HOMEWORK 3

1. Let R be the ring of all continuous functions on $\mathbb{R}$. Let I be the subset of all functions in R such that $f(0) = f(1) = 0$. Prove that I is an ideal in R. Is I a prime ideal?

2. Let R be a commutative ring, let I and J be ideals in R and P a prime ideal containing $I \cap J$. Prove that either I or J is contained in P.

3. Let R be a finite commutative ring. Prove that every prime ideal in R is maximal.

4. A commutative ring R is called *local* if it has a unique maximal ideal M.
a) Prove that $R^\times = R \setminus M$.
b) Show that R has no nontrivial idempotents.
c) Prove that the set of all fractions $\frac{n}{m}$, $n, m \in \mathbb{Z}$, m is odd, is a local subring of $\mathbb{Q}$.
d) Determine all n such that the ring $\mathbb{Z}/n\mathbb{Z}$ is local.

5. Prove that every field is isomorphic to the fraction field of itself.

6. Prove that every subfield of $\mathbb{R}$ contains $\mathbb{Q}$.

7. Let $R = \mathbb{Z}[i]$ be the ring of Gauss integers. Find a generator of the intersection of the two principal ideals $2R$ and $(3 + i)R$.

8. Determine the group $\mathbb{Z}[i]^\times$.

9. Prove that $\mathbb{Z}[X]$ is not a PID. (Hint: Consider the ideal generated by X and 2.)

10. Prove that the ring $\mathbb{Z}[\sqrt{2}]$ is Euclidean.