Homework 3 (Due: We, 11/07)

Problem 1: Let μ be a positive Borel probability measure on the unit circle $\partial\mathbb{D}$, and define

$$p(z) := \int_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta) \quad \text{for } z \in \mathbb{D}. \tag{1}$$

a) Show that p is holomorphic on $\mathbb{D}$, and satisfies $p(0) = 1$ and $\operatorname{Re} p(z) \geq 0$ for $z \in \mathbb{D}$.

b) Fix $r \in (0,1)$, let $f \colon \partial\mathbb{D} \to \mathbb{C}$ be continuous, and define

$$f_r(\xi) := \int_{\partial\mathbb{D}} \operatorname{Re}\left[\frac{\zeta + r\xi}{\zeta - r\xi}\right] f(\zeta)\, d\sigma(\zeta) \quad \text{for } \xi \in \partial\mathbb{D},$$

where σ is Lebesgue measure on $\partial\mathbb{D}$ normalized so that $\sigma(\partial\mathbb{D}) = 1$. Show that f_r is continuous on $\partial\mathbb{D}$ and that $f_r \to f$ uniformly on $\partial\mathbb{D}$ as $r \to 1$.

c) Show that μ in (1) is uniquely determined by p. Hint: Suppose ν is another Borel probability measure on $\partial\mathbb{D}$ that gives the function p in (1) (if we replace μ by ν). Show that then $\int_{\partial\mathbb{D}} f\, d\mu = \int_{\partial\mathbb{D}} f\, d\nu$ for all continuous functions $f \colon \partial\mathbb{D} \to \mathbb{C}$.

Problem 2: Let $\varphi \colon \mathbb{D} \to \mathbb{C}$ be a holomorphic function satisfying $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, $\varphi(0) = 0$, and $\varphi'(0) = c \in (0,1]$.

a) Show that then

$$|\varphi(z) - z| \leq (1-c)|z|\frac{1+|z|}{1-c|z|} \quad \text{for } z \in \mathbb{D}. \tag{2}$$

Prove that this inequality is best possible in the sense that for given $z \in \mathbb{D}$, there exists a function φ satisfying the above hypotheses for which we have equality in (2). Hint: First prove that φ can be written in the form

$$\varphi(z) = z\frac{\psi(z) + c}{1 + c\psi(z)} \quad \text{for } z \in \mathbb{D},$$

where $\psi \colon \mathbb{D} \to \mathbb{C}$ is a holomorphic function satisfying $\psi(\mathbb{D}) \subseteq \mathbb{D}$ and $\psi(0) = 0$.

b) Find the best possible upper bound for $|\varphi''(0)|$ (as a function of c).