HOMEWORK 9

1. Show that a submodule of a cyclic module over a PID is also cyclic.

2. Let a and b be nonzero elements of a PID R. Prove that $R/aR \oplus R/bR \simeq R/cR \oplus R/dR$, where c is a least common multiple and d is a greatest common divisor of a and b.

3. Find the invariant factors of the factor group \mathbb{Z}^3/N , where N is generated by (-4, 4, 2), (16, -4, -8), (12, 0, -6) and (8, 4, 2).

4. Find the rational canonical form over \mathbb{Q} of the matrix

$$\begin{pmatrix} -2 & 0 & 0 \\ -1 & -4 & -1 \\ 2 & 4 & 0 \end{pmatrix}$$

5. Find the Jordan canonical form over \mathbb{C} of the matrix

$$\begin{pmatrix} 2i & 1 \\ 1 & 0 \end{pmatrix}$$

6. a) Prove that two 2×2 matrices that are not scalar matrices are similar if and only if they have the same characteristic polynomials.

b) Prove that two 3×3 matrices are similar if and only if they have the same characteristic and the same minimal polynomials.

7. Show that the minimal polynomial of an $n \times n$ -matrix A has the same irreducible divisors as the characteristic polynomial of A.

8. Prove that an $n \times n$ -matrix A is similar to a diagonal matrix if and only if the elementary divisors of A are all linear polynomials.

9. Let A be a nilpotent $n \times n$ -matrix (that is $A^N = 0$ for some N > 0). Show that the invariant factors of A are powers of X. Prove that $A^n = 0$.

10. Prove that any $n \times n$ -matrix A is similar to its transpose A^t .