# HOMEWORK 8

1. Let $G$ act on a set $X$. Prove that if $x, y \in X$ satisfy $ax = y$ for some $a \in G$ then $G_y = a \cdot G_x \cdot a^{-1}$.

2. Let $G$ be a group, $a \in G$. Show that the number of elements in the conjugacy class $\{bab^{-1}, b \in G\}$ divides $|G|$.

3. An action of a group $G$ on a set $X$ is called *double transitive* if for any two pairs $(x_1, x_2)$ and $(y_1, y_2)$ of elements of $X$ such that $x_1 \neq x_2$ and $y_1 \neq y_2$ there is $a \in G$ such that $ax_1 = y_1$ and $ax_2 = y_2$. Prove that $|G| \geq |X|^2 - |X|$. (Hint: Consider an action of $G$ on $X \times X$.)

4. (a) Let $H$ be a subgroup of a finite group $G$. Prove that the number of different conjugate subgroups $xHx^{-1}$, $x \in G$, is at most $[G : H]$.
(b) Let $H$ be a subgroup of a finite group $G$. Prove that if $G$ is the union of $xHx^{-1}$ over all $x \in G$, then $H = G$.
(c) Let a finite group $G$ act transitively on a set $X$ consisting of at least two elements. Prove that there exists a $g \in G$ fixing no element of $X$.

5. Determine all Sylow $p$-subgroups of $A_5$.

6. Find the number of all Sylow $p$-subgroups of $S_p$ ($p$ is prime).

7. Prove the following Useful Counting Result. Let $H$ be a subgroup of a finite group $G$. Suppose that $|G|$ does not divide $[G : H]\,!$. Then $G$ contains a nontrivial normal subgroup $N$ such that $N$ is a subgroup of $H$. In particular, $G$ is not simple.

8. Prove that all groups of order $2p^n$ and $3p^n$ ($p$ is prime, $n \geq 1$) are not simple.

9. a) Let $H \subset G$ be a subgroup. Prove that if $H$ is contained in the center of $G$ and the factor group $G/H$ is cyclic, then $G$ is abelian.
(b) Prove that any group of order $p^2$ is abelian ($p$ is prime).

10. Prove that a subgroup $H \subset A_n$ of index $n$ is isomorphic to $A_{n-1}$. (Hint: Consider the action of $A_n$ on $A_n/H$ by left translations.)