PROBLEMS, MATH 214A

AFFINE AND QUASI-AFFINE VARIETIES

$k$ is an algebraically closed field

Basic notions

1. Let $X \subset \mathbb{A}^2$ be defined by $x^2 + y^2 = 1$ and $x = 1$. Find the ideal $I(X)$.
2. Prove that the subset in $\mathbb{A}^2$ consisting of all points of the form $(t^2, t^3)$, $t \in k$ is closed.
3. Let $X$ and $X'$ be subsets of $\mathbb{A}^n$. Show that $I(X) \subset I(X')$ if and only if $X' \subset \overline{X}$.
4. Let $S$ and $S'$ be subsets of $k[T_1, \ldots, T_n]$. Show that $Z(S) \subset Z(S')$ if and only if $S' \subset \sqrt{\langle S \rangle}$.
5. Prove that a subset $X \subset \mathbb{A}^n$ is quasi-affine if and only if $X = \overline{Z}_1 \setminus \overline{Z}_2$, where $Z_1$ and $Z_2$ are closed subsets of $\mathbb{A}^n$.
6. Prove that the intersection of two quasi-affine subsets is quasi-affine.
7. Let $Z = Z(x)$ and $U = \mathbb{A}^2 \setminus Z(y)$ be respectively closed and open subsets of $\mathbb{A}^2$ with the coordinates $(x, y)$. Prove that the set $Z \cup U$ is not quasi-affine.
8. Let $X$ be a quasi affine subset of $\mathbb{A}^n$. Prove that every closed or open subset of $X$ is quasi-affine.
9. Prove that every quasi-affine subset of $\mathbb{A}^1$ is either open or closed.
10. Give an example of a quasi-affine subset of $\mathbb{A}^n$, $n \geq 2$ that is neither open nor closed.
11. Prove that every quasi-affine set (in particular, an open set!) is compact in Zariski topology.

Regular functions and maps

12. Let $X \subset \mathbb{A}^2$ be defined by $x(y^2 - x) = 0$. Show that the function $f : X \to k$ defined by

$$f(x, y) = \begin{cases} 0 & \text{if } x = 0, \\ y & \text{otherwise} \end{cases}$$

is not regular. Prove that $f(x, y)^2$ is regular.
13. Prove that every open subset of $\mathbb{A}^1$ is principal.
14. Show that the open subset $\mathbb{A}^2 \setminus (0, 0)$ is not principal in $\mathbb{A}^2$.
15. Suppose that $X$ consists of $n$ points. Prove that the ring $k[X]$ is isomorphic to the direct product of $n$ copies of $k$. 
16. Establish a bijection between an closed set $X$ and the set of all $k$-algebra homomorphisms $k[X] \to k$.

17. Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be given by $f(x, y) = (xy, y)$. Show that the image of $f$ is not quasi-affine.

18. Let $f : X \to Y$ be a regular maps of quasi-affine sets and let $I \subset k[Y]$, $J \subset k[X]$ be two ideals. Prove that 
$$f^{-1}(Z(I)) = Z(f^*(I))$$
and 
$$f(Z(J)) = Z(f^{-1}(J)).$$

19. Prove that two curves $X$ and $Y$ given in $\mathbb{A}^2$ by equations $xy = 0$ and $x^2 = x$ respectively are not isomorphic.

20. Let $x$ be a point in $\mathbb{A}^n$ and let $X = \mathbb{A}^n \setminus \{x\}$. Prove that if $n \geq 2$, then $k[X] = k[\mathbb{A}^n]$.

21. Let $f : \mathbb{A}^1 \to k$ be a regular function. Prove that the image of $f$ is either closed or open in $\mathbb{A}^1$.

22. Let $X$ be a quasi-affine set such that $k[X] = k$. Prove that $X$ is a one point set.

23. Let $S$ be a subset of $k[X]$, $X$ a closed set. Prove that the principal open sets $D(f)$, for all $f \in S$, cover $X$ if and only if the ideal generated by $S$ is equal to $k[X]$.

24. Let $X$ be a point and let $Y = \mathbb{A}^2 \setminus \{(0,0)\}$. Show that the natural map 
$$Mor(X,Y) \to Hom_{k-alg}(k[Y],k[X])$$
is not surjective. ($Mor(X,Y)$ is the set of all regular maps $X \to Y$.)

25. Let $X$ be a quasi-affine set and $Y$ be a closed set. Prove that the natural map 
$$Mor(X,Y) \to Hom_{k-alg}(k[Y],k[X])$$
is a bijection.

26. Prove that $\mathbb{A}^1 \setminus \{0,1\}$ and $\mathbb{A}^1 \setminus \{a,b\}$ are isomorphic for all $a \neq b \in k$.

27. Prove that $\mathbb{A}^1$ is not isomorphic to any proper quasi-affine subset of $\mathbb{A}^1$.

28. Prove that the curves $Z(x^2 + y^2 - 1)$ and $Z(xy - 1)$ in $\mathbb{A}^2$ are isomorphic (assume char $k \neq 2$). Are these curves isomorphic to $\mathbb{A}^1$?

Categories. Products and coproducts

29. Let $\mathcal{A}$ be a category and let $X,Y \in Ob(\mathcal{A})$. The product of $X$ and $Y$ in $\mathcal{A}$ is an object $X \times Y$ together with two morphisms $p : X \times Y \to X$ and $q : X \times Y \to Y$ such that for every two morphisms $f : Z \to X$ and $g : Z \to Y$ there exists a unique morphism $h : Z \to X \times Y$ with $f = ph$ and $g = qh$. Prove that the product is unique up to canonical isomorphism. Determine products in the categories of sets, rings, commutative rings, (left) modules over a ring, closed sets, quasi-affine sets.
30. Formulate the dual notion of the coproduct in a category. Determine coproducts in the categories of sets, rings, commutative rings, (left) modules over a ring, closed sets, quasi-affine sets.

31. An initial (resp. final) object of a category $\mathcal{A}$ is an object $X \in \text{Ob}(\mathcal{A})$ such that for any object $Y \in \text{Ob}(\mathcal{A})$ there is a unique morphism $X \to Y$ (resp. $Y \to X$). Prove that initial and final objects are unique up to canonical isomorphism. Determine initial and final objects in the categories of sets, rings, commutative rings, (left) modules over a ring, closed sets, quasi-affine sets.

32. Prove that for quasi-affine sets $X$ and $Y$, the canonical homomorphism $k[X] \otimes k[Y] \to k[X \times Y]$ is an isomorphism.

33. Let $X_1, X_2, \ldots, X_n$ be the irreducible components of a Noetherian topological space $X$.
   a) Let $U \subset X$ be an open subset. Prove that the nonempty intersections $U \cap X_i$ are the irreducible components of $U$.
   b) Prove that an open subset $U \subset X$ is dense if and only if $U \cap X_i \neq \emptyset$ for every $i$.
   c) Prove that every irreducible subset of $X$ is contained in the $X_i$ for some $i$.
   d) Prove that the irreducible components of $X$ can be defined as maximal elements in the set of all closed irreducible subsets of $X$.

34. Let $f : X \to Y$ be a continuous map of topological spaces and let $Z \subset X$ be an irreducible subset. Prove that $f(Z)$ is also irreducible.

35. Find irreducible components of the variety given by $x^2 = yz, xz = x$ in $\mathbb{A}^3$.

36. Find irreducible components of the variety given by $xy = z^3, xz = y^3$ in $\mathbb{A}^3$.

37. Let $X$ be a quasi-affine variety and let $Y \subset X$ be a closed irreducible subset. Prove that $I(Y)$ is a prime ideal in $k[X]$.

38. Show that every nonempty quasi-affine variety admits a cover by open affine dense subsets.

39. Prove that any quasi-affine variety is isomorphic to a dense open subset of an affine variety.

40. Prove that the algebra $k[X]$ for a quasi-affine variety $X$ is isomorphic to a subalgebra of a finitely generated $k$-algebra.

41. Let $X$ be an affine variety, $X = Z_1 \cup Z_2$, where $Z_i$ are closed and disjoint. Show that there is a function $e \in k[X]$ such that $e|_{Z_1} = 0$ and $e|_{Z_2} = 1$. Prove that $k[X]$ is isomorphic to the product $k[Z_1] \times k[Z_2]$.

42. Let $X$ be an affine variety and let $e \in k[X]$ be a function such that $e^2 = e$. Prove that $X = Z_1 \cup Z_2$, where $Z_i$ are closed, disjoint and $e|_{Z_1} = 0$, $e|_{Z_2} = 1$.

43. Prove that an quasi-affine variety $X$ is connected if and only if the ring $k[X]$ has no idempotents other than 0 and 1.
44. Prove that a Noetherian topological space is Hausdorff if and only if it is finite with discrete topology.

Rational functions

45. Find the domain of definition of the rational function \((1 - y)/x\) on the curve given by \(x^2 + y^2 = 1\) in \(\mathbb{A}^2\).

46. Prove that the variety \(\mathbb{A}^1\) satisfies the following property: If \(f \in k(\mathbb{A}^1)\) and \(f^2 \in k[\mathbb{A}^1]\), then \(f \in k[\mathbb{A}^1]\). Does the variety \(Z(y^2 - x^2 - x^3) \subset \mathbb{A}^2\) satisfy this property?

47. Prove the theorem \(\alpha : \mathbb{A}^2 \rightarrow \mathbb{A}^2\) given by \(\alpha(x, y) = (x, xy)\) is a birational isomorphism. Find domain of definition of \(\alpha^{-1}\).

48. Prove that the field \(k(\mathbb{A}^n)(\sqrt{1 - x^2})\) is purely transcendental over \(k\).

49. Prove that the plane curves given by the equations \(y^2 = x^3\) and \(y^2 = x^3 + x^2\) respectively are rational varieties.

50. Let \(X\) be ”sphere” given in \(\mathbb{A}^n\) by the equation \(x_1^2 + x_2^2 + \cdots + x_n^2 = 1\) \((n \geq 2)\). Prove that \(X\) is a rational variety.

Local ring of a subvariety

51. Let \(X\) and \(X'\) be quasi-affine varieties and let \(Y \subset X\) and \(Y' \subset X'\) be closed irreducible subvarieties. Prove that the local rings \(\mathcal{O}_{X,Y}\) and \(\mathcal{O}_{X',Y'}\) are isomorphic as \(k\)-algebras if and only if there are neighborhoods \(U\) and \(U'\) of \(Y\) and \(Y'\) in \(X\) and \(X'\) respectively and an isomorphism \(f : U \rightarrow U'\) such that \(f(U \cap Y) = U' \cap Y'\).

52. Let \(Y\) and \(Z\) be closed irreducible subvarieties of \(X\) such that \(Y \subset Z\). Let \(P \subset \mathcal{O}_{X,Y}\) be the prime ideal corresponding to \(Z\). Prove that \(\mathcal{O}_{X,Y}/P \simeq \mathcal{O}_{Z,Y}\) and \((\mathcal{O}_{X,Y})_P \simeq \mathcal{O}_{X,Z}\).

53. The completion \(\hat{R}\) of a local commutative ring \(R\) with maximal ideal \(M\) is the inverse limit of the rings \(R/M^i\) over all \(i \geq 1\). Show that \(\hat{R}\) is a local ring. Let \(X\) and \(Y\) be plane curves given by equations \(xy = 0\) and \(x^3 + y^3 + xy = 0\) respectively. Let \(z = (0, 0)\) be the origin. Prove that the local rings \(\mathcal{O}_{X,z}\) and \(\mathcal{O}_{Y,z}\) are not isomorphic but have isomorphic completions.

Quasi-projective varieties

Basic notions

54. Let \(I \subset k[S_0, S_1, \ldots, X_n]\) be an ideal. Prove that the following are equivalent:
   (i) \(I\) is generated by homogeneous polynomials;
   (ii) If \(F_0 + F_1 + \cdots + F_d \in I\), where \(F_i\) is a homogeneous polynomial of degree \(i\), then \(F_i \in I\) for all \(i\).

55. Let \(F\) be a homogeneous polynomial. Prove that every divisor of \(F\) is also homogeneous.
56. Let \( I \subset k[S_0, S_1, \ldots, X_n] \) be a homogeneous ideal. Prove that the ideal \( \sqrt{I} \) is also homogeneous.

**Regular functions and maps**

57. Prove that the image the Veronese map is closed.

58. Is there a surjective regular map \( \mathbb{A}^1 \to \mathbb{P}^1 \)?

59. Let \( x \in \mathbb{P}^2 \). Prove that \( \mathbb{P}^2 \setminus \{x\} \) is neither projective nor quasi-affine variety.

60. Determine all regular maps \( \mathbb{P}^1 \to \mathbb{A}^1 \).

61. Let \( F \in k[S_0, S_1, \ldots, S_n] \) be a homogeneous polynomial of degree \( d \) and let \( U \) be the principal open set \( D(F) \) in \( \mathbb{P}^n \). Prove that the \( k \)-algebra \( k[\mathcal{O}_U] \) is canonically isomorphic to the \( k \)-algebra \( k[S_0, S_1, \ldots, S_n](F) \) of all rational functions \( G_{F_m} \), where \( G \) is a homogeneous polynomial of degree \( md \).

62. Prove that every regular map \( f : \mathbb{P}^n \to \mathbb{P}^m \) is of the form \( x \mapsto [F_0(x) : F_1(x) : \cdots : F_m(x)] \), where \( F_0, F_1, \ldots, F_m \in k[S_0, S_1, \ldots, S_n] \) are homogeneous polynomials of the same degree having no nontrivial common zeros. Prove that every regular map \( \mathbb{P}^n \to \mathbb{P}^m \) with \( m < n \) is constant.

63. Prove that a regular map \( f : X \to Y \) of quasi-projective varieties is an isomorphism if and only if \( f \) is a homeomorphism and for any point \( x \in X \) the induced local ring homomorphism \( f^* : \mathcal{O}_{Y,f(y)} \to \mathcal{O}_{X,x} \) is an isomorphism.

**Rational functions and maps**

64. Find the domain of definition of the regular function \( f = S_1/S_0 \) on \( \mathbb{P}^2 \).

65. Let \( f : \mathbb{P}^2 \to \mathbb{P}^2 \) be the rational map defined by \( f([S_0 : S_1 : S_2]) = [S_1S_2 : S_0S_2 : S_0S_1] \). Find the domain of definition of \( f \). Prove that \( f \) is a birational isomorphism and \( f^2 = \text{id} \).

66. Prove that every rational map \( \mathbb{P}^1 \to \mathbb{P}^n \) is regular.

67. Let \( f : X \to Y \) be a rational map of quasi-projective varieties with \( X \) irreducible. Prove that there is a regular map \( g : X' \to X \) for some quasi-projective \( X' \) such that \( g \) is a birational isomorphism and the composition \( f \circ g \) is regular.

**Product of quasi-projective varieties**

68. Prove that \( (X \times Y) \times Z \cong X \times (Y \times Z) \).

69. Prove that \( \mathbb{P}^n \times \mathbb{P}^m \) is not isomorphic to \( \mathbb{P}^{n+m} \).

70. Prove that \( \mathbb{A}^1 \times \mathbb{P}^1 \) is not isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \).

71. Let \( f : X \to S \) and \( g : Y \to S \) be two regular maps. Show that the set \( \{(x, y) \in X \times Y : f(x) = g(y)\} \) is a quasi-projective variety.
Proper maps

72. Prove that if \( f_1 : X_1 \to Y_1 \) and \( f_2 : X_2 \to Y_2 \) are proper maps, then \( f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2 \) is proper.

73. Prove that the map \( X \to pt \) is proper if and only if \( X \) is a projective variety. A morphism \( f : X \to Y \) is proper if and only if \( f \) factors as \( X \to Y \times \mathbb{P}^n \to Y \) for some \( n \) with the first map a closed embedding and the second one the projection.

74. Let \( f : X \to Y \) and \( g : Y \to Z \) be two regular maps. Prove that if \( g \circ f \) is proper, then \( f \) is proper.

75. Prove that a map \( f : X \to Y \) is proper if and only if \( Y \) can be covered by open subsets \( U_i \) such that \( f^{-1}(U_i) \to U_i \) is proper for each \( i \).