HOMEWORK 2

1. Prove that every (left) ideal of the product $R \times S$ of two rings is a product $I \times J$, where $I \subset R$ and $J \subset S$ are (left) ideals.

2. a) Find all idempotents in $\mathbb{Z}/105\mathbb{Z}$.
b) Prove that $\mathbb{Z}/p^n\mathbb{Z}$, p a prime, has no nontrivial idempotents.

3. Let e be a central idempotent of a ring R (i.e. e commutes with any $a \in R$).
a) Prove that Re is a ring. Is Re a subring of R?
b) Prove that $f = 1 - e$ is also an idempotent. Show that R is isomorphic to the product of two rings $Re \times Rf$.

4. Describe all homomorphisms from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$. In each case describe the kernel and the image.

5. Let R be an integral domain and let $a, b \in R$. Prove that $Ra = Rb$ if and only if $a = bu$ for some $u \in R^\times$.

6. Prove that an element a of a commutative ring R is invertible if and only if a does not belong to any maximal ideal of R.

7. Determine all maximal and prime ideals of $\mathbb{Z}/n\mathbb{Z}$.

8. Let R be a commutative ring. The *radical* $Rad(R)$ of R is the intersection of all maximal ideals in R.
a) Determine $Rad(\mathbb{Z})$ and $Rad(\mathbb{Z}/12\mathbb{Z})$.
b) Prove that $Rad(R)$ consists of all elements $a \in R$ such that $1+ab$ is invertible for all $b \in R$.

9. a) Prove that the nilradical $Nil(R)$ of a commutative ring R is contained in every prime ideal of R.
b) Prove that $Nil(R) \subset Rad(R)$.

10. Let A be an abelian group (written additively). Define a product on the (additive) group $R = \mathbb{Z} \oplus A$ by $(n, a) \cdot (m, b) = (nm, nb + ma)$. Prove that R is a ring. Determine all prime and maximal ideals in R.