**Homework 3** (Due: We, 11/07)

**Problem 1:** Let $\mu$ be a positive Borel probability measure on the unit circle $\partial \mathbb{D}$, and define

\[
 p(z) := \int_{\partial \mathbb{D}} \frac{\zeta + z}{\zeta - z} \, d\mu(\zeta) \quad \text{for } z \in \mathbb{D}.
\]

a) Show that $p$ is holomorphic on $\mathbb{D}$, and satisfies $p(0) = 1$ and $\text{Re} \, p(z) \geq 0$ for $z \in \mathbb{D}$.

b) Fix $r \in (0, 1)$, let $f: \partial \mathbb{D} \to \mathbb{C}$ be continuous, and define

\[
 f_r(\xi) := \int_{\partial \mathbb{D}} \text{Re} \left[ \frac{\xi + r \zeta}{\xi - r \zeta} \right] f(\zeta) \, d\sigma(\zeta) \quad \text{for } \xi \in \partial \mathbb{D},
\]

where $\sigma$ is Lebesgue measure on $\partial \mathbb{D}$ normalized so that $\sigma(\partial \mathbb{D}) = 1$. Show that $f_r$ is continuous on $\partial \mathbb{D}$ and that $f_r \to f$ uniformly on $\partial \mathbb{D}$ as $r \to 1$.

c) Show that $\mu$ in (1) is uniquely determined by $p$. Hint: Suppose $\nu$ is another Borel probability measure on $\partial \mathbb{D}$ that gives the function $p$ in (1) (if we replace $\mu$ by $\nu$). Show that then $\int_{\partial \mathbb{D}} f \, d\mu = \int_{\partial \mathbb{D}} f \, d\nu$ for all continuous functions $f: \partial \mathbb{D} \to \mathbb{C}$.

**Problem 2:** Let $\varphi: \mathbb{D} \to \mathbb{C}$ be a holomorphic function satisfying $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, $\varphi(0) = 0$, and $\varphi'(0) = c \in (0, 1]$.

a) Show that then

\[
 |\varphi(z) - z| \leq (1 - c)|z| \frac{1 + |z|}{1 - c|z|} \quad \text{for } z \in \mathbb{D}.
\]

Prove that this inequality is best possible in the sense that for given $z \in \mathbb{D}$, there exists a function $\varphi$ satisfying the above hypotheses for which we have equality in (2). Hint: First prove that $\varphi$ can be written in the form

\[
 \varphi(z) = z \frac{\psi(z) + c}{1 + c\psi(z)} \quad \text{for } z \in \mathbb{D},
\]

where $\psi: \mathbb{D} \to \mathbb{C}$ is a holomorphic function satisfying $\psi(\mathbb{D}) \subseteq \mathbb{D}$ and $\psi(0) = 0$.

b) Find the best possible upper bound for $|\varphi''(0)|$ (as a function of $c$).