PROBLEMS, MATH 212

Categories and functors

1. Find initial objects in $Set^\circ \times Set$.

2. Give an example of a subcategory $\mathcal{A} \subset \mathcal{B}$ and a morphism f in $\mathcal{A}$ such that f is an isomorphism in $\mathcal{B}$ but not an isomorphism in $\mathcal{A}$. Prove that if $\mathcal{A}$ is a full subcategory of $\mathcal{B}$ and f is an isomorphism in $\mathcal{B}$, then f is an isomorphism in $\mathcal{A}$.

3. Prove that every category can be embedded (as a full subcategory) into a category with initial and terminal objects.

4. Prove that the category Set° is not equivalent to Set.

5. Let $Vect(k)$ be the category of finite dimensional vector spaces over k.
a) Prove that the forgetful functor $Vect(k) \to Set$ is representable.
b) Let $W \in \mathcal{O}b Vect(k)$. Prove that the functor

$$Vect(k) \to Set, \quad V \mapsto V \otimes_k W$$

is representable.

6. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Suppose that F has a left adjoint functor. Prove that F takes terminal objects of $\mathcal{A}$ to terminal objects of $\mathcal{B}$. Formulate and prove the dual statement.

7. Prove that the forgetful functor $Gr \to Set$ has no right adjoint.

8. Let $F : \mathcal{A} \to \mathcal{B}$ be an equivalence of categories and let $G : \mathcal{B} \to \mathcal{A}$ be a quasi-inverse of F. Show that G is left and right adjoint to F.

9*. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Prove that for every functor $G : \mathcal{I} \to \mathcal{A}$ there is a natural morphism $\alpha : F(\lim G) \to \lim(FG)$. We say that F *takes limits to limits* if α is an isomorphism for every functor $G : \mathcal{I} \to \mathcal{A}$. Prove that if F has a left adjoint functor then F takes limits to limits. Formulate and prove dual statements.

10. Let $F : Gr \to Set$ be the forgetful functor. Which property of F implies that $F(G \times H) \simeq F(G) \times F(H)$?

11*. Let R be a nonzero ring. Prove that the category $(R\text{-}mod)^\circ$ is not equivalent to S-*mod* for any ring S.

12. Let $\mathcal{A}$ be a category such that for every two objects X and Y the set $\mathrm{Mor}(X, Y)$ is equipped with a structure of an abelian group such that the composition of morphisms is bi-additive. Prove that $\mathcal{A}$ can be embedded into an additive category as a full subcategory.

13. Let f be a morphism in an abelian category such that $\mathrm{Ker}(f) = 0$ and $\mathrm{Coker}(f) = 0$. Prove that f is an isomorphism.

14*. Let $\mathcal{A}$ be the following category. The objects of $\mathcal{A}$ are pairs (A', A), where A is an abelian group and A' is a subgroup of A. A morphism $(A', A) \to (B', B)$ is a group homomorphism $f : A \to B$ such that $f(A') \subset B'$. Prove that $\mathcal{A}$ has a structure of an additive category having kernel and cokernel of every morphism. Is $\mathcal{A}$ an abelian category?

15. Let $f : A \to B$ be a morphism in an abelian category.
a) Prove that the natural morphisms $\mathrm{Ker}(f) \to A$ and $\mathrm{Im}(f) \to B$ are monomorphisms.
b) Suppose that f is a monomorphism. Prove that $A = \mathrm{Im}(f)$.

16. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a complex in an abelian category. Prove that the natural morphism $\mathrm{Im}(f) \to \mathrm{Ker}(g)$ is a monomorphism.

17. Prove that a sequence $0 \to A \to B \xrightarrow{g} C$ in an abelian category is exact if and only if $A = \mathrm{Ker}(g)$.

18. Prove that an additive functor is exact if and only if it is left and right exact.

19. Prove that the homology H of a complex $A \xrightarrow{f} B \xrightarrow{g} C$ is canonically isomorphic to each of the following three objects: $\mathrm{Ker}\big(\mathrm{Coker}(f) \to C\big)$, $\mathrm{Ker}\big(\mathrm{Coker}(f) \to \mathrm{Coim}(g)\big)$ and $\mathrm{Coker}\big(A \to \mathrm{Ker}(g)\big)$.

20. Let R-$fmod$ be the category of *finitely generated* left modules over a ring R. Let $S \subset R$ be a multiplicative system of a commutative ring R and let T be the class of all morphisms f in R-$fmod$ such that $S^{-1}f$ is an isomorphism. Prove that T is a localizing class. Construct an equivalence of $S^{-1}R$-$fmod$ and R-$fmod[T^{-1}]$.

Complexes

21. Prove that for an additive category $\mathcal{A}$ the composition functor $\mathcal{A} \to Com(\mathcal{A}) \to K(\mathcal{A})$ is full and faithful.

22. Prove that a complex $A^\bullet$ is a zero object in $Com(\mathcal{A})$ if and only if $A^n = 0$ for all n.

23. (a) Prove that a complex $A^\bullet$ is a zero object in $K(\mathcal{A})$ if and only if the identity morphism $\mathrm{id} : A^\bullet \to A^\bullet$ is homotopic to zero.
(b) Prove that a complex $0 \to A \to 0$ is zero in $K(\mathcal{A})$ if and only if $A = 0$.
(c) Prove that a complex $0 \to A \xrightarrow{f} B \to 0$ is zero in $K(\mathcal{A})$ if and only if the complex is acyclic (i.e., f is an isomorphism).
(d) A short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is called *split* is there is a morphism $h : C \to B$ such that $g \circ h = \mathrm{id}_C$. Prove that a complex $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is zero in $K(\mathcal{A})$ if and only if the sequence is split

exact. Give an example of a short exact sequence that represents a nonzero object in $K(\mathcal{A})$.

24. Let $F : Ab \to \mathcal{A}$ be an additive functor.
(a) Prove that $R^iF = 0$ and $L_iF = 0$ for $i > 1$.
(b) Prove that if F is right exact then $R^1F = 0$.

25* (a) Let N be an abelian group. Determine R^0F for the functor $F : Ab \to Ab$, $A \mapsto A \otimes N$.
(b) Let A_t be the subgroup of finite order elements of an abelian group A. Determine left and right derived functors for the functors $F, G : Ab \to Ab$, $F(A) = A_t$ and $G = A/A_t$.

26. Prove that for an abelian group A there is a canonical isomorphism between A_t and $Tor_1^{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$.

27*. Let R be a commutative ring and let $S = R[X, Y]$ be the polynomial ring in two variables. Consider R as an S-module via the ring homomorphism $S \to R$, $X \mapsto 0$, $Y \mapsto 0$. Calculate $Ext^n_S(R, R)$ for all n.

28*. Suppose that for an abelian group A we have $nA_t = 0$ for some n. Prove that there is a subgroup $B \subset A$ such that $A = A_t \oplus B$. (Hint: Show that $Ext^1_{\mathbb{Z}}(A/A_t, A_t) = 0$.)

29. Let R be a ring. A right R-module M is called *flat* if the functor

$$F : R\text{-}mod \to Ab; \quad N \mapsto M \otimes_R N$$

is exact.
(a) Prove that a projective module is flat.
(b) Prove that an abelian group M is flat (as a $\mathbb{Z}$-module) if and only if M is torsion-free. (Hint: The functor F takes colimits to colimits.)

Triangulated categories

30. Let $\mathcal{A}$ be an abelian category and let $u : A^\bullet \to B^\bullet$ be a morphism in $K(\mathcal{A})$. Prove that u is isomorphic to a morphism $[f] : A_1^\bullet \to B_1^\bullet$ such that f is a monomorphism (resp. epimorphism) in $Com(\mathcal{A})$.

31. Let $\mathcal{A}$ be an abelian category and let $A^\bullet \in Com(\mathcal{A})$ such that $H^n(A^\bullet) = 0$ for $n >> 0$. Prove that $A^\bullet$ is quasi-isomorphic to a complex in $Com^-(\mathcal{A})$.

32*. Prove that two triangles $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ and $A' \xrightarrow{u'} B' \xrightarrow{v'} C' \xrightarrow{w'} A'[1]$ in a triangulated category are exact if and only if the triangle $A \oplus A' \xrightarrow{u \oplus u'} B \oplus B' \xrightarrow{v \oplus v'} C \oplus C' \xrightarrow{w \oplus w'} A \oplus A'[1]$ is exact.

33. Prove that a morphism $f : A \to B$ in a triangulated category is an isomorphism if and only if the triangle $A \xrightarrow{f} B \to 0 \to A[1]$ is exact.

34. Let $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ and $A' \xrightarrow{u'} B' \xrightarrow{v'} C' \xrightarrow{w'} A'[1]$ be two exact triangles in a triangulated category and let $g : B \to B'$ be a morphism such that $v'gu = 0$. Prove that g extends to a morphism of the triangles.

35. (a) Prove that for every two objects A and C of a triangulated category the triangle $A \xrightarrow{i_A} A \oplus C \xrightarrow{p_C} C \xrightarrow{0} A[1]$ is exact.
(b) Let $A \to B \to C \xrightarrow{0} A[1]$ be an exact triangle in a triangulated category. Prove that this triangle is isomorphic to the triangle $A \xrightarrow{i_X} A \oplus C \xrightarrow{p_C} C \xrightarrow{0} A[1]$.

36*. Give an example of an exact triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ such that the triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{-w} A[1]$ is not exact.

37*. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence in an abelian category. Let $X^\bullet$ be the complex $\dots 0 \to A \xrightarrow{f} B \to 0 \to \dots$ with B of degree 0.
(a) Show that the obvious roof $C \leftarrow X^\bullet \to A[1]$ defines a morphism $h \in \mathrm{Mor}_{D(\mathcal{A})}(C, A[1])$.
(b) Let $0 \to A \xrightarrow{f'} B' \xrightarrow{g'} C \to 0$ and $0 \to A \xrightarrow{f''} B'' \xrightarrow{g''} C \to 0$ be two exact sequences. Prove that these sequences are isomorphic (that is there exists an isomorphism $k : B' \to B''$ such that $kf' = f''$ and $g''k = g'$) if and only if the corresponding morphisms h' and h'' in $\mathrm{Mor}_{D(\mathcal{A})}(C, A[1])$ are equal.

38. Prove that a complex $A^\bullet$ is isomorphic in $D(\mathcal{A})$ to a single object complex if and only if $H^n(A^\bullet) = 0$ for all $n \neq 0$.

39*. Let $A^\bullet$ and $B^\bullet$ be two complexes such that $H^n(A^\bullet) = 0$ for $n \geq 0$ and $H^n(B^\bullet) = 0$ for $n < 0$. Prove that $\mathrm{Mor}_{D(\mathcal{A})}(A^\bullet, B^\bullet) = 0$. Is the group $\mathrm{Mor}_{D(\mathcal{A})}(B^\bullet, A^\bullet)$ also trivial?

40. Let $A^\bullet$ be a complex such that $A^n = 0$ for $n < 0$. Prove that for every object B in $\mathcal{A}$ the group $\mathrm{Mor}_{D(\mathcal{A})}(B, A^\bullet)$ is isomorphic to the kernel of the homomorphism $\mathrm{Mor}_{\mathcal{A}}(B, A^0) \to \mathrm{Mor}_{\mathcal{A}}(B, A^1)$.

41. A triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ in a triangulated category is called *split* if one of the three morphisms u, v or w is zero. Prove that if every short exact sequence in an abelian category $\mathcal{A}$ splits, then every triangle in $D(\mathcal{A})$ splits.

42*. Let $A^\bullet$ and $B^\bullet$ be complexes of abelian groups such that $A^n = 0$ for $n \leq 0$ and $B^n = 0$ for $n \geq 0$. Prove that $\mathrm{Mor}_{D(Ab)}(A^\bullet, B^\bullet) = 0$.

43. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of additive categories and let $A^\bullet$ be a zero object in $K(\mathcal{A})$. Prove that $F(A^\bullet)$ is zero in $K(\mathcal{B})$.

44. The complex $\dots \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \dots$ is acyclic and hence it is a zero object in $D(Ab)$. Is the complex zero in $K(Ab)$?