HOMEWORK 5

1. (a) Let $H \subset G$ be a subgroup. Show that H is the image of a homomorphism from some group to G.
(b) Let $N \subset G$ be a normal subgroup. Show that N is the kernel of a homomorphism from G to some group.

2. Let n be a natural number. Show that the map

$$f : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}, \qquad f(a + \mathbb{Z}) = na + \mathbb{Z}$$

is a well defined homomorphism. Find $Ker(f)$ and $Im(f)$.

3. Let $K \subset H \subset G$ be subgroups. Show that if K has finite index in G then $[G : K] = [G : H][H : K]$.

4. Let $H \subset G$ be a subgroup. Show that the correspondence $Ha \mapsto a^{-1}H$ is a bijection between the sets of right and left cosets.

5. Let $f : G \to H$ be a surjective group homomorphism.
(a) Let H' be a subgroup of H. Show that $G' = f^{-1}(H')$ is a subgroup of G. Prove that the correspondence $H' \mapsto G'$ is a bijection between the set of all subgroups of H and the set of all subgroups of G containing $Ker(f)$.
(b) Let H' be a normal subgroup of H. Show that $G' = f^{-1}(H')$ is a normal subgroup of G. Prove that $G/G' \simeq H/H'$ and the correspondence $H' \mapsto G'$ is a bijection between the set of all normal subgroups of H and the set of all normal subgroups of G containing $Ker(f)$.

6. Show that every subgroup of index 2 is normal.

7. Let $H \subset G$ be a subgroup. Suppose that for any $a \in G$ there exists $b \in G$ such that $aH = Hb$. Show that H is normal in G.

8. Show that the group $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$, $m > 1$, can be generated by three elements and cannot be generated by two elements.

9. Let p be an odd prime. Prove that the congruence $x^2 \equiv -1$ $(mod\ p)$ has an integer solution if and only if $p \equiv 1$ $(mod\ 4)$. (Hint: use Fermat's Little Theorem assuming known that the group $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic.)

10. Prove that if a group G contains a subgroup H of finite index, then G contains a normal subgroup of finite index. (Hint: Consider the homomorphism of G to the symmetric group of all left cosets of H in G taking any $x \in G$ to f_x defined by $f_x(aH) = xaH$.)