1. Let $P$ be a module over a commutative ring $A$. Prove that $P$ is a finitely generated projective $A$-module if and only if there are elements $f_1, f_2, \ldots, f_n$ in $A$ generating the unit ideal and such that $P_{f_i}$ is a free $A_{f_i}$-module of finite rank for every $i$.

2. Let $V$ be a finite dimensional vector space over a field $K$. Prove that every section of the tautological line bundle $L_t$ over $\mathbb{P}_k(V)$ is trivial.

3. Let $k$ be a field. The projective line $\mathbb{P}^1_k$ is covered by two open sets $U_1$ and $U_2$, both isomorphic to $\mathbb{A}^1_k$, in a standard way. The intersection $U_1 \cap U_2$ is $\text{Spec } k[t, t^{-1}]$. Let $\alpha \in \text{GL}_n(k[t, t^{-1}])$ for some $n$. Write $E_\alpha$ for the vector bundle over $\mathbb{P}^1_k$ which is obtained by gluing the trivial rank $n$ vector bundles over $U_1$ and $U_2$ along the isomorphism over $U_1 \cap U_2$ given by the matrix $\alpha$. Prove that $E_\alpha \simeq E_{\alpha'}$ if and only if there are $\beta \in \text{GL}_n(k[t])$ and $\gamma \in \text{GL}_n(k[t^{-1}])$ such that $\alpha' = \beta \alpha \gamma$.

4. Prove that every vector bundle over $\mathbb{P}^1_k$, $k$ a field, is isomorphic to a direct sum of tensor powers of the tautological line bundle.

5. Classify line bundles over the affine line with double origin.

6. Let $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n$ be line bundles over a scheme $X$. Prove that

$$\Lambda^n(\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_n) \simeq \mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \cdots \otimes \mathcal{L}_n.$$ 

7. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be line bundles over a scheme $X$ such that $\mathcal{L}_1 \oplus \mathbb{A}^n_X \simeq \mathcal{L}_2 \oplus \mathbb{A}^n_X$ for some $n$. Prove that $\mathcal{L}_1 \simeq \mathcal{L}_2$.

8. Let $E \to X$ be a vector bundle, $s : X \to E$ a section. Show that $s$ is a closed embedding.

9. Let $L \to X$ be a line bundle, $z : X \to L$ the zero section. Prove that the bundle is trivial if and only if there is a section $s : X \to L$ such that $s(X) \cap z(X) = \emptyset$.

10. Let $E$ and $E'$ be two vector bundles over $X = \text{Spec } A$. Prove that there is a scheme $I = \text{Iso}(E', E)$ over $X$ such that for every commutative $A$-algebra $R$ the set of $R$-points of $I$ is the set of isomorphisms between $E' \times_X \text{Spec } R$ and $E \times_X \text{Spec } R$.

11. Prove that a morphism of vector bundles $f : E \to F$ is an admissible monomorphism (i.e., $E$ is a subbundle of $F$) if and only if $f$ is a closed embedding.

12. Let $E \to X$ be a vector bundle. Prove that there is a line bundle morphism $f : \mathbb{P}_X(E \oplus 1) \setminus \mathbb{P}_X(1) \to \mathbb{P}_X(E)$ dual to the tautological line bundle over $\mathbb{P}_X(E)$ (thus $f$ is a canonical line bundle over $\mathbb{P}_X(E)$).
13. Let \( E \to X \) be a vector bundle, where \( X \) is a scheme over \( K \). Determine the set \( \mathbb{P}_X(E)(R) \) for a commutative \( K \)-algebra \( R \).

14. For an integer \( n \geq 2 \), let \( F_n : K-\text{Alg} \to \text{Sets} \) be the functor
\[
F_n(R) = U_n(R)/R^X,
\]
where \( U_n(R) \) is the set of unimodular \( n \)-rows over \( R \). Prove that the functor \( F_n \) admits an open cover by affine schemes, but \( F_n \) is not local.

15. A subfunctor \( F' \subset F \) is called closed if for every morphism \( \text{Spec } R \to F \) the morphism \( \text{Spec } R \times_F F' \to \text{Spec } R \) is a closed embedding of schemes. Prove that if \( X' \) is a closed subscheme of a scheme \( X \), then \( X' \) is a closed subfunctor of the functor \( X \).

16. a) Let \( R \) be a commutative \( K \)-algebra. Let \( p_i : R^n \to R \) and \( q_j : R^m \to R \) be the projections and \( s_{ij} : R^n \otimes R^m \to R \) defined by \( s_{ij}(x \otimes y) = p_i(x)q_j(y) \). Prove that a direct summand \( M \subset R^n \otimes R^m \) of rank 1 is of the form \( P \otimes Q \), where \( P \) and \( Q \) are direct summands of \( R^n \) and \( R^m \) of rank 1, respectively, if and only if \( s_{ij}(m)s_{kl}(m) = s_{il}(m)s_{kj}(m) \) for all \( m \in M \) and all \( i, j, k, l \).

b) Prove that the Segre morphism \( \mathbb{P}_{K}^{n-1} \times_K \mathbb{P}_{K}^{m-1} \to \mathbb{P}_{K}^{nm-1} \) taking a pair \( (P, Q) \) of submodules to \( P \otimes Q \) is a closed embedding.

17. a) Let \( R \) be a commutative algebra over a field \( K \) and \( V \) a vector space of dimension \( n \) over \( K \). Prove that every right ideal \( I \) of \( \text{End}_R(V \otimes_K R) \) that is a direct summand of rank \( n \) as an \( R \)-submodule is of the form \( \text{Hom}_R(V \otimes_K R; P) \) for a unique direct summand \( P \) of \( V \otimes_K R \) of rank 1.

b) Prove that the Severi-Brauer variety of the \( K \)-algebra \( \text{End}_K(V) \) is isomorphic to the projective space \( \mathbb{P}_K(V) \).

18. Prove that a local functor covered by open subschemes (not necessarily affine) is a scheme.

19. Let \( f \in K \). Prove that the functor \( F : K-\text{Alg} \to \text{Sets} \) defined by
\[
F(R) = \begin{cases} 
R, & \text{if } f \in R^\times; \\
\emptyset, & \text{otherwise},
\end{cases}
\]
is an affine scheme.

20. Let \( K \to K' \) be a commutative ring homomorphism and \( F : K-\text{Alg} \to \text{Sets} \) a functor. Define a functor \( F' : K'-\text{Alg} \to \text{Sets} \) by \( F'(R) = F(R) \). Prove that if \( F \) is a scheme, then so is \( F' \).

21. Let \( X \) be an integral variety over a field \( K \), \( \xi \in X \) the generic point and \( L = K(X) \).

a) Let \( F \) be a sheaf of sets on \( X \) such that for every nonempty open subset \( U \subset X \) the natural map \( F(U) \to F_\xi \) to the generic stalk is injective. Prove that for every point \( x \in X \) the induced map \( F_x \to F_\xi \) is injective and \( F(X) = \bigcap_{x \in X} F_x \) in \( F_\xi \).
b) Let $Y$ be a variety over $K$. Prove that the map $\text{Mor}_K(X,Y) \to Y(L)$, induced by the morphism $\text{Spec} L \to X$, is injective. Show that $\text{Mor}_K(X,Y) = \cap_{x \in X} Y(O_{X,x})$ in $Y(L)$.

22. Prove that the class of flat morphisms is closed under compositions, base changes and local on the target.

23. Prove that the variety $\text{Spec} \left( K[x, y, z]/(z^2 - xy) \right)$ is normal.

24. Compute $\text{Pic}(X)$ for $X = \text{Spec} \left( K[x, y]/(y^2 - x^3 - x^2) \right)$.

25. Compute $\text{CH}_0(X)$ for $X = \text{Spec} \left( K[x, y]/(y^2 - x^3 - x^2) \right)$.

26. Compute the image of the class of the tautological line bundle under the isomorphism $\text{Pic}(\mathbb{P}^n_K) \to \text{CH}^1(\mathbb{P}^n_K)$. 