HOMEWORK 4

1. Prove that every field is a Euclidean domain.

2. Prove that the ideal in $\mathbb{Z}[\sqrt{-5}]$ generated by 2 and $1+\sqrt{-5}$ is not principal.

3. Let $R = \mathbb{Z}[\sqrt{-1}]$ be the ring of Gauss integers.
a) Prove that 3 is prime in R.
b) Prove that 2 is not prime in R. Factor 2 into product of primes.

4. Let R be a P.I.D. and let a be a prime element in R. Prove that R/aR is a field.

5. Prove that an element of a U.F.D. is prime if and only if it is irreducible.

6. Show that $R = \mathbb{Z}[\sqrt{-13}]$ is not a U.F.D. Is R a P.I.D.?

7. Prove that the product of two Noetherian rings is also Noetherian.

8. An integral domain in which every ideal generated by two elements is principal is called a *Bezout domain*. Prove that a ring R is a P.I.D. if and only if R is a Noetherian Bezout domain.

9. Define a chain of rings $R_1 \subset R_2 \subset R_3 \subset \cdots$ inductively as follows: $R_1 = \mathbb{Q}$, $R_{n+1} = R_n[X_n]$ (X_n is a variable). Prove the union of all R_n is a commutative ring that is not Noetherian.

10. Let R be a commutative ring. We define the *product* $I \cdot J$ of two ideals I and J in R as the ideal generated by the products xy for all $x \in I$ and $y \in J$. Let $R = \mathbb{Z}[\sqrt{-5}]$, $I = 2R + (1+\sqrt{-5})R$, $J = 3R + (1+\sqrt{-5})R$. Prove that $I \cdot J$ is the principal ideal $(1+\sqrt{-5})R$.