HOMEWORK 6

1. Assume that a subset $S \subset G$ of a group G satisfies $gSg^{-1} \subset S$ for all $g \in G$. Prove that the subgroup $\langle S \rangle$ is normal in G.

2. Prove that for any $n \in \mathbb{N}$ there exists a unique cyclic subgroup $H_n \subset \mathbb{Q}/\mathbb{Z}$ of order n.

3. Let $H_n \subset G = \mathbb{Q}/\mathbb{Z}$ be a cyclic subgroup of order n. Prove that G/H_n is isomorphic to G. (Hint: Consider the homomorphism $f : G \to G$, $f(x) = nx$.)

4. Find all elements of finite order in $\mathbb{R}/\mathbb{Z}$.

5. Show that $Aut(\mathbb{Z})$ is a cyclic group of order 2. What is $Inn(\mathbb{Z})$?

6. Prove that for a group G, the subgroup $Inn(G)$ in $Aut(G)$ is normal.

7. Prove that $Aut(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ is isomorphic to the symmetric group S_3. (Hint: Notice that every automorphism of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ permutes all three nonzero elements of the group.)

8. Show that the center of S_n is trivial if $n \geq 3$.

9. (a) A subgroup H in G is called *characteristic*, if $f(H) = H$ for any automorphism f of G. Show that a characteristic subgroup H is normal in G.
(b) Prove that if K is a characteristic subgroup in H and H is a characteristic subgroup in G, then K is characteristic in G.
(c) Prove that if K is a characteristic subgroup in H and H is normal in G, then K is normal in G.

10. Let N be an abelian normal subgroup of a finite group G. Assume that the orders $|G/N|$ and $|Aut(N)|$ are relatively prime. Prove that N is contained in the center of G. (Hint: Consider the conjugation homomorphism $f : G \to Aut(N)$, $f(g)(n) = gng^{-1}$.)