HOMEWORK 4

1. Prove that the group $\mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z} \times \cdots \times \mathbb{Z}/m_k\mathbb{Z}$ is cyclic if and only if the m_i's are pairwise relatively prime. (Hint: Let n be the least common multiple of the m_i's. Show that $nx = 0$ for any x in the group.)

2. Let K and H be two subgroups of a group G. Prove that the union $K \cup H$ is a subgroup if and only if either $K \subset H$ or $H \subset K$.

3. Show that if K and H are two finite subgroups in G of relatively prime order, then $K \cap H = \{e\}$.

4. Show that if a group G has only finite number of subgroups, then G is finite. (Hint: Note that G is a union of cyclic subgroups.)

5. Prove that if a is an element of a finite group G such that $ord(a) = |G|$, then G is cyclic.

6. Find a non-cyclic group of the smallest order.

7. Find all subgroups of S_3 and determine which ones are normal.

8. Prove that if H is a subgroup of G then $\langle H \rangle = H$.

9. Show that the groups $GL_2(\mathbb{Z}/2\mathbb{Z})$ and S_3 are isomorphic. (Hint: Compare multiplication tables.)

10. Prove that every homomorphism $f : \mathbb{Q} \to \mathbb{Z}/m\mathbb{Z}$ is trivial, i.e. $f(x) = [0]_m$ for all $x \in \mathbb{Q}$.