1. Let $P$ be a module over a commutative ring $A$. Prove that $P$ is a finitely generated projective $A$-module if and only if there are elements $f_1, f_2, \ldots, f_n$ in $A$ generating the unit ideal and such that $P_{f_i}$ is a free $A_{f_i}$-module of finite rank for every $i$.

2. Let $V$ be a finite dimensional vector space over a field $K$. Prove that every section of the tautological line bundle $L_t$ over $\mathbb{P}(V)$ is trivial.

3. Let $k$ be a field. The projective line $\mathbb{P}(V)$ is covered by two open sets $U_1$ and $U_2$, both isomorphic to $\mathbb{A}^1_k$, in a standard way. The intersection $U_1 \cap U_2$ is $\text{Spec } k[t, t^{-1}]$. Let $\alpha \in \text{GL}_n(k[t, t^{-1}])$ for some $n$. Write $E_\alpha$ for the vector bundle over $\mathbb{P}_k$ which is obtained by gluing the trivial rank $n$ vector bundles over $U_1$ and $U_2$ along the isomorphism over $U_1 \cap U_2$ given by the matrix $\alpha$. Prove that $E_\alpha \simeq E_{\alpha'}$ if and only if there are $\beta \in \text{GL}_n(k[t])$ and $\gamma \in \text{GL}_n(k[t^{-1}])$ such that $\alpha' = \beta \alpha \gamma$.

4. Prove that every vector bundle over $\mathbb{P}^1_k$, $k$ a field, is isomorphic to a direct sum of tensor powers of the tautological line bundle.

5. Classify line bundles over the affine line with double origin.

6. Let $L_1, L_2, \ldots, L_n$ be line bundles over a scheme $X$. Prove that
$$\Lambda^n(L_1 \oplus L_2 \oplus \cdots \oplus L_n) \simeq L_1 \otimes L_2 \otimes \cdots \otimes L_n.$$
13. Let $E \to X$ be a vector bundle, where $X$ is a scheme over $K$. Determine the set $\mathbb{P}_X(E)(R)$ for a commutative $K$-algebra $R$.

14. For an integer $n \geq 2$, let $F_n : K\text{-Alg} \to \text{Sets}$ be the functor

$$F_n(R) = U_n(R)/R^\times,$$

where $U_n(R)$ is the set of unimodular $n$-rows over $R$. Prove that the functor $F_n$ admits an open cover by affine schemes, but $F_n$ is not local.

15. A subfunctor $F' \subset F$ is called closed if for every morphism $\text{Spec } R \to F$ the morphism $\text{Spec } R \times_F F' \to \text{Spec } R$ is a closed embedding of schemes. Prove that if $X'$ is a closed subscheme of a scheme $X$, then $X'$ is a closed subfunctor of the functor $X$.

16. a) Let $R$ be a commutative $K$-algebra. Let $p_i : R^n \to R$ and $q_j : R^m \to R$ be the projections and $s_{ij} : R^n \otimes R^m \to R$ defined by $s_{ij}(x \otimes y) = p_i(x)q_j(y)$. Prove that a direct summand $M \subset R^n \otimes R^m$ of rank 1 is of the form $P \otimes Q$, where $P$ and $Q$ are direct summands of $R^n$ and $R^m$ of rank 1, respectively, if and only if $s_{ij}(m)s_{kl}(m) = s_{il}(m)s_{kj}(m)$ for all $m \in M$ and all $i, j, k, l$.

b) Prove that the Segre morphism $\mathbb{P}^{n-1}_K \times \mathbb{P}^{m-1}_K \to \mathbb{P}^{nm-1}_K$ taking a pair $(P, Q)$ of submodules to $P \otimes Q$ is a closed embedding.

17. a) Let $R$ be a commutative algebra over a field $K$ and $V$ a vector space of dimension $n$ over $K$. Prove that every right ideal $I$ of $\text{End}_R(V \otimes_K R)$ that is a direct summand of rank $n$ as an $R$-submodule is of the form $I = \text{Hom}_R(V \otimes_K R, P)$ for a unique direct summand $P$ of $V \otimes_K R$ of rank 1.

b) Prove that the Severi-Brauer variety of the $K$-algebra $\text{End}_K(V)$ is isomorphic to the projective space $\mathbb{P}_K(V)$.

18. Prove that a local functor covered by open subschemes (not necessarily affine) is a scheme.

19. Let $f \in K$. Prove that the functor $F : K\text{-Alg} \to \text{Sets}$ defined by

$$F(R) = \begin{cases} R, & \text{if } f \in R^\times; \\ \emptyset, & \text{otherwise}, \end{cases}$$

is an affine scheme.

20. Let $K \to K'$ be a commutative ring homomorphism and $F : K\text{-Alg} \to \text{Sets}$ a functor. Define a functor $F' : K'\text{-Alg} \to \text{Sets}$ by $F'(R) = F(R)$. Prove that if $F$ is a scheme, then so is $F'$.

21. Let $X$ be an integral variety over a field $K$, $\xi \in X$ the generic point and $L = K(X)$.

a) Let $F$ be a sheaf of sets on $X$ such that for every nonempty open subset $U \subset X$ the natural map $F(U) \to F_\xi$ to the generic stalk is injective. Prove that for every point $x \in X$ the induced map $F_x \to F_\xi$ is injective and $F(X) = \cap_{x \in X} F_x$ in $F_\xi$. 
b) Let $Y$ be a variety over $K$. Prove that the map $\text{Mor}_K(X,Y) \to Y(L)$, induced by the morphism $\text{Spec} \ L \to X$, is injective. Show that $\text{Mor}_K(X,Y) = \cap_{x \in X} Y(\mathcal{O}_{X,x})$ in $Y(L)$.

22. Prove that the class of flat morphisms is closed under compositions, base changes and local on the target.

23. Prove that the variety $\text{Spec} \left( K[x,y,z]/(z^2 - xy) \right)$ is normal.

24. Compute $\text{Pic}(X)$ for $X = \text{Spec} \left( K[x,y]/(y^2 - x^3 - x^2) \right)$.

25. Compute $\text{CH}_0(X)$ for $X = \text{Spec} \left( K[x,y]/(y^2 - x^3 - x^2) \right)$.

26. Compute the image of the class of the tautological line bundle under the isomorphism $\text{Pic}(\mathbb{P}_K^n) \to \text{CH}^1(\mathbb{P}_K^n)$.

27. Let $W_1$ and $W_2$ be linear subspaces of the same dimension of a finite dimensional vector space $V$. Prove that the classes $[\mathbb{P}(W_1)]$ and $[\mathbb{P}(W_2)]$ in $\text{CH}(\mathbb{P}(V))$ coincide.

28. Let $X$ be a closed smooth curve in $\mathbb{A}_K^2$. Prove that the tangent bundle of $X$ is trivial.

29. Let $X$ be a locally factorial integral variety of dimension $n$ over $K$. Construct an exact sequence

\[ 1 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \prod_{x \in X^{(n-1)}} (i_x)_*(\mathbb{Z}) \to 0, \]

where $i_x : \text{Spec} \ K(x) \to X$ is a natural morphism.

30. Compute $\text{CH}(\mathbb{P}_K^n \times_K \mathbb{P}_K^n)$.

31. Let $f : \mathbb{P}_K^n \to \mathbb{P}_K^n$ be a linear automorphism. Prove that the pull-back homomorphism $f^* : \text{CH}(\mathbb{P}_K^n) \to \text{CH}(\mathbb{P}_K^n)$ is the identity.