# HOMEWORK 8

1. Let $R$ be a PID and let $M$ be a torsion finitely generated $R$-module with the invariant factors $d_1|d_2|\dots|d_k$. Set

$$I = \{a \in R \text{ such that } aM = 0\}.$$

Prove that $I = d_k R$.

2. Classify all abelian groups of order 300.

3. Find the rank of the subgroup in $\mathbb{Z}^3$ generated by $(2,-2,0)$, $(0,4,-4)$ and $(5,0,-5)$.

4. Determine the invariant factors of the factor group $\mathbb{Z}^3/N$, where $N$ is generated by $(3,-3,3)$, $(0,6,-12)$ and $(9,0,-9)$.

5. Let $M$ be a finitely generated torsion module over a PID $R$. Prove that $M$ is cyclic if and only if every two elementary divisors of $M$ are relatively prime.

6. Find an example of a non-free submodule $N \subset M$ of a free module $M$ over some ring $R$.

7. Let $n$ be an integer. Prove that every abelian group $A$ with $nA = 0$ has the structure of a $\mathbb{Z}/n\mathbb{Z}$-module.

8. Classify all finite $\mathbb{Z}/n\mathbb{Z}$-modules up to isomorphism. (Hint: Use the classification of finite abelian groups.)

9. Find two non-free modules $M$ and $N$ over $\mathbb{Z}/6\mathbb{Z}$ such that $M \oplus N$ is free.

10. Show that a submodule $N$ of a module $M$ generated by $n$ elements over a PID also can be generated by $n$ elements. (Hint: Consider a surjection $F \to M$ with free $F$ of rank $n$.)