HOMEWORK 9

1. Let $H \subset G = \mathbb{Z} \times \mathbb{Z}$ be the cyclic subgroup generated by $(2, 4)$. Is the factor group G/H isomorphic to $\mathbb{Z}$? (Hint: Consider elements of finite order of G/H.)

2. Determine all subgroups of the alternating group A_4. Notice that A_4 has no subgroup of order 6.

3. a) Let $g \in S_n$ be an odd element. Show that the map $f_n : A_n \to A_n$ given by $f_n(x) = gxg^{-1}$, is an automorphism.
b) Prove that the automorphism f_n of A_n is not inner for $n \geq 3$.

4. Show that the group $\mathbb{Q}/\mathbb{Z}$ cannot be generated by a finite set of elements.

5. a) Let G be an (additively written) abelian group. An element $a \in G$ is called *torsion* is $na = 0$ for some $n \in \mathbb{N}$. Prove that the set G_{tors} of all torsion elements in G is a subgroup of G.
b) Determine $(\mathbb{R}/\mathbb{Z})_{tors}$.
c) Determine $(\mathbb{Q}^{\times})_{tors}$.

6. Prove that A_n is the only nontrivial normal subgroup of S_n if $n \geq 5$. (Hint: Use simplicity of A_n.)

7. Describe all conjugacy classes in A_4 and S_4.

8. A *commutator* of G is an element of the form $xyx^{-1}y^{-1}$ where $x, y \in G$. Let G' be the subgroup of G generated by all commutators. We call G' the *commutator subgroup* of G. Show all the following are true.
a) G' is normal in G.
b) G/G' is abelian.
c) If N is a normal subgroup of G and G/N is abelian then $G' \subset N$.

9. A group G is called *perfect* if the commutator subgroup G' coincides with G. Find all n such that the alternating group A_n is perfect.

10. Let N be a normal subgroup of G and let K be a subgroup of G such that the restriction $K \to G/N$ of the canonical homomorphism $G \to G/N$ to K is an isomorphism. Prove that G is the semidirect product of N and K.