# HOMEWORK 5

1. Show that over any field there exist infinitely many non-associate irreducible polynomials.

2. Prove that if $p$ is a prime integer such that $p \equiv 3$ $(mod\ 4)$, then $p$ is a prime element in $\mathbb{Z}[i]$.

3. A *Laurent polynomial* over a field $F$ is a rational function $\frac{f(X)}{X^n}$ where $f(X)$ is a polynomial over $F$ and $n \geq 0$.
a) Show that Laurent polynomials form a subring $R$ of the field of all rational functions $F(X)$.
b) Prove that $R$ is a P.I.D.

4. Show that the ring $R = \mathbb{Z}[X_1, X_2, \dots]$ (the union of $\mathbb{Z}[X_1, X_2, \dots, X_n]$ for all $n$) is a U.F.D.

5. Let $d = \gcd(a, b)$ in a P.I.D. $R$. Prove that $dR = aR + bR$.

6. Find $\gcd(2, 5 + i)$ in $\mathbb{Z}[i]$.

7. Prove that the factor ring $\mathbb{Z}[i]/(1 + i)\mathbb{Z}[i]$ is a field of two elements.

8. Let $f, g \in \mathbb{Q}[X]$ with $fg \in \mathbb{Z}[X]$. Prove that there is $a \in \mathbb{Q}$ such that $af \in \mathbb{Z}[X]$ and $a^{-1}g \in \mathbb{Z}[X]$.

9. Let $S$ be a set of (not necessarily all) prime integers. Let $R$ be the set of all rational numbers $\frac{a}{b}$ such that all prime divisors of $b$ belong to $S$.
a) Prove that $R$ is a subring of $\mathbb{Q}$.
b) Show that $R$ is a P.I.D. Find all irreducible elements in $R$.

10. Let $F$ be a field. Prove that the set $R$ of all polynomials in $F[X]$ whose $X$-coefficient is equal to 0 is a subring of $F[X]$ and that $R$ is not a U.F.D. (Hint: Use $X^6 = (X^2)^3 = (X^3)^2$.)