HOMEWORK 1

1. Let $f : X \to Y$ and $g : Y \to Z$ be two maps. Prove that if f and g are injective (resp. surjective), then so is the composition $g \circ f$.

2. Prove that $(1 + 2 + \cdots + n)^2 = 1^3 + 2^3 + \cdots + n^3$.

3. Prove that 13 divides $14^n - 1$ for any $n \in \mathbb{N}$.

4. Show that if $a^n - 1$ is prime and $n > 1$, then $a = 2$ and n is prime. If $2^n + 1$ is prime, what can you say about n?

5. Find all integer solutions of $93x + 39y = -6$.

6. Let a, b, c be non-zero integers and let $d = gcd(a, b)$. Prove that the equation $ax + by = c$ has a solution x, y in integers if and only if $d|c$. Moreover, if $d|c$ and x_0, y_0 is a solution in integers then the general solution in integers is $x = x_0 + \frac{b}{d}k$, $y = y_0 - \frac{a}{d}k$ for all integers k.

7. Show that if for $a, b \in \mathbb{N}$, ab is a square of an integer and $(a, b) = 1$, then a and b are squares.

8. Prove that if $(a, n) = 1$ and $(b, n) = 1$ then $(ab, n) = 1$

9. Is $2^{10} + 5^{12}$ a prime? (Hint: use the identity $4x^4 + y^4 = (2x^2 + y^2)^2 - (2xy)^2$.)

10. Show that there are infinitely many primes $p \equiv 2 \pmod 3$. (Hint: consider $3p_1p_2 \dots p_n - 1$.)