1. Show that if 1 = 0 in a ring $R$, then $R$ is the zero ring.

2. (a) For any ring $R$ define a ring structure on the abelian group $\tilde{R} = R \oplus \mathbb{Z}$ such that $(0, 1)$ is the identity of $\tilde{R}$ and the inclusion map $R \to \tilde{R}$, $r \mapsto (r, 0)$ is a ring homomorphism.
(b) Let $\mathcal{C}$ be the category of rings without identity. Show that the functor $F : \mathcal{C} \to \text{Rings}$ such that $F(R) = \tilde{R}$ is a left adjoint to the forgetful functor $G : \text{Rings} \to \mathcal{C}$ taking a ring with identity $R$ to $R$ considered as an object of $\mathcal{C}$.

3. Prove that a finite nonzero ring with no zero divisors is a division ring and a finite integral domain is a field.

4. Let $R$ be the set of all $2 \times 2$ matrices over $\mathbb{C}$ of the form
$$
\begin{pmatrix}
u & v \\
-\overline{v} & \overline{u}
\end{pmatrix},
$$
where “bar” denotes the complex conjugation. Show that $R$ is a subring with identity in $M_2(\mathbb{C})$ that is isomorphic to the ring of real quaternions $\mathbb{H}$.

5. Show that the subset $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ in the real quaternion ring $\mathbb{H}$ is a group (with respect to the multiplication) isomorphic to the quaternion group (of order 8).

6. (a) Prove that a nonzero matrix $a \in M_n(F)$, where $F$ is a field, is a zero divisor if and only if $\det(a) = 0$.
(b) Prove that a nonzero matrix $a \in M_n(R)$, where $R$ is a commutative ring, is a zero divisor if $\det(a) = 0$.

7. Let $S = M_n(R)$ where $R$ is a ring with identity. Show that for any ideal $J \subset S$ there is a unique ideal $I \subset R$ such that $J$ is the set of all $n \times n$ matrices with elements in $I$.

8. (a) Let $f : R \to S$ be a ring homomorphism, $I$ an ideal in $R$, $J$ an ideal in $S$. Show that $f^{-1}(J)$ is an ideal in $R$ that contains $\text{Ker}(f)$.
(b) If $f$ is surjective, then $f(I)$ is an ideal in $S$. Is $f$ is not surjective, $f(I)$ need not be an ideal in $S$. 
9. (a) An element $a$ of a ring $R$ is called \textit{nilpotent}, if $a^n = 0$ for some $n \in \mathbb{N}$. Show that if $R$ is a commutative ring, then the set $\text{Nil}(R)$ of all nilpotent elements in $R$ is an ideal (\textit{nilradical of $R$}). Prove that the factor ring $R/\text{Nil}(R)$ has no nonzero nilpotent elements.

(b) Prove that a polynomial $f(X) = a_0 + a_1X + \ldots + a_nX^n \in R[X]$ ($R$ is a commutative ring) is nilpotent if and only if all $a_i$ are nilpotent in $R$.

10. (a) Prove that if $a$ is a nilpotent element of a ring $R$, then the element $1 + a$ is invertible.

(b) Prove that a polynomial $f(X) = a_0 + a_1X + \ldots + a_nX^n \in R[X]$ ($R$ is a commutative ring) is invertible in $R[X]$ if and only if all $a_0$ is invertible and $a_i$ are nilpotent in $R$ for $i \geq 1$. 