HOMEWORK 3

1. Prove that for an element a of a group, $a^n \cdot a^m = a^{n+m}$ and $(a^{-1})^n = (a^n)^{-1}$ for every $n, m \in \mathbb{Z}$.

2. Show that $((ab)c)d = a(b(cd))$ for all elements a, b, c, d of a group.

3. Show that if G is a group in which $(ab)^2 = a^2b^2$ for all $a, b \in G$, then G is abelian.

4. Find all elements of order 3 in $\mathbb{Z}/18\mathbb{Z}$.

5. Prove that the composite of two homomorphisms (resp. isomorphisms) is also a homomorphism (resp. isomorphism).

6. Prove that the group $(\mathbb{Z}/9\mathbb{Z})^\times$ is isomorphic to $\mathbb{Z}/6\mathbb{Z}$.

7. Let G be an abelian group and let $a, b \in G$ have finite order n and m respectively. Suppose that n and m are relatively prime. Show that ab has order nm.

8. a) Prove that for every natural integer n the set of all complex n-th roots of unity is a cyclic group of order n with respect to the complex multiplication.

   b) Prove that if G is a cyclic group of order n and k divides n, then G has exactly one subgroup of order k.

9. Prove that if G is a finite group of even order, then G contains an element of order 2. (Hint: Consider the set of pairs (a, a^{-1}).)

10. Find the order of $GL_n(\mathbb{Z}/p\mathbb{Z})$ for a prime integer p.