HOMEWORK 2

1. Let $G$ be a group and $a, b \in G$.
   (a) Prove that $a^n \cdot a^m = a^{n+m}$ and $(a^n)^m = a^{nm}$.
   (b) Prove that $\text{ord}(a^n) = \frac{\text{ord}(a)}{\gcd(n, \text{ord}(a))}$ if $\text{ord}(a) < \infty$.
   (c) Prove that $\text{ord}(ab) = \text{ord}(a) \cdot \text{ord}(b)$ if $a$ and $b$ commute and $\gcd\{\text{ord}(a) \ \text{ord}(b)\} = 1$.

2. Let $H \subseteq G$ be a subgroup. Show that the correspondence $Ha \mapsto (Ha)^{-1} = a^{-1}H$ is a bijection between the sets of right and left cosets.

3. Let $H \subseteq G$ be a subgroup. Suppose that for any $a \in G$ there exists $b \in G$ such that $aH = Hb$. Show that $H$ is normal in $G$.

4. Let $f : G \to H$ be a surjective group homomorphism.
   (a) Let $H'$ be a subgroup of $H$. Show that $G' = f^{-1}(H')$ is a subgroup of $G$. Prove that the correspondence $H' \mapsto G'$ is a bijection between the set of all subgroups of $H$ and the set of all subgroups of $G$ containing $\text{Ker}(f)$.
   (b) Let $H'$ be a normal subgroup of $H$. Show that $G' = f^{-1}(H')$ is a normal subgroup of $G$. Prove that $G/G' \simeq H/H'$ and the correspondence $H' \mapsto G'$ is a bijection between the set of all normal subgroups of $H$ and the set of all normal subgroups of $G$ containing $\text{Ker}(f)$.

5. (a) Let $N$ be a subgroup in the center $Z(G)$ of $G$. Show that $N$ is normal in $G$. Prove that if the factor group $G/N$ is cyclic, then $G$ is abelian.
   (b) Prove that every group of order $p^2$ (for a prime $p$) is abelian.

6. Prove that if a group $G$ contains a subgroup $H$ of finite index, then $G$ contains a normal subgroup $N$ of finite index such that $N \subseteq H$. (Hint: Consider the homomorphism of $G$ to the symmetric group of all left cosets of $H$ in $G$ taking any $x \in G$ to $f_x$ defined by $f_x(aH) = xaH$.)

7. (a) Show that the group $\text{Inn}(G)$ of all inner automorphisms of a group $G$ (given by $a \mapsto gag^{-1}$ for some $g \in G$) is a normal subgroup in $\text{Aut}(G)$.
   (b) Find all automorphisms of all (finite and infinite) cyclic groups.

8. Prove that if $G$ has no non-trivial automorphisms, then $G$ is abelian and $g^2 = e$ for all $g \in G$.

9. Let $x$ and $x'$ be two elements in the same orbit under some action of a group $G$ on a set. Show that the stabilizers $G_x$ and $G_{x'}$ are conjugate in $G$. 
10. Let a group $G$ act on two sets $X$ and $Y$. We say that $X$ and $Y$ are $G$-isomorphic if there is a bijection $f : X \to Y$ such that $f(gx) = g(f(x))$ for every $x \in X$ and $g \in G$. Prove that if $G$ acts on $X$ transitively, then $X$ is $G$-isomorphic to the set of left cosets $G/H$ for some subgroup $H \subset G$ (with the action of $G$ on $G/H$ by left translations).