HOMEWORK 7

1. a) Let M be a left R-module. For any $a \in R$ consider the group endomorphism $f_a : M \to M$ defined by $f_a(x) = ax$. Prove that the map $f : R \to \mathrm{End}(M)$ taking $a \in R$ to f_a is a ring homomorphism.
b) Conversely, let M be an abelian group, R a ring and $f : R \to \mathrm{End}(M)$ a ring homomorphism. Show that the formula $a \cdot m = f(a)(m)$ defined a left R-module structure on M.

2. Let M be a (left) R-module generated by one element. Prove that M is isomorphic to the factor module R/I where I is a (left) ideal of R.

3. Let R be a commutative ring. Show that for every two R-modules M and N, the group $\mathrm{Hom}_R(M, N)$ has a structure of an R-module.

4. Let M be a (left) R-module, $N \subset M$ a submodule. Prove that if N and M/N are finitely generated, then so is M.

5. Prove that for any (left) R-module M the groups $\mathrm{Hom}_R(R, M)$ and M are isomorphic. (Hint: show that an R-module homomorphism $f : R \to M$ is uniquely determined by the value $f(1)$.)

6. Let $f : R^n \to R^m$ be a right R-module homomorphism. Show that there is aunique $n \times m$-matrix A such that $f(x) = A \cdot x$ for any $x \in R^n$.

7. Show that $\mathbb{Q}$ is not a free abelian group ($\mathbb{Z}$-module). (Hint: For any $x, y \in \mathbb{Q}$ there are nontrivial $a, b \in \mathbb{Z}$ such that $ax + by = 0$, i.e. any set of at least two elements in $\mathbb{Q}$ is linearly dependent.)

8. Prove that a free finitely generated (left) R-module has a finite basis.

9. Let M be a (left) R-module, $I \subset R$ an ideal. Denote by IM the submodule of M generated by the products am for all $a \in I$ and $m \in M$.
a) Assume that $IM = 0$. Show that M admits a structure of a (left) module over the factor ring R/I.
b) Show that M/IM admits a structure of a (left) module over the factor ring R/I.
c) Prove that if M is a free R-module then M/IM is a free R/I-module. (Hint: Show that if S is a basis for M then the set of cosets $\{s + IM, s \in S\}$ is a basis for M/IM.)
d) Let R be a nonzero commutative ring. Prove that if (left) R-modules R^n and R^m are isomorphic, then $n = m$. Deduce that every two bases for a free finitely generated R-module have the same number of elements. (Hint: Consider modules over the factor ring R/I where I is a maximal ideal of R.)

10. Let A be an abelian group, $f \in \mathrm{End}(A)$. Show that A admits a $\mathbb{Z}[X]$-module structure such that $X \cdot a = f(a)$ for all $a \in A$.