## HOMEWORK 9

1. Show that a submodule of a cyclic module over a PID is also cyclic.
2. Let $a$ and $b$ be nonzero elements of a PID $R$. Prove that $R / a R \oplus R / b R \simeq$ $R / c R \oplus R / d R$, where $c$ is a least common multiple and $d$ is a greatest common divisor of $a$ and $b$.
3. Find the invariant factors of the factor group $\mathbb{Z}^{3} / N$, where $N$ is generated by $(-4,4,2),(16,-4,-8),(12,0,-6)$ and $(8,4,2)$.
4. Find the rational canonical form over $\mathbb{Q}$ of the matrix

$$
\left(\begin{array}{ccc}
-2 & 0 & 0 \\
-1 & -4 & -1 \\
2 & 4 & 0
\end{array}\right)
$$

5. Find the Jordan canonical form over $\mathbb{C}$ of the matrix

$$
\left(\begin{array}{cc}
2 i & 1 \\
1 & 0
\end{array}\right)
$$

6. a) Prove that two $2 \times 2$ matrices that are not scalar matrices are similar if and only if they have the same characteristic polynomials.
b) Prove that two $3 \times 3$ matrices are similar if and only if they have the same characteristic and the same minimal polynomials.
7. Show that the minimal polynomial of an $n \times n$-matrix $A$ has the same irreducible divisors as the characteristic polynomial of $A$.
8. Prove that an $n \times n$-matrix $A$ is similar to a diagonal matrix if and only if the elementary divisors of $A$ are all linear polynomials.
9 . Let $A$ be a nilpotent $n \times n$-matrix (that is $A^{N}=0$ for some $N>0$ ). Show that the invariant factors of $A$ are powers of $X$. Prove that $A^{n}=0$.
9. Prove that any $n \times n$-matrix $A$ is similar to its transpose $A^{t}$.
