# HOMEWORK 6

1. Prove that the factor ring $R[X]/(X \cdot R[X])$ is isomorphic to $R$.

2. Factor the following polynomials:
a) $X^2 + X + 1$ in $(\mathbb{Z}/2\mathbb{Z})[X]$.
b) $X^3 + X + 1$ in $(\mathbb{Z}/3\mathbb{Z})[X]$.
c) $X^4 + 1$ in $(\mathbb{Z}/5\mathbb{Z})[X]$.

3. Find all monic irreducible polynomials of degree $\leq 3$ in $(\mathbb{Z}/2\mathbb{Z})[X]$.

4. Let $f \in \mathbb{Z}[X]$, $a, b \in \mathbb{Z}$, $a \neq b$. Prove that $b - a$ divides $f(b) - f(a)$. (Hint: $b - a$ divides $b^n - a^n$.)

5. Prove that $X^n - 13$ is irreducible in $\mathbb{Z}[i][X]$. (Hint: Use Eisenstein's Criterion.)

6. Prove that $X^2 + Y^2 - 1$ is irreducible in $\mathbb{Z}[X, Y]$.

7. Let $f$ be a monic polynomial in $\mathbb{Z}[X]$. Prove that if $a \in \mathbb{Q}$ is a root of $f$ then $a \in \mathbb{Z}$. (Hint: Write $a = \frac{b}{c}$ with relatively prime $b$, $c$ and show that $c = \pm 1$.)

8. Prove that $X^3 - 3X + 1$ is irreducible in $\mathbb{Q}[X]$ and $\mathbb{Z}[X]$.

9. Find all roots of $f = X^p - X$ in $(\mathbb{Z}/p\mathbb{Z})[X]$ ($p$ prime) and factor $f$ into a product of irreducibles. (Hint: Use Fermat's Little Theorem.)

10. Determine whether $X^4 + 64$ is irreducible in $\mathbb{Q}[X]$.