HOMEWORK 1

1. Show that if $1 = 0$ in a ring R, then R is the zero ring.

2. Find an example of a subring of $\mathbb{Q}$ different from $\mathbb{Z}$ and $\mathbb{Q}$.

3. Find all zero divisors in $\mathbb{Z}/m\mathbb{Z}$.

4. Prove that the ring $\mathrm{End}(\mathbb{Z})$ is isomorphic to $\mathbb{Z}$.

5. Show that a subring of an integral domain is an integral domain. Is it true that a subring of a field is a field?

6. Prove that a finite integral domain is a field.

7. (a) Find a ring A such that for any ring R there is exactly one ring homomorphism $A \to R$.
(b) Find a ring B such that for any ring R there is exactly one ring homomorphism $R \to B$.

8. (a) Let $f : R \to S$ be a ring homomorphism, I an ideal of R, J an ideal of S. Show that $f^{-1}(J)$ is an ideal of R that contains $Ker(f)$.
(b) Prove that if f is surjective, then $f(I)$ is an ideal of S. Show that if f is not surjective, $f(I)$ need not be an ideal of S.

9. (a) An element a of a ring R is called *nilpotent*, if $a^n = 0$ for some $n \in \mathbb{N}$. Show that if R is a commutative ring, then the set $Nil(R)$ of all nilpotent elements in R is an ideal (called the *nilradical of* R).
(b) Prove that a polynomial $f(X) = a_0 + a_1X + \cdots + a_nX^n \in R[X]$ (R is a commutative ring) is nilpotent if and only if all a_i are nilpotent in R.

10. (a) Prove that if a is a nilpotent element of a ring R, then the element $1+a$ is invertible. (Hint: Use the identity $1 - X^n = (1 - X)(1 + X + \cdots X^{n-1})$.)
(b) Prove that a polynomial $f(X) = a_0 + a_1X + \cdots + a_nX^n \in R[X]$ (R is a commutative ring) is invertible in $R[X]$ if and only if a_0 is invertible and all a_i are nilpotent in R for $i \geq 1$. (Hint: Let $g(X) = b_0 + b_1X + \cdots + b_mX^m \in R[X]$ be the inverse of $f(X)$. Prove first that $a_n^{m+1} = 0$. Then use induction.)