $eFe = eF \simeq F$ and $[eAe] = [A]$ in $\mathrm{Br}(F)$. (Hint: Let $A \simeq M_n(D)$ for a division $F$-algebra $D$. Show that $e$ is conjugate to a diagonal matrix in $M_n(D)$.)

7. Let $L/F$ be a finite Galois field extension with Galois group $G$ and $s, t \in Z^2(G, L^\times)$ two 2-cocycles. Prove that $[L/F, s] + [L/F, t] = [L/F, st]$ in $\mathrm{Br}(L/F)$. (Hint: Let $e \in L \otimes_F L$ be the idempotent as in Problem 5 and $A = (L/F, s) \otimes_F (L/F, t)$. Consider $L \otimes_F L$ as a subalgebra in $A$. Show that $eAe \simeq (L/F, st)$ and use Problem 6.)

8. Let $L/F$ be a finite Galois field extension with Galois group $G$, $H \subset G$ a normal subgroup and $K = L^H$. Prove that the diagram

$$\begin{array}{ccc} H^2(G/H, K^\times) & \xrightarrow{\sim} & \mathrm{Br}(K/F) \\ {\scriptstyle \mathrm{Inf}}\downarrow & & \downarrow \\ H^2(G, L^\times) & \xrightarrow{\sim} & \mathrm{Br}(L/F) \end{array}$$

is commutative. (Hint: Let $s \in Z^2(G/H, K^\times)$ be a 2-cocycle, $B = (K/F, s)$ and $A = (L/F, \mathrm{Inf}(s))$. Consider $K$ as a subfield of $B$ and $M_n(K)$ as a subalgebra of $M_n(B)$, where $n = [L : K]$. Show that there is a $K$-algebra embedding

$f : L \hookrightarrow M_n(B)$ and $n \times n$ matrices $J_\sigma \in M_n(B)$ for all $\sigma \in G$ such that $J_\sigma \cdot \sigma(f(x)) = f(\sigma x) \cdot J_\sigma$ and $J_\sigma \cdot \sigma(J_\tau) = J_{\sigma\tau}$ for all $x \in L$ and $\sigma, \tau \in G$. Prove that there is an isomorphism $A \xrightarrow{\sim} M_n(B)$ taking $x \in L$ to $f(x)$ and $u_\sigma$ to $J_\sigma \cdot u_{\sigma H}$ for all $\sigma \in G$. Here $u_\sigma$ and $u_{\sigma H}$ are the canonical generators of $A$ and $B$ respectively.)

9. Let $L/F$ be a finite Galois field extension with Galois group $G$, $H \subset G$ a subgroup and $K = L^H$. Prove that the diagram

$$\begin{array}{ccc} H^2(G, L^\times) & \xrightarrow{\sim} & \mathrm{Br}(L/F) \\ \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \cdot\otimes_F K} \\ H^2(H, L^\times) & \xrightarrow{\sim} & \mathrm{Br}(L/K) \end{array}$$

is commutative. (Hint: Let $s \in Z^2(G, L^\times)$ be a 2-cocycle and $A = (L/F, s)$. View $K$ as a subfield of $A$ and let $B = C_A(K)$. Prove that $B \simeq (L/K, \mathrm{Res}(s))$. Using Problem 4 show that $[A_K] = [B]$ in $\mathrm{Br}(K)$.

10. Generalize Problems 8 and 9 as follows. Let $L'/F$ be a field extension and let $F'$ and $L$ be intermediate fields between $F$ and $L'$ such that $L/F$ is a finite Galois field extension with Galois group $G$ and $L'/F'$ is a finite Galois field extension with Galois group $G'$. The restriction homomorphism $G' \to G$ and the inclusion $L^\times \hookrightarrow L'^\times$ yield a homomorphism $H^n(G, L^\times) \to H^n(G', L'^\times)$. Prove that the diagram

$$\begin{array}{ccc} H^2(G, L^\times) & \xrightarrow{\sim} & \mathrm{Br}(L/F) \\ \downarrow & & \downarrow{\scriptstyle \cdot\otimes_F F'} \\ H^2(G', L'^\times) & \xrightarrow{\sim} & \mathrm{Br}(L'/F') \end{array}$$

is commutative.

11. Prove that $K_2(F) = 0$ for a finite field $F$.

12. Let $s : F(t)^\times \to F^\times$ be the map defined by $s\left(\frac{at^n + \text{lower terms}}{bt^m + \text{lower terms}}\right) = \frac{a}{b}$. Prove that the map $K_2(F(t)) \to K_2(F)$, taking $\{f, g\}$ to $\{s(f), s(g)\}$ is a well-defined homomorphism.

13. Let $(F_i)_{i \in I}$ be a direct system of fields, where $I$ is a directed partially ordered set and $F = \mathrm{colim}_{i \in I} F_i$. Prove that the natural map

$$\mathrm{colim}_{i \in I} K_2(F_i) \to K_2(F)$$

is an isomorphism.

14. Let $A$ be a central simple algebra of degree $n$ over $F$ and $L \subset A$ a subfield of degree $n$ over $F$. Prove that the composition

$$K_1(L) \to K_1(A) \xrightarrow{\mathrm{Nrd}} K_1(F)$$

coincides with the norm homomorphism.

15. Let $L$ be a splitting field of a central simple algebra $A$ over $F$, so that $A \otimes_F L \simeq M_n(L)$. Prove that the composition

$$K_1(A) \to K_1(A \otimes_F L) \simeq K_1(M_n(L)) \simeq K_1(L)$$

coincides with the composition

$$K_1(A) \xrightarrow{\mathrm{Nrd}} K_1(F) \to K_1(L).$$

16. Let $Q$ be a quaternion algebra. Prove that $\mathrm{Nrd}(q) = q \cdot \tau(q)$ for every $q \in Q$, where $\tau$ is the canonical involution on $Q$.

17. Let $p$ and $q$ be nondegenerate quadratic forms of even dimension. Prove that

$$M_2(C(p \perp q)) \simeq C(p) \otimes C(q) \otimes Q,$$

where $Q$ is the quaternion algebra $(\mathrm{disc}(p), \mathrm{disc}(q))$.

18. Let $(V, q)$ be a quadratic form over $F$ and $a \in F^\times$. Prove that

$$C_0(\langle a \rangle \perp q) \simeq C(-aq).$$

19. Let $q$ be a nondegenerate quadratic form of dimension $n$. Prove that every element of the orthogonal group $O(q)$ is the product of at most $n$ reflections.

20. Let $q$ be an anisotropic quadratic form over $F$ and $a \in F^\times$. Prove that the form $\langle -a \rangle \perp q$ is isotropic if and only if $a \in D(q)$.

21. Let $q$ be a Pfister form over $F$. Prove that the kernel of the natural homomorphism $W(F) \to W(F(q))$ is the principal ideal generated by $[q]$.

22. Let $q$ be a $2^n$-dimensional anisotropic form in $I^n(F)$. Prove that $q$ is a multiple of an $n$-fold Pfister form.