Homework 4 (Due: We, 11/14)

Problem 1: Suppose that for $n \in \mathbb{N} \cup \{\infty\}$ the set $\Omega_n \subseteq \mathbb{C}$ is a simply connected region with $0 \in \Omega_n$ and $\Omega_n \neq \mathbb{C}$, and that $f_n \colon \mathbb{D} \to \Omega_n$ is the unique conformal map with $f_n(0) = 0$ and $f_n'(0) > 0$. Assume that $\Omega_\infty \subseteq \Omega_n$ for all $n \in \mathbb{N}$ or $\Omega_n \subseteq \Omega_\infty$ for all $n \in \mathbb{N}$.

Show that $\Omega_n \to \Omega_\infty$ in the sense of kernel convergence with respect to 0 if and only if $f_n'(0) \to f_\infty'(0)$ as $n \to \infty$.

Problem 2: Let $f \in \mathcal{S}$ and $f(z) = z + a_2 z^2 + \dots$ be the Taylor expansion of f at 0. Suppose that $M \geq 0$ and $|f(z)| \leq M$ for $z \in \mathbb{D}$ (note that necessarily $M \geq 1$).

a) Show that $|a_2| \leq 2(1 - 1/M)$. Hint: Post-compose f by a suitable conformal map onto a slit domain.

b) Find the largest constant $R_M > 0$ only depending on M (and not on f) such that $B(0, R_M) \subseteq f(\mathbb{D})$.