MATH 212A PROBLEMS

1. Let $\mathcal{A}$ be an additive (respectively, abelian) category. Prove that the category of complexes over $\mathcal{A}$ is also additive (respectively, abelian).

2. Prove that an object X of an additive category is zero if and only if $1_X = 0_X$.

3. Let $\mathcal{A}$ be the following category. The objects of $\mathcal{A}$ are pairs (A', A), where A is an abelian group and A' is a subgroup of A. A morphism $(A', A) \to (B', B)$ is a group homomorphism $f : A \to B$ such that $f(A') \subset B'$. Prove that $\mathcal{A}$ has a structure of an additive category so that every morphism has kernel and cokernel. Is $\mathcal{A}$ an abelian category?

4. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. Prove that for every exact sequence $0 \to X \to Y \to Z$ in $\mathcal{A}$ the sequence $0 \to F(X) \to F(Y) \to F(Z)$ is also exact in $\mathcal{B}$.

5. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories. Prove that F is exact if and only if for every exact sequence $X \to Y \to Z$ in $\mathcal{A}$ the sequence $F(X) \to F(Y) \to F(Z)$ is exact in $\mathcal{B}$.

6. (a) Prove that an abelian group A is injective in the category of abelian groups if and only if A is divisible, i.e., $nA = A$ for all nonzero integers n.
(b) Prove that the category of abelian groups has enough injective objects.

7. Let R be a ring
(a) Let M be a left R-module. For an abelian group A the group $\mathrm{Hom}_{\mathbb{Z}}(R, A)$ has the structure of a left R-module via $(rf)(x) = f(xr)$ for $r, x \in R$ and $f \in \mathrm{Hom}_{\mathbb{Z}}(R, A)$. Prove that $\mathrm{Hom}_{\mathbb{Z}}(R, A)$ is an injective R-module if the group A is divisible.
(b) Prove that the category R-Mod has enough injective modules.

8. Prove that the homology H of a complex $A \xrightarrow{f} B \xrightarrow{g} C$ is canonically isomorphic to each of the following three objects: $\mathrm{Ker}\big(\mathrm{Coker}(f) \to C\big)$, $\mathrm{Ker}\big(\mathrm{Coker}(f) \to \mathrm{Coim}(g)\big)$ and $\mathrm{Coker}\big(A \to \mathrm{Ker}(g)\big)$.

9. Let $0 \to A \to B \to C \to 0$ be a short exact sequence in an abelian category $\mathcal{A}$ and let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories. Prove that the connecting morphisms $\partial^n : (R^n F)(C) \to (R^{n+1} F)(A)$ are well defined. Prove that for every $n \geq 0$ the correspondence

$$\big[0 \to A \to B \to C \to 0\big] \mapsto \big[(R^n F)(C) \xrightarrow{\partial} (R^{n+1} F)(A)\big]$$

extends to a functor from the category of short exact sequences in $\mathcal{A}$ to the category of morphisms in $\mathcal{B}$.

10. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories. Show that there is a natural morphism of functors $F \to R^0F$. Prove that F is left exact if and only if $F \to R^0F$ is an isomorphism.

11. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories. Prove that $R^n(R^mF) = 0$ for all $n \geq 0$ and $m > 0$.

12. Prove that for an abelian group A there is a canonical isomorphism between the subgroup A_{tors} of finite order elements of A and $\operatorname{Tor}_1^{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$.

13. Let M be an abelian group and let $F : Ab \to Ab$ be the functor defined by $F(A) = A \otimes M$ and $F(f) = f \otimes 1_M$. Prove that

$$(R^nF)(A) = \begin{cases} A \otimes (M/M_{\text{tors}}), & \text{if } n = 0; \\ 0, & \text{if } n > 0. \end{cases}$$

14. Prove that an object A of an abelian category is projective if and only if $\operatorname{Ext}^1(A, B) = 0$ for all objects B.

15. Prove that the following properties of an abelian category are equivalent:
1. A subobject of a projective object is projective;
2. A factor object of an injective object is injective;
3. $\operatorname{Ext}^2(A, B) = 0$ for all objects A and B.

16. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories and let $A \to X^\bullet$ be a resolution of an object A in $\mathcal{A}$ such that $(R^nF)(X^i) = 0$ for all $n > 0$ and $i \geq 0$. Prove that $H^n(F(X^\bullet)) \simeq (R^nF)(A)$ for all $n \geq 0$.

17. (a) Prove that the ring $\operatorname{End}(\mathbb{Q}/\mathbb{Z})$ is naturally isomorphic to the ring of profinite integers $\widehat{\mathbb{Z}} := \lim \mathbb{Z}/n\mathbb{Z}$.

(b) Prove that $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \simeq \widehat{\mathbb{Z}}/\mathbb{Z}$. Give an explicit example of a non-split short exact sequence $0 \to \mathbb{Z} \to A \to \mathbb{Q} \to 0$ of abelian groups.

18. (a) Let A be a torsion free abelian group and B an abelian group such that $nB = 0$ for some nonzero integer n. Show that $\operatorname{Ext}^1_{\mathbb{Z}}(A, B) = 0$.

(b) Suppose that for an abelian group A we have $nA_{\text{tors}} = 0$ for some nonzero integer n. Prove that there is a subgroup $B \subset A$ such that $A = A_{\text{tors}} \oplus B$.

19. Let R be a commutative ring and let $S = R[X, Y]$ be the polynomial ring in two variables. Consider R as an S-module via the ring homomorphism $S \to R$, $X \mapsto 0$, $Y \mapsto 0$. Calculate $\operatorname{Ext}^n_S(R, R)$ for all n.

20. Let $f : A \to A'$ and $g : C \to C'$ be two morphisms in an abelian category. Prove that the diagrams

$$\begin{tikzcd} \mathcal{E}(A, C) \arrow[r, "\sim"] \arrow[d, "f_\circ"'] & \operatorname{Ext}^1(C, A) \arrow[d, "f_*"] \\ \mathcal{E}(A', C) \arrow[r, "\sim"] & \operatorname{Ext}^1(C, A') \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E}(A, C') \arrow[r, "\sim"] \arrow[d, "f^\circ"'] & \operatorname{Ext}^1(C', A) \arrow[d, "g^*"] \\ \mathcal{E}(A, C) \arrow[r, "\sim"] & \operatorname{Ext}^1(C, A), \end{tikzcd}$$

where $f_\circ$ and f° are given by the pushout and pullback constructions.

21. Let A and B be two abelian group. Prove that the natural homomorphism

$$\mathrm{Tor}_1^{\mathbb{Z}}(A_{\mathrm{tors}}, B_{\mathrm{tors}}) \to \mathrm{Tor}_1^{\mathbb{Z}}(A, B)$$

is an isomorphism.

22. Let R be a ring.

(b) Let $(N_i)_{i\in I}$ be a direct system of left R-modules, where I is a directed set. Prove that for any right R-module M the natural map

$$\mathrm{colim}_{i\in I}\,\mathrm{Tor}_n^R(M, N_i) \to \mathrm{Tor}_n^R(M, \mathrm{colim}_{i\in I}(N_i))$$

is an isomorphism for every n.

(b) Prove that a right R-module M is flat if and only if $\mathrm{Tor}_1^R(M, N) = 0$ for every finitely generated left R-module N.

23. Let R be a ring. Prove that a right R-module M is flat if and only if $\mathrm{Tor}_1^R(M, R/I) = 0$ for every left ideal $I \subset R$.

24. Let $\sigma : A \to A$ be an automorphism of an abelian group A. We can view A as a G-module, where G is an infinite cyclic group with a chosen generator acting on A via σ. Prove that

$$H^n(G, A) = \begin{cases} \mathrm{Ker}(\sigma - 1), & \text{if } n = 0; \\ \mathrm{Coker}(\sigma - 1), & \text{if } n = 1; \\ 0, & \text{if } n \geq 2. \end{cases}$$

25. Let $H \subset G$ be a normal subgroup and A a G-module such that $H^i(H, A) = 0$ for $i = 1, 2, \ldots, n-1$ for some $n \geq 1$. Prove that the inflation map $H^i(G/H, A^H) \to H^i(G, A)$ is an isomorphism for $i = 1, 2, \ldots, n-1$ and the sequence

$$0 \to H^n(G/H, A^H) \xrightarrow{\mathrm{Inf}} H^n(G, A) \xrightarrow{\mathrm{Res}} H^n(H, A)$$

is exact.

26. Let $H \subset G$ be a normal subgroup and let $0 \to A \to B \to C \to 0$ be a short exact sequence of G-modules. Prove that the connecting map $H^n(H, C) \to H^{n+1}(H, A)$ is a homomorphism of G-modules.

27. Let G be a finite group and A a finitely generated G-module. Prove that the groups $H^n(G, A)$ are finite for all $n \geq 1$.

28. Let G be a finite group, let I be the kernel of the augmentation map $\varepsilon : \mathbb{Z}[G] \to \mathbb{Z}$ and $J = \mathrm{Coker}(\mathbb{Z} \xrightarrow{N_G} \mathbb{Z}[G])$. Prove that the G-modules I and J are isomorphic if and only if G is cyclic.

29. Let G, I and J be as in Problem 28 and let A be a G-module.

(a) Prove that the s.e.s. of G-modules $0 \to A \otimes_{\mathbb{Z}} I \to A \otimes_{\mathbb{Z}} \mathbb{Z}[G] \to A \to 0$ yields an isomorphism $H^n(G, A) \xrightarrow{\sim} H^{n+1}(G, A \otimes_{\mathbb{Z}} I)$ for every $n \geq 1$.

(b) Prove that the s.e.s. of G-modules $0 \to A \to A \otimes_{\mathbb{Z}} \mathbb{Z}[G] \to A \otimes_{\mathbb{Z}} J \to 0$ yields an isomorphism $H^{n+1}(G, A) \xrightarrow{\sim} H^n(G, A \otimes_{\mathbb{Z}} J)$ for every $n \geq 1$.

(c) Prove that there is a G-module homomorphism $f : I \otimes_{\mathbb{Z}} J \to \mathbb{Z}$ such that the composition

$$H^n(G, A) \xrightarrow{\sim} H^{n+1}(G, A \otimes_{\mathbb{Z}} I) \xrightarrow{\sim} H^n(G, A \otimes_{\mathbb{Z}} I \otimes_{\mathbb{Z}} J) \xrightarrow{f_*} H^n(G, A)$$

is the identity.

30. Let A be a G-module that is free as an abelian group. Prove that the G-module $A \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ (with the diagonal G-action) is a free $\mathbb{Z}[G]$-module.

31. Let G be a group, let R be a ring and A an $R[G]$-module. Prove that

$$H^n(G, A) \simeq \operatorname{Ext}^n_{R[G]}(R, A)$$

for all $n \geq 0$.

32. Give an example of a group G, its subgroup H and an H-module B such that the G-modules $\operatorname{Ind}_H^G(B)$ and $\operatorname{CoInd}_H^G(B)$ are not isomorphic.

33. Let $0 \to B_1 \to B_2 \to B_3 \to 0$ be an exact sequence of G-modules, $0 \to A_1 \to A_2 \to A_3 \to 0$ an exact subsequence and $0 \to C_1 \to C_2 \to C_3 \to 0$ the exact factor sequence. Prove the composition of connecting homomorphisms

$$H^n(G, C_3) \to H^{n+1}(G, C_1) \to H^{n+2}(G, A_1)$$

is equal to the negative of the composition of connecting homomorphisms

$$H^n(G, C_3) \to H^{n+1}(G, A_3) \to H^{n+2}(G, A_1).$$

34. Let A and B be G-modules, where G is a finite group. Let $\alpha \in H^1(G, A)$ be represented by a 1-cocycle $a : G \to A$ and $\beta \in H^{-1}(G, B) = {}_N B/IB$ represented by $b \in B_N$. Prove that the cup-product $\alpha \cup \beta$ in $H^0(G, A \otimes B) = (A \otimes B)^G / N(A \otimes B)$ is represented by

$$\sum_{g \in G} a(g) \otimes gb \in (A \otimes B)^G.$$

35. Let G be a finite group. We identify $\hat{H}^2(G, \mathbb{Z})$ with $G^* = \operatorname{Hom}(G, \mathbb{Q}/\mathbb{Z})$ and $\hat{H}^{-2}(G, \mathbb{Z})$ with $I/I^2 = G^{ab}$. Prove that the cup-product map

$$\cup : \hat{H}^2(G, \mathbb{Z}) \otimes \hat{H}^{-2}(G, \mathbb{Z}) \to \hat{H}^0(G, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z} = \frac{1}{n}\,\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$$

satisfies $f \cup g = f(g)$ for all $f \in G^*$ and $g \in G^{ab}$.

36. Let G be a group of order n and let $s \in \hat{H}^m(G, \mathbb{Z})$ be an element of order n. Prove that $\hat{H}^m(G, \mathbb{Z})$ is generated by s. (Hint: Show by duality that there is $t \in \hat{H}^{-m}(G, \mathbb{Z})$ such that $s \cup t = [1] \in \mathbb{Z}/n\mathbb{Z} = \hat{H}^0(G, \mathbb{Z})$.)

37. Let G be a finite group. A G-lattice P is called *permutation* if there exists a G-invariant $\mathbb{Z}$-basis for P.

a) Prove that a permutation G-lattice is isomorphic to a direct sum of G-lattices of the form $\mathbb{Z}[G/H]$, where H is a subgroup of G. (Here G/H is the G-set of left cosets of H in G.)
b) Prove that the groups $\hat{H}^{-1}(H, P)$ and $\hat{H}^{1}(H, P)$ are trivial for a permutation G-lattice P and every subgroup $H \subset G$.

38. Let G be a finite group. A G-lattice C is called *coflasque* if $\hat{H}^{1}(H, C) = 0$ for every subgroup $H \subset G$. Prove that every G-lattice M admits an exact sequence

$$0 \to C \to P \to M \to 0$$

with P a permutation lattice and C a coflasque lattice.

39. Let G be a finite group. A G-lattice F is called *flasque* if $\hat{H}^{-1}(H, F) = 0$ for every subgroup $H \subset G$. Prove that every G-lattice N admits an exact sequence

$$0 \to N \to P \to F \to 0$$

with P a permutation lattice and F a coflasque lattice.

40. Let $H \subset G$ be a subgroup. Determine left and right adjoint functors to the restriction functor

$$Res : G\text{–}Mod \to H\text{–}Mod.$$