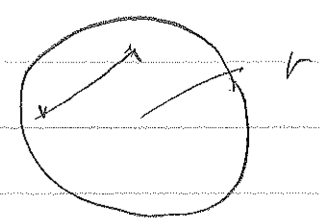
(62) Proof: Estimates.

1. $|h(z_1)-h(z_2)| \le \max\limits_{|u|\le r} |h'(u)| \cdot |z_1-z_2|$

$h\in H(\mathbb{D})$,

$z_1,z_2\in \overline{B}(0,r)$, $r<1$.



2. $|e^u - e^v| \le |u-v|$, $u,v\in\mathbb{C}$,

$\operatorname{Re} u, \operatorname{Re} v \le 0$

3. $\varphi\in \operatorname{Aut}(\mathbb{D})$. Then

$$|\varphi'(z)| \underset{\text{Schwarz-Pick}}{\le} \frac{1-|\varphi(z)|^2}{1-|z|^2} \le \frac{1}{1-|z|}, \quad z\in\mathbb{D}.$$

---

i) $s\le t$: $f_t\circ\varphi_{s,t} = f_s$; so

$\varphi_{s,t}(0)=0$, $\varphi_{s,t}(\mathbb{D})\subseteq\mathbb{D}$ $|\varphi_{s,t}(z)|\le|z|$

$f_t'(\underbrace{\varphi_{s,t}(0)}_{0})\cdot\varphi_{s,t}'(0) = f_s'(0)$ equiv

$e^t\cdot\varphi_{s,t}'(0) = e^s$; so $\varphi_{s,t}'(0)=e^{s-t}\le 1$.

Define

$$\Phi_{s,t}(z) = \log\left(\frac{z}{\varphi_{s,t}(z)}\right)$$

$$= \log\frac{z}{e^{s-t}z+\cdots} = \log(e^{t-s}+\cdots)$$

$$=(t-s)+\cdots \quad \text{holomorphic in } \mathbb{D}$$

$\Phi_{s,t}(0)=t-s$

$\left|\frac{z}{\varphi_{s,t}(z)}\right|\ge 1$ so $\operatorname{Re}\Phi_{s,t}\ge 0$, and

$\frac{1}{t-s}\Phi_{s,t}\in\mathcal{P}$.

(63) Hence (Thm. 4.9)

$$\left|\Phi_{s,t}(z)\right| \leq |t-s| \frac{1+|z|}{1-|z|} \leq |t-s| \frac{2}{1-|z|}.$$

$$\varphi_{s,t}(z) = z \cdot e^{-\phi_{s,t}(z)}, \quad \operatorname{Re} \phi_{s,t}(z) \geq 0.$$

$$\left|\varphi_{s,t}(z) - z\right| = |z| \left| e^{-\phi_{s,t}(z)} - e^{0} \right|$$

$$\leq |z| \cdot \left|\phi_{s,t}(z)\right| \leq |t-s| \frac{2|z|}{1-|z|}.$$

ii) $$\left|f_t(z) - f_s(z)\right| = \left|f_t(z) - f_t(\varphi_{s,t}(z))\right|$$

$$\leq \max_{|u| \leq |z|} \left|f_t'(u)\right| \cdot \left|z - \varphi_{s,t}(z)\right|$$

$$\overset{\text{Koebe + (i)}}{\leq} e^t \cdot \frac{1+|z|}{(1-|z|)^3} \cdot |t-s| \frac{2|z|}{(1-|z|)}$$

$$\leq e^t |t-s| \frac{4|z|}{(1-|z|)^4}.$$

iii) $$\left|\varphi_{s,u}(z) - \varphi_{t,u}(z)\right| =$$

$$\left|\varphi_{t,u}(\varphi_{s,t}(z)) - \varphi_{t,u}(z)\right|$$

$$\leq \max_{|u| \leq |z|} \left|\varphi_{t,u}'(u)\right| \cdot \left|\varphi_{s,t}(z) - z\right|$$

$$\overset{(i)}{\leq} \frac{1}{1-|z|} \cdot |t-s| \cdot \frac{2}{1-|z|} = |t-s| \cdot \frac{2}{(1-|z|)^2}.$$

iv) $$\left|\varphi_{s,t}(z) - \varphi_{s,u}(z)\right| = \left|\varphi_{s,t}(z) - \varphi_{t,u}(\varphi_{s,t}(z))\right|$$

$$\overset{(i)}{\leq} |u-t| \cdot \frac{2|w|}{1-|w|} \leq |u-t| \cdot \frac{2|z|}{1-|z|}.$$

$$w = \varphi_{s,t}(z) \qquad |w| \leq |z|$$

64 <u>Def. 4.11.</u> $\Omega \subseteq \mathbb{C}$ region, $I \subseteq \mathbb{R}$ interval.
$HL(\Omega \times I)$ is the set of all function
$f : \Omega \times I \to \mathbb{C}$ s.t.

i) $f(\cdot, t)$ is holomorphic on Ω for all $t \in I$,

"uniform Lip. property"

ii) $f(z, \cdot)$ is uniformly Lipschitz on compact sets, i.e.,

whenever $K \subseteq \Omega$ comp., $J \subseteq I$ comp. interval,
then there ex. $L \geq 0$ s.t.

$$|f(z,s) - f(z,t)| \leq L|s-t| \text{ for all } z \in K,\ s,t \in J.$$

$\{f_t\}$ normalized Loewner chain on $I = [a, \infty)$

Then $(z,t) \mapsto f_t(z) \in HL(\mathbb{D} \times [a, \infty))$

$(z,t) \mapsto \varphi_{s,t}(z) \in HL(\mathbb{D} \times [s, \infty))$

$(z,s) \mapsto \varphi_{s,t}(z) \in HL(\mathbb{D} \times [a, t])$

$\varphi_{s,t} = f_t^{-1} \circ f_s$

<u>Prop. 4.12.</u> $\Omega \subseteq \mathbb{C}$ region, $I \subseteq \mathbb{R}$ interval,
$f \in HL(\Omega \times I)$.

Then:

i) f is continuous on $\Omega \times I$,

ii) there ex. a set $E \subseteq I$, $|E| = 0$ s.t.

1-dim. Lebesgue measure

$\frac{\partial f}{\partial t}(z,t)$ ex. for all $z \in \Omega$, $t \in I \setminus E$;

moreover,

$\frac{\partial f}{\partial t}(\cdot, t)$ is holomorphic on Ω for all $t \in I \setminus E$,

(65) $\frac{\partial f}{\partial t}$ is measurable and unif. bdd. on compact subsets (i.e., whenever $K \subseteq \Omega$ comp., $J \subseteq I$ comp. interval, then there ex. $M \geq 0$ s.t.

$$\left|\frac{\partial f}{\partial t}(z,t)\right| \leq M \quad \text{for all } z \in K,\ t \in J \setminus E.$$

iii) f is differentiable at each point $(z,t) \in \Omega \times I \setminus E$; more precisely,

$$f(z',t') = f(z,t) + \frac{\partial f}{\partial z}(z,t)(z'-z) + \frac{\partial f}{\partial t}(z,t)(t'-t) + o(|z'-z| + |t'-t|) \text{ as } (z',t') \longrightarrow (z,t).$$

iv) $\frac{\partial^n f}{\partial z^n} \in HL(\Omega \times I)$ for all $n \in \mathbb{N}$; moreover,

$$\frac{\partial}{\partial t}\left(\frac{\partial^n f}{\partial z^n}\right)(z,t) = \frac{\partial^n}{\partial z^n}\left(\frac{\partial f}{\partial t}\right)(z,t) \qquad (*)$$

for all $(z,t) \in \Omega \times I \setminus E$

v) Let $z_0 \in \Omega$ and

$$f(z,t) = \sum_{n=0}^{\infty} a_n(t)(z-z_0)^n$$

be the Taylor expansion of $f(\cdot,t)$ at z_0. Then for each $n \in \mathbb{N}_0$, a_n is unif. Lipschitz on comp. intervals $J \subseteq I$; moreover, $\dot{a}_n(t) = \frac{da_n}{dt}(t)$ ex. for all $t \in I \setminus E$, and for $t \in I \setminus E$ the function $\frac{\partial f}{\partial t}(\cdot,t)$ has the Taylor expansion

$$\frac{\partial f}{\partial t}(z,t) = \sum_{n=0}^{\infty} \dot{a}_n(t)(z-z_0)^n. \qquad (**)$$

(66) <u>Proof</u>: i)

$|f(z',t') - f(z,t)|$

$\leq \underbrace{|f(z',t') - f(z',t)|}_{\text{unif. small}} + \underbrace{|f(z',t) - f(z,t)|}_{\text{small}}$

small if $|z'-z|$ & $|t'-t|$ small.

ii) Pick a sequence $\{u_k\}$ in Ω of distinct pts. s.t. $\{u_k\}$ has a limit point in Ω (e.g. $u_k = u_0 + \delta/k$, $u_0 \in \Omega$, $\delta > 0$ small).

each function $t \longmapsto f(u_k, t)$ is loc. Lipschitz on I, and so diff. a.e. on I. So there ex. a set $E_k \subseteq I$ with $|E_k| = 0$ s.t. $\frac{\partial f}{\partial t}(z_k, t)$ ex. for each $t \in I \setminus E_k$.

Let $E = \bigcup_{k \in \mathbb{N}} E_k \subseteq I$ $\cup$ endpts. of I. Then $|E| = 0$.

<u>Claim</u> $\frac{\partial f}{\partial t}(z,t)$ ex. for all $(z,t) \in \Omega \times I \setminus E$.

It suffices to show that if $\{\delta_n\}$ is a seq. in $\mathbb{R}$ with $\delta_n \neq 0$ and $\delta_n \to 0$, then

(*) $\lim_{n\to\infty} \frac{f(z, t+\delta_n) - f(z,t)}{\delta_n}$ exists (then the limit is ind. of $\{\delta_n\}$)

Define

$F_n(z') := \frac{f(z', t+\delta_n) - f(z',t)}{\delta_n}$ for $z' \in \Omega$.

Then $\{F_n\}$ is a seq. of holomorphic functions on Ω that are loc. unif bdd.

(67) on Ω, and so form a normal family

$$F_n(a_k) \longrightarrow \frac{\partial f}{\partial t}(a_k, t) \quad \text{as } n \longrightarrow \infty$$

for each $k \in \mathbb{N}$.

By Vitali's Thm. 4.8.

$\{F_n(z')\}$ converges for each $z' \in \Omega$, and so also for $z' = z$; so the limit (*) exists.

So $\frac{\partial f}{\partial t}(z,t)$ ex. for all $(z,t) \in \Omega \times I \setminus E$.

actually, by Vitali

$$F_n \longrightarrow \frac{\partial f}{\partial t}(\cdot, t) \quad \text{loc. unif. on } \Omega \quad (t \in I \setminus E \text{ fixed}).$$

So $\frac{\partial f}{\partial t}(\cdot, t)$ is holomorphic on Ω (Weierstrass)

$\frac{\partial f}{\partial t}$ is measurable as a pointwise limit of continuous functions, and the boundedness property follows from unif. Lipschitz property of f.

iii) $(z,t) \in \Omega \times I \setminus E$ arb.

$(z_n, t_n) \in \Omega \times I \longrightarrow (z,t)$ as $n \longrightarrow \infty$

with $t_n \neq t$

$$\frac{f(\cdot, t_n) - f(\cdot, t)}{t_n - t} \longrightarrow \frac{\partial f}{\partial t}(\cdot, t),$$

loc. unif. on Ω, and so

$$\frac{f(z_n, t_n) - f(z_n, t)}{t_n - t} - \frac{\partial f}{\partial t}(z_n, t) = o(1)$$

$$(t_n - t) \to 0$$

(6d) So

$f(z_n,t_n) - f(z,t) =$ $\quad f(\cdot,t)$ holomorphic

$f(z_n,t_n) - f(z_n,t) + f(z_n,t) - f(z,t)$

$$= \frac{\partial f}{\partial t}(z_n,t)\cdot(t_n-t) + o(|t_n-t|)$$

$$+ \frac{\partial f}{\partial z}(z,t)(z_n-z) + o(|z_n-z|)$$

$$= \frac{\partial f}{\partial t}(z,t)(t_n-t) + \frac{\partial f}{\partial z}(z,t) + o(|t_n-t| + |z_n-z|).$$

iv) $n \in \mathbb{N}$.

$\frac{\partial^n f}{\partial z^n}(\cdot,t)$ holomorphic on Ω for $t \in I$.

Suppose $\bar{B}(a,R) \subseteq \Omega$, $\gamma(t) = a + Re^{it}$.

Then

$$\frac{\partial^n f}{\partial z^n}(z,t) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\xi,t)}{(\xi-z)^{n+1}}\,d\xi$$

for $z \in B(a,R)$, $t \in I$

by the Residue Thm.

If $z \in \bar{B}'(a,R/2)$, $s,t \in J \subseteq I$ comp.,

then by the unif. Lip. property of f:

$$\left|\frac{\partial^n f}{\partial z^n}(z,s) - \frac{\partial^n f}{\partial z^n}(z,t)\right| \le L|s-t|$$

$$\le \frac{n!}{2\pi}\cdot 2\pi R \sup_{\xi\in\partial B(a,R)} |f(\xi,s) - f(\xi,t)| \frac{1}{(R/2)^{n+1}}$$

$\le C|s-t|$; so $t \longmapsto \frac{\partial^n f}{\partial z^n}(z,t)$ is

unif. Lip. on $\bar{B}(a,R/2)\times J$.

The unif. Lip. property of $\frac{\partial^n f}{\partial z^n}$ follows

from a covering argument.

(69) $f \in I \setminus E$, $\{\delta_n\}$ seq. in $\mathbb{R}$ with $\delta_n \neq 0$, $\delta_n \to 0$.

Then

$$\frac{f(\cdot, t+\delta_n) - f(\cdot, t)}{\delta_n} \longrightarrow \frac{\partial f}{\partial t}(\cdot, t)$$

loc. unif. on Ω; hence for $z \in B(a, R)$

$$\frac{1}{\delta_n}\left[\frac{\partial^n f}{\partial z^n}(z, t+\delta_n) - \frac{\partial^n f}{\partial z^n}(z,t)\right]$$

$$= \frac{n!}{2\pi i}\int_\gamma \frac{f(\xi, t+\delta_n) - f(\xi, t)}{\delta_n} \frac{d\xi}{(\xi - z)^{n+1}}$$

$\downarrow$

$\frac{\partial f}{\partial t}(\xi, t)$ unif. for $\xi \in \partial B(a, R)$

$$\longrightarrow \frac{n!}{2\pi i}\int_\gamma \frac{\partial f(\xi, t)}{\partial t} \frac{d\xi}{(\xi - z)^{n+1}}$$

$$= \frac{\partial^n}{\partial z^n}\left(\frac{\partial f}{\partial t}\right)(z,t).$$

This shows that $\frac{\partial}{\partial t}\left(\frac{\partial^n f}{\partial z^n}\right)(z,t)$ exists, and (*) holds.

v.)

$$a_n(t) = \frac{1}{n!}\frac{\partial^n f}{\partial z^n}(z_0, t) \quad \text{for } t \in I,$$

so a_n is unif. Lip. on compact $J \subseteq I$ for each $n \in \mathbb{N}$ by (iv).

Moreover,

$$\dot{a}_n(t) = \frac{1}{n!}\frac{\partial}{\partial t}\left(\frac{\partial^n f}{\partial z^n}\right)(z_0, t) =$$

70 $= \frac{1}{n!} \frac{\partial^n}{\partial z^n}\left(\frac{\partial f}{\partial t}\right)(z_0,t)$ for $t \in I \setminus E$.

So for $t \in I\setminus E$ the n-th Taylor coeff. of the holomorphic function at z_0 $\frac{\partial f}{\partial t}(\cdot,t)$ is given by $\dot{a}_n(t)$; (**) follows. ∎

Thm. 4.13 (Main Thm. of Loewner Theory)

Let $\{f_t\}_{t\in I}$, $I = [a,\infty)$, be a normalized Loewner chain, $\varphi_{s,t} := f_t^{-1} \circ f_s$, $f(z,t) := f_t(z)$.

Then there ex. $E \subseteq I$, $|E| = 0$, s.t.

a) $V(z,t) := \lim_{\varepsilon\to 0^+} \frac{\varphi_{t,t+\varepsilon}(z) - z}{\varepsilon}$ (*)

ex. for all $z \in \mathbb{D}$, $t \in I\setminus E$.

b) $\frac{\partial f}{\partial t}(z,t)$ ex. for all $z \in \mathbb{D}$, $t \in I \setminus E$, and

$$\boxed{\frac{\partial f}{\partial t}(z,t) = - V(z,t)\,\frac{\partial f}{\partial z}(z,t)}$$

Loewner-Kufarev eq.

Moreover, $V(z,t)$ has the following properties:

i) $V(\cdot,t)$ is holomorphic on $\mathbb{D}$ for each $t \in I\setminus E$,

ii) V is measurable on $\Omega \times I$, and has the unif. bdd. property: whenever $K \subseteq \mathbb{D}$, $J \subseteq I$ comp., then there ex. $M \geq 0$ s.t. $|V(z,t)| \leq M$ for $(z,t) \in K \times J\setminus E$.

(71) iii) V can be written in the form
$V(z,t) = -z\, p(z,t)$, where
$p(\cdot,t) \in \mathcal{P}$ for $t \in I \setminus E$, i.e.,
$p(\cdot,t)$ is holomorphic on $\mathbb{D}$, $\operatorname{Re} p(\cdot,t) \geq 0$,
and $p(0,t) = 1$.

Proof: $f \in HL(\mathbb{D} \times I)$, so there ex.
$E \subseteq I$, $|E| = 0$, s.t.
$\frac{\partial f}{\partial t}(z,t)$ ex. for $(z,t) \in \mathbb{D} \times I \setminus E$.

Pick $(z,t) \in \mathbb{D} \times I \setminus E$ and $\varepsilon > 0$. Then
$f_{t+\varepsilon}(\varphi_{t,t+\varepsilon}(z)) = f_t(z)$. equiv.
$f(\varphi_{t,t+\varepsilon}(z), t+\varepsilon) = f(z,t)$.

Differentiating w.r.t. $\varepsilon > 0$ and setting $\varepsilon = 0$ we obtain by the chain rule

$$0 = \frac{d}{d\varepsilon} f(\varphi_{t,t+\varepsilon}(z), t+\varepsilon)\Big|_{\varepsilon=0}$$

$$= \frac{\partial f}{\partial z}(z,t) \cdot \frac{\partial \varphi_{t,t+\varepsilon}(z)}{\partial \varepsilon}\Big|_{\varepsilon=0} + \frac{\partial f}{\partial t}(z,t).$$

Actually, this is true for any sublimit of

$$\frac{\partial \varphi_{t,t+\varepsilon}(z)}{\partial \varepsilon}\Big|_{\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{\varphi_{t,t+\varepsilon}(z) - z}{\varepsilon}$$

Since $\frac{\partial f}{\partial z}(z,t) \neq 0$ (f_t is conformal!)

(72) such a sublimit is unique;
since $\varepsilon \longmapsto \varphi_{t,t+\varepsilon}$ is Lipschitz,
the existence of

$$V(z,t) = \lim_{\varepsilon \to 0} \frac{\varphi_{t,t+\varepsilon}(z) - z}{\varepsilon} \quad \text{follows}$$

$(z \in \mathbb{D},\ t \in I \setminus E)$,

and $\frac{\partial f}{\partial z}(z,t)\, V(z,t) = \frac{\partial f}{\partial t}(z,t)$

equiv. to Loewner - Kufarev eq.

By Vitali:

$$\frac{\varphi_{t,t+\varepsilon_n}(z) - z}{\varepsilon_n} \longrightarrow V(z,t)$$

loc. unif. for $z \in \mathbb{D}$ where $t \in I \setminus E$ fixed.

So $V(\cdot,t)$ holomorphic on $\mathbb{D}$;
V is measurable (ptw. limit of cont. functions), and has the unif. bddness property as follows from the unif. Lip. property of $(z,t) \longmapsto \varphi_{s,t}(z)$

$$f(z,t) = a_0(t) + a_1(t)\, z + a_2(t)\, z^2 + \cdots$$

Taylor exp.

$$V(z,t) = c_0(t) + c_1(t)\, z + \cdots$$

for fixed $t \in I \setminus E$.

$$a_0(t) \equiv w_0, \quad a_1(t) \equiv e^t$$

$$\frac{\partial f}{\partial z}(z,t) = a_1(t) + 2a_2(t)\, z + \cdots$$

(Prop. 4.12.iv)

$$\frac{\partial f}{\partial t}(z,t) = \dot{a}_1(t)\, z + \dot{a}_2(t)\, z^2 + \cdots$$

(73) So

$$\frac{\partial f}{\partial z}(z,t) = \dot{a}_1 z + \dot{a}_2 z^2 + \cdots$$

$$= -V(z,t)\,\frac{\partial f}{\partial z}(z,t)$$

$$= -(c_0 + c_1 z + \cdots)(a_1 + 2a_2 z + \cdots)$$

So $0 = -c_0 a_1 = c_0 \cdot e^{t}$ equiv.

$c_0 = 0$

$\dot{a}_1 = -c_1 a_1$ equiv. $e^{t} = c_1(t)\cdot e^{t}$

equiv. $c_1(t) = -1$, i.e.

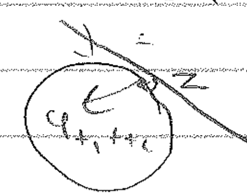
$$V(z,t) = -z\,p(z,t),$$

where $p(\cdot,t)$ holomorphic and $p(0,t) = 1$.

By Schwarz's Lemma:

$$|\varphi_{t,t+\varepsilon}(z)| \le |z| \; ; \text{ so for } z \ne 0$$

$$\mathrm{Re}\left(\frac{\varphi_{t,t+\varepsilon}(z) - z}{z}\right) \le 0,$$



and

Re ≤ 0

$$\mathrm{Re}(p(z,t))$$

$$= -\mathrm{Re}\left(\frac{V(z,t)}{z}\right) = -\lim_{\varepsilon \to 0^+} \mathrm{Re}\left[\frac{\varphi_{t,t+\varepsilon}(z) - z}{\varepsilon z}\right]$$

≥ 0, $z \ne 0$ (also true for $z = 0$:

$\mathrm{Re}\,p(0,t) = 1$). □

(74) <u>Cor. 4.14.</u> $\{f_t\}$ normalized Loewner chain on $I = [a, \infty)$, $\varphi_{s,t} = f_t^{-1} \circ f_s$, $E \subseteq I$, $|E| = 0$, $V(z,t)$ as in Thm. 4.13.

Then

i) $$V(z,t) := \lim_{\varepsilon \to 0^+} \frac{\varphi_{t,t+\varepsilon}(z) - z}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0^+} \frac{\varphi_{t-\varepsilon,t}(z) - z}{\varepsilon} \quad \text{for } z \in \mathbb{D}, t \in I \setminus E$$

ii) $$\frac{\partial \varphi_{s,t}(z)}{\partial t} = V(\varphi_{s,t}(z), t)$$

for $z \in \mathbb{D}$, $t \in [s, \infty) \setminus E$, $s \in I$

(left-hand derivative for $t = s$!)

iii) $$\frac{\partial \varphi_{s,t}(z)}{\partial s} = -\varphi_{s,t}'(z) \cdot V(z,s),$$

for $z \in \mathbb{D}$, $s \in [a,t] \setminus E$, $t \in I$,

(right-hand derivative for $s = t$!)

Existence of limits part of the statement!

<u>Proof:</u> W $f_t \circ \varphi_{s,t} = f_s$, $a \le s \le t < \infty$,

$$\dot{f}_t(z) = -V(z,t) \cdot f_t'(z), \quad z \in \mathbb{D}, t \in I \setminus E.$$

$t \in I \setminus E$: $\varepsilon \ge 0$: $\to 0$:

i $f_t \circ \varphi_{t-\varepsilon,t}(z) = f_{t-\varepsilon}(z)$

Differentiating w.r.t. ε and setting $\varepsilon = 0$:

$$f_t'(\cancel{f_{t+\varepsilon}}(z)) \cdot \underbrace{\frac{d}{d\varepsilon} \varphi_{t-\varepsilon,t}(z)\Big|_{\varepsilon=0}}_{\text{true for any sublimit!}} = -\dot{f}_t(z)$$

(75) $= V(z,t) \cdot f_t'(z)$;

Hence

$$\lim_{\varepsilon \to 0^+} \frac{\varphi_{t-\varepsilon,t}(z) - z}{\varepsilon}$$ ex. and is equal

to $= V(z,t)$.

ii) $s \in I$, $t \in [s,\infty) \setminus E$: $f_t \circ \varphi_{s,t} = f_s$

Differentiating w.r.t. t a.e.:

equiv.

$$f(\varphi_{s,t}(z), t) = f(z,s)$$

$$\Rightarrow \quad f_t'(\varphi_{s,t}(z)) \cdot \frac{\partial \varphi_{s,t}}{\partial t}(z) + \dot{f}_t(\varphi_{s,t}(z)) = 0$$

equiv.

$$\frac{\partial \varphi_{s,t}(z)}{\partial t} = -\frac{\dot{f}_t \circ \varphi_{s,t}}{f_t' \circ \varphi_{s,t}} = V(\varphi_{s,t}(z), t).$$

iii) $t \in I$, $s \in [a,t] \setminus E$

$$f_t \circ \varphi_{s,t} = f_s$$

Differentiate w.r.t. to s a.e.

$$f(\varphi_{s,t}(z), t) = f(z,s)$$

$$\Rightarrow \quad f_t'(\varphi_{s,t}(z)) \cdot \frac{\partial \varphi_{s,t}}{\partial s}(z) = \dot{f}_s(z)$$

$$= -V(z,s) \cdot f_s'(z) = -V(z,s) \cdot \varphi_{s,t}'(z) \cdot f'(\varphi_{s,t}(z))$$

Differentiate w.r.t. z,

$$f_t' \circ \varphi_{s,t}(z) \cdot \varphi_{s,t}'(z) = f_s'(z)$$

$$\Rightarrow \quad \frac{\partial \varphi_{s,t}(z)}{\partial s} = -\varphi_{s,t}'(z) \cdot V(z,s).$$

In all cases, existence of limits follows from

(76) uniqueness of sublimits. □

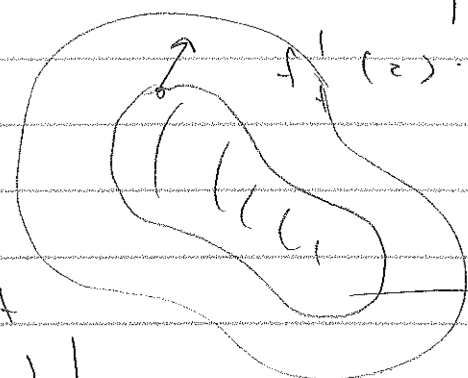
4.15. Geometric interpretations

$$\dot f_t(z) = -V(z,t)\cdot f_t'(z) = z\,p(z,t)\, f_t'(z)$$

$-V(z,t) = z\,p(z,t)$

$$V(z,t) = -z\,p(z,t)$$

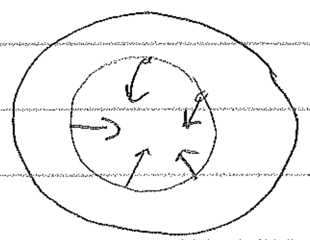
f_t



So $\dot f_t(z)$ points out of $f_t(\bar B(0,|z|))$

$$\frac{\partial \varphi_{s,t}(z)}{\partial t} = V(\varphi_{s,t}(z), t);$$

so $t \longmapsto \varphi_{s,t}(z)$ integral curve of vector field $V(z,t)$



$z \longmapsto \varphi_{s,t}(z)$ h.p from time s to time t shrinks $\mathbb{D}$ for large t:
$\varphi_{s,t}'(0) = e^{s-t}$

$$\frac{\partial \varphi_{s,t}(z)}{\partial s} = -\varphi_{s,t}'(z)\cdot V(z,s)$$

$\varphi_{s,t}$; $\varphi_{s,t}(z)$

$\varphi_{s-\varepsilon,t}(z) \simeq \varphi_{s,t}(z) + \varepsilon\, \varphi_{s,t}'(z)\cdot V(z,s)$

z ; $\varphi_{s,t}$

$\varphi_{s-\varepsilon,s}(z) \simeq z + \varepsilon V(z,s)$

(77) 5. Existence results for Loewner chains and applications

Prop. 5.1. $\{f_t^n\}$ sequence of normalized Loewner chains on $I=[a,\infty)$, $f_t^n(0)=w_0\in\mathbb{C}$, $(f_t^n)'(0)=e^t$, $t\in I$. Then $\{f_t^n\}$ subconverges to a Loewner chain as $n\to\infty$; more precisely, there ex. a sequence $\{n_k\}$ with $n_k\to\infty$ as $k\to\infty$ and a normalized Loewner chain $\{f_t\}_{t\in I}$ s.t.

$$f_t^{n_k}\longrightarrow f_t \quad \text{loc. unif. on } \mathbb{D} \text{ as } k\to\infty$$
for all $t\in[a,\infty)$.

Proof: Let $z_l := 1/l$, $l\geq 2$. wlog $w_0=0$
$z_l\to 0\in\mathbb{D}$ as $l\to\infty$.
For fixed $l\in\mathbb{N}$ the maps
$$t\in[a,\infty)\longrightarrow f_t^n(z_l),\quad n\in\mathbb{N},$$
are unif. Lipschitz (cf. Lem. 4.10) and unif. bdd., (Koebe) on compact set $J\subseteq I$.
In particular, the family $\{t\mapsto f_t^n(z_l)\}_{n\in\mathbb{N}}$ is equicont. and unif. bdd. at each $t_0\in I$.
Hence by Arzela-Ascoli there ex. a subsequence that converges loc. unif. on I and in particular pointwise on I.

(78) Applying this successively for each $l = 2, 3, \ldots$ and passing to a diagonal subsequence, we find a sequence $\{n_k\}$ in $\mathbb{N}$ with $n_k \to \infty$ as $k \to \infty$ s.t.

$$\{f_t^{n_k}(z_l)\} \text{ converges as } k \to \infty \text{ for all } t \in I,\ l \geq 2.$$

Fix $t \in I$. Then $e^{-t} f_t^{n_k} \in \mathcal{S}$, and so these functions form a normal family. Since we have pointwise convergence at each $z_l \in \mathbb{D}$, $l \geq 2$, by Vitali $f_t^{n_k}$ converges loc. unif. on $\mathbb{D}$ to some limit function $f_t \in H(\mathbb{D})$;

so

$$f_t^{n_k} \longrightarrow f_t \text{ loc. unif. on } \mathbb{D} \text{ as } k \to \infty \text{ for each } t \in I.$$

It suffices to show that $\{f_t\}_{t \in I}$ is a normalized Loewner chain.

$$f_t(0) = \lim_{k \to \infty} f_t^{n_k}(0) = w_0 = 0$$

$$(1)\quad f_t'(0) = \lim_{k \to \infty} (f_t^{n_k})'(0) = e^t \neq 0 \quad \text{for } t \in I.$$

By Hurwitz f_t is a conformal map $f: \mathbb{D} \leftrightarrow \Omega_t := f_t(\mathbb{D})$.

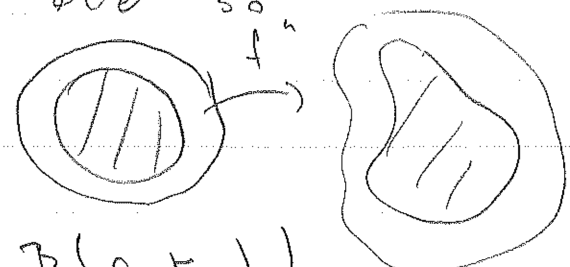
If $s, t \in I$, $s \leq t$, then

$$\begin{array}{ccccc} \Omega_s^{n_k} := f_s^{n_k}(\mathbb{D}) & \longrightarrow & \Omega_s & & \text{w.r.t. } w_0 \\ \cap\!\mid & & & & \text{So} \\ \Omega_t^{n_k} := f_t^{n_k}(\mathbb{D}) & \longrightarrow & \Omega_t & & \Omega_s \subseteq \Omega_t. \ (2) \end{array}$$

79 (1)+(2) imply the Lipschitz estimates for $t \longmapsto f_t(z)$ as in Lem. 4.10. (ii) ($\varphi_{t,s} := f_t^{-1} \circ f_s$ is def. etc.)
Hence $f_{t_n} \longrightarrow f_t$ loc. unif. on $\mathbb{D}$ whenever $t_n \in I \longrightarrow t \in I$.
So $\{f_t\}$ is a Loewner chain.

<u>Cor. 5.2</u> Let $f \in \mathcal{S}$ be arb.
Then there ex. a normalized Loewner chain $\{f_t\}_{t \in [0,\infty)}$ with $u_0 = 0$
s.t. $f_0 = f$.

<u>Proof:</u> For $n \in \mathbb{N}$, $n \geq 2$, let $r_n = (1 - \frac{1}{n}) \in (0,1)$, and
$$f^n(z) := \frac{1}{r_n} \cdot f(r_n z), \quad z \in \mathbb{D}.$$
Then $f^n(0) = 0$, $(f^n)'(0) = 1$, and so $f^n \in \mathcal{S}$.



f^n is a conf. map of $\mathbb{D}$ onto the bdd. Jordan region $\Omega^n := f^n(\mathbb{D}) = f(B(0,r_n))$.
So Ω^n can be embedded in a Loewner chain; equivalently, there ex. a normalized Loewner chain $\{f_t^n\}_{t \in [0,\infty)}$ with $f_t^n(0) = 0$, $(f_t^n)'(0) = e^t$, for $t \in I$,
and $f_0^n = f^n$.
By Prop. 5.1 the sequence $\{f_t^n\}$ of Loewner

(80) chain subconverges to a normalized Loewner chain $\{f_t\}$; i.e. for some sequence n_k with $n_k \longrightarrow \infty$ we have

$$f_t^{n_k} \longrightarrow f_t \text{ loc. unif. on } \mathbb{D}$$

for each $t \in I = [0,\infty)$

In particular,

$$f_0^{n_k} = f^{n_k} \longrightarrow f_0 \text{ loc. unif. on } \mathbb{D}.$$

On the other hand,

$$f^{n_k}(z) = \frac{1}{r_{n_k}} f(r_{n_k} z) \longrightarrow f(z)$$

loc. unif. for $z \in \mathbb{D}$.

So $f_0 = f$. The claim follows. □

The same proof shows that every conf. map f on $\mathbb{D}$ can be embedded in an analytic Loewner chain, or equiv., every simply connected region $\Omega \subsetneq \mathbb{C}$ can be embedded in a geometric Loewner chain.

5.3. Loewner chains and Taylor coefficients

$f \in \mathcal{S}$ arb., $f : \mathbb{D} \longleftrightarrow \Omega = f(\mathbb{D})$ conf.

$f(0) = 0$, $f'(0) = 1$.

By Cor. 5.2. there ex. normalized Loewner chain $\{f_t\}_{t \in [0,\infty)}$ s.t. $f_0 = f$

$f_t(0) = 0$,

$f_t'(0) = e^t$

$$f_t(z) = \sum_{n=1}^{\infty} a_n(t) z^n, \quad t \in [0,\infty), \quad z \in \mathbb{D}.$$

(81) $a_1(t) = e^t$. $\quad f(z,t) = f_t(z)$

Ex. $E \subseteq [0,\infty)$ s.t. $\quad I = [0,\infty)$

$$\frac{\partial f}{\partial t}(z,t) = z\, p(z,t)\, \frac{\partial f}{\partial z}(z,t), \quad z \in \mathbb{D}, \; t \in I \setminus E.$$

$f \in HL(\mathbb{D} \times I)$.

$p(\cdot,t) \in \mathcal{P}$ for $t \in I \setminus E$,

i.e., $p(\cdot,t) \in H(\mathbb{D})$, $p(0,t) = 1$, $\operatorname{Re} p(\cdot,t) \geq 0$.

$$p(z,t) = 1 + \sum_{n=1}^{\infty} c_n(t) z^n, \quad z \in \mathbb{D},\; t \in I \setminus E$$

$$\frac{\partial f}{\partial t}(z,t) = \sum_{n=0}^{\infty} \dot{a}_n(t) z^n \qquad \text{(c.f. Prop. 4.12)}, \; t \in I \setminus E,\; z \in \mathbb{D}.$$

Fix $t \in I \setminus E$. Then

$$\sum_{n=1}^{\infty} \dot{a}_n(t) z^n = z \cdot \underbrace{\left(1 + \sum_{n=1}^{\infty} c_n(t) z^n\right)}^{p(z,t)} \underbrace{\left(\sum_{n=1}^{\infty} n\, a_n(t) z^n\right)}_{z \cdot f'(z,t)}$$

$$= \sum_{n=1}^{\infty} \left(n\, a_n(t) + \sum_{k=1}^{n-1} k\, a_k(t)\, c_{n-k}(t) \right) z^n.$$

Comparing coefficients:

$$\boxed{\dot{a}_n(t) = n\, a_n(t) + \sum_{k=1}^{n-1} k\, a_k(t)\, c_{n-k}(t)}$$

$t \in I \setminus E$, $n \in \mathbb{N}$.

Each a_n is locally Lipschitz (c.f. Prop. 4.12.);
c_n is measurable (Homework!)
Moreover, $|c_n(t)| \leq 2$ for $n \in \mathbb{N}$, $t \in I \setminus E$ (Thm 4.9. (iv))

(82) $e^{-t} f_t^{h_t} \in \mathcal{F}$; $\mathcal{F}$ normal family

Hence, there ex. $C_n \geq 0$ s.t.

$$\left| e^{-t} a_n(t) \right| = \left| \frac{h_t^{(n)}(0)}{n!} \right| \leq C_n \quad \text{for } t \in I$$

$$e^{-nt} a_n(t) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty \quad \text{for } n \geq 2.$$

$$\frac{d}{dt}\left[e^{-nt} a_n(t) \right] = e^{-nt} \dot{a}_n(t) - e^{-nt} n \dot{a}_n(t)$$

$$= \sum_{k=1}^{n} e^{-nt} k a_k(t) c_{n-k}(t) \quad \text{for } t \in I \setminus E.$$

$0 \leq s$; $n \geq 2$.

$$-e^{ns} a_n(s) = \lim_{u \to \infty} \int_s^u \frac{d}{dt}\left[e^{-nt} a_n(t) \right] dt$$

$$= \sum_{k=1}^{n-1} k \int_s^\infty e^{-nt} a_k(t) c_{n-k}(t) dt$$

So

$$\boxed{a_n(s) = -e^{ns} \sum_{k=1}^{n-1} k \int_s^\infty e^{-nt} a_k(t) c_{n-k}(t) dt \qquad s \geq 0, \ n \geq 2.}$$

$s = 0$; $n = 2$

$$a_2 = a_2(0) = -\int_0^\infty e^{-2t} \underbrace{a_1(t)}_{= e^t} c_1(t) dt$$

$$= -\int_0^\infty e^{-t} c_1(t) dt$$

$s = 0$, $n = 3$.

$$a_3 = -\sum_{k=1}^{2} k \int_0^\infty e^{-3t} a_k(t) c_{n-k}(t) dt$$

$= a_3(0) =$

$$(83) = -\int_0^\infty e^{-2t} c_2(t)\,dt - 2\int_0^\infty e^{-3t} a_2(t)\, c_1(t)\,dt$$

$$= -\int_0^\infty e^{-2t} c_2(t)\,dt + 2\int_0^\infty e^{-3t} e^{2t}\left(\int_t^\infty e^{-u} c_1(u)\,du\right) c_1(t)\,dt$$

$$= -\int_0^\infty e^{-2t} c_2(t)\,dt + 2\int_0^\infty e^{-t} c_1(t)\left(\int_t^\infty e^{-u} c_1(u)\,du\right) dt$$

$$= -\int_0^\infty e^{-2t} c_2(t)\,dt + \int_0^\infty\int_0^\infty e^{-t} c_1(t)\, e^{-u} c_1(u)\,dt\,du$$

$$= -\int_0^\infty e^{-2t} c_2(t)\,dt + \left(\int_0^\infty e^{-t} c_1(t)\,dt\right)^2.$$

<u>Cor. 5.4.</u> Let $f \in S$,

$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$

Then $|a_2| \le 2$, $|a_3| \le 3$.

<u>Proof:</u> Using notation from 5.3. we have

$$a_2 = -\int_0^\infty e^{-t} c_1(t)\,dt. \quad \text{Now } |c_1(t)| \le 2,$$

and so

$$|a_2| \le \int_0^\infty e^{-t}|c_1(t)|\,dt \le 2\int_0^\infty e^{-t}\,dt = 1.$$

(Case of equality can be analyzed!)

By rotational invariance ($f \in S \Rightarrow e^{i\theta} f(z e^{-i\theta}) \in S$)

wlog $a_3 \ge 0$

Then

$$a_3 = \operatorname{Re} a_3$$
$$\leq - \int_0^\infty e^{-2t}(\operatorname{Re} c_2(t))\,dt + \left(\int_0^\infty e^{-t}(\operatorname{Re} c_1(t))\,dt\right)^2$$
$$\underset{CS}{\leq} - \int_0^\infty e^{-2t}(\operatorname{Re} c_2(t))\,dt + \int_0^\infty (\operatorname{Re} c_1(t))^2 e^{-t}\,dt$$

$(\operatorname{Re} c_1)^2 \leq 2 + \operatorname{Re} c_2$ (cf. Thm. 4.9 (iv))

$$\leq 2\int_0^\infty e^{-t}\,dt + \int_0^\infty (\operatorname{Re} c_2(t))\underbrace{(e^{-t} - e^{-2t})}_{\geq 0}\,dt$$

$$\overset{|c_2|\leq 2}{\leq} 2 + 2\int_0^\infty (e^{-t} - e^{-2t})\,dt$$
$$= 2 + 2 + 2\left[\tfrac{1}{2}e^{-2t}\right]_0^\infty = 2 + 2 - 2\tfrac{1}{2} = 3. \quad \square$$

<u>Lem. 5.5.</u> Let $p \in \mathcal{P}$. Then

(i) $|p'(z)| \leq \frac{2}{(1-|z|)^{-2}}$, $z \in \mathbb{D}$,

(ii) $|p(u) - p(v)| \leq \frac{2|u-v|}{(1-r)^2}$ $u, v \in \overline{B}(0,r)$, $r \in (0,1)$.

<u>Proof:</u> $z_0 \in \mathbb{D}$, $r \in (0,1)$, $r > |z_0|$,
$\gamma(t) = r \cdot e^{it}$, $t \in [0, 2\pi]$.

$$p'(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{p(\xi)}{(\xi - z_0)^2}\,d\xi$$

On the other hand, there ex. prob. measure μ on $\partial\mathbb{D}$ s.t.

$$p(\xi) = \int_{\partial\mathbb{D}} \underbrace{\frac{\eta + \xi}{\eta - \xi}}_{K_\eta(\xi)}\,d\mu(\eta) \quad \text{for } \xi \in \mathbb{D}.$$

(5) By Fubini

$$p'(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{p(\xi)}{(\xi - z_0)^2} d\xi$$

$$= \frac{1}{2\pi i} \int_\gamma \int_{\partial \mathbb{D}} \frac{K_\eta(\xi)}{(\xi - z_0)^2} d\mu(\eta)\, d\xi$$

$$= \int_{\partial \mathbb{D}} \left[\frac{1}{2\pi i} \int_\gamma \frac{K_\eta(\xi)}{(\xi - z_0)^2} d\xi \right] d\mu(\eta)$$

$$= \int_{\partial \mathbb{D}} K'_\eta(z_0)\, d\mu(\eta)$$ (So we can differentiate under the integral sign in the Herglotz formula).

Here $K'_\eta(z_0) = \frac{d}{dz} \left(\frac{\eta + z}{\eta - z} \right) \Big|_{z = z_0}$

$= \frac{2}{(\eta - z_0)^2}$ $\qquad |K'_\eta(z_0)| \le \frac{2}{(1 - |z_0|)^2}$,

and

$$|p'(z_0)| \le \int_{\partial \mathbb{D}} \frac{2}{(1 - |z_0|)^2} d\mu(\eta) = \frac{2}{(1 - |z_0|)^2}.$$

ii) Follows from (i).

Lem. 5.6. (Uniqueness statement)

$I = [a, \infty)$, $p : \mathbb{D} \times I \to \mathbb{C}$ a.e. def.,
p measurable, $p(\cdot, t) \in \mathcal{P}$ for a.e. $t \in I$.
Let $J = [a, b] \subseteq I$, and suppose
$u, v : J \to \mathbb{D}$ are abs. cont. and
solutions of the ODE

(86)

(*) $\dot{w}(t) = -w(t)\, p(w(t), t)$ for a.e. t.

If $u(t_0) = v(t_0)$ for some $t_0 \in J$, then $u = v$.

Proof: 1) For a solution $w : J \to \mathbb{D}$, $t \mapsto |w(t)|$ is decreasing:

$$\frac{d}{dt}|w(t)|^2 = \frac{d}{dt} w(t) \cdot \overline{w(t)}$$

$$= \dot{w}(t)\,\overline{w(t)} + w(t) \cdot \overline{\dot{w}(t)}$$

$$= -|w(t)|^2\, p(w(t),t) - |w(t)|^2\, \overline{p(w(t),t)}$$

$$= -|w(t)|^2 \operatorname{Re} p(w(t),t) \le 0 \text{ for a.e. } t.$$

So $|u(t)|, |v(t)| \le r := \max\{|u(a)|, |v(a)|\} < 1$.

2) $|u(t)\, p(u(t),t) - v(t)\, p(v(t),t)|$

$$\le |u(t)|\,|p(u(t),t) - p(v(t),t)|$$

$$+ |u(t) - v(t)|\,|p(v(t),t)|$$

Lem. 5.5
$$\le 1 \cdot \frac{2}{(1-r)^2}|u(t)-v(t)| + \frac{2}{1-r}|u(t)-v(t)|$$

$$\le K|u(t)-v(t)|$$ for a.e. t, where K ind. of t.

Let
$$D(t) := |u(t)-v(t)|^2, \quad t \in J.$$

Then D is abs. cont. and

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$$\left|\frac{d}{dt} D(t)\right| \leq 2\,|\dot{u}(t) - \dot{v}(t)|\,|u(t)-v(t)|$$

$$= 2\,|u(t)\,p(u(t),t) - v(t)\,p(v(t),t)| \cdot |u(t)-v(t)|$$

$$\leq 2k\,|u(t)-v(t)|^2 = k' D(t).$$

Hence

$$D(t) \leq e^{k''|t-t_0|} D(t_0) \quad \text{for } t \in J.$$

(special case of Gronwall's Ineq.)

Since $D(t_0) = 0$ we conclude

$D(t) \equiv 0$ and so $u \equiv v$. $\square$

<u>Thm. 5.7.</u> $I = [a, \infty) \subseteq \mathbb{R}$,

$V : \mathbb{D} \times I \to \mathbb{C}$ a.e. def. measurable function s.t.

i) $V(z, \cdot)$ a.e. def. + measurable for each $z \in \mathbb{D}$,

ii) $V(\cdot, t)$ holomorphic on $\mathbb{D}$ for a.e. $t \in \mathbb{D}$ and

$V(z,t) = -z\,p(z,t)$ for $z \in \mathbb{D}$,

where $p(\cdot, t) \in \mathcal{B}$.

Then for each $z \in \mathbb{D}$, $s \in I$ there ex. a unique map $w : [s, \infty) \to \mathbb{D}$ s.t.

i) w is Lipschitz on $[s, \infty)$. $[s,\infty)$ ... local

ii) $w(s) = z$ (initial condition)

iii) $\dot{w}(t) = V(w(t), t)$ for a.e. $t \in I$.

(88) Proof: Need a technical lemma that will be formulated afterwards!

Idea of proof:

Picard - Lindelöf iteration scheme!

$z \in \mathbb{D}$, $s \in I$ fixed.

Define $w_0(t) \equiv 0$

and

$$w_{n+1}(t) = z \cdot \exp\left(-\int_s^t \underbrace{p(w_n(u), u)}_{\text{measurable}}\, du\right)$$

$w_1(t) = z \cdot e^{s-t}$

$n \in \mathbb{N}_0$, $t \geq s$.

i) $|w_n(t)| \leq r := |z|$, $t \geq s$, $n \in \mathbb{N}$

(note $\operatorname{Re} p \geq 0$)

ii) w_n is L - Lipschitz on $[s, \infty)$ with $L = \frac{2r}{1-r}$:

$|e^{-a} - e^{-b}| \leq |a-b|$, $\operatorname{Re} a, \operatorname{Re} b \geq 0$.

$|w_{n+1}(t_2) - w_{n+1}(t_1)|$

$$= |z| \left| \exp\left(-\int_s^{t_2} \ldots\right) - \exp\left(-\int_s^{t_1} \ldots\right)\right|$$

$$\leq |z| \left| \int_s^{t_2} \ldots - \int_s^{t_1} \right|$$

$$= |z| \cdot \left| \int_{t_1}^{t_2} \underbrace{p(w_n(u), u)}_{\leq \frac{2}{1-|w_n(u)|} \leq \frac{2}{1-r}}\, du \right|$$

$$= \frac{2r}{1-r} |t_2 - t_1|, \quad t_2 \geq t_1 \geq s.$$

(89) iii)

$$\left| w_{n+1}(t) - w_n(t) \right| \le \frac{2^n (t-s)^n}{(1-r)^{2n} n!}$$

$n \in \mathbb{N}_0$, $t \ge s$

By induction, $n=0$.

$$\left| w_1(t) - w_0(t) \right| = e^{s-t} \cdot |z| \le 1 \quad \checkmark.$$

$n \longrightarrow n+1$:

$$\left| w_{n+1}(t) - w_n(t) \right|$$

$$= |z| \cdot \left| \exp\left(- \int_s^t p(w_n(u), u)\, du \right) - \exp\left(- \int_s^t p(w_{n-1}(u), u)\, du \right) \right|$$

$$\le |z| \left| \int_s^t \underbrace{\left| p(w_n(u), u) - p(w_{n-1}(u), u) \right|}_{\text{Lou. 5.5. } \frac{2}{(1-r)^2} \cdot |w_n(u) - w_{n-1}(u)|} du \right|$$

$$\le |z| \frac{2}{(1-r)^2} \cdot \int_s^t \frac{2^n (u-s)^n}{(1-r)^{2n} n!} du$$

$$= \frac{2^{n+1}}{(1-r)^{2n+2}} \cdot \frac{(t-s)^{n+1}}{(n+1)!}$$

So $w(t) := \lim_{n \to \infty} w_n(t)$

$$= w_0(t) + \sum_{n=1}^{\infty} (w_n(t) - w_{n-1}(t))$$

ex. for each $t \in I$; convergence unif.

90 on compact subsets $J \subseteq I$, i.e.,
$w_n \longrightarrow w$ loc. unif. on I.

Then:

w is L-Lipschitz on I,

$|w(t)| \leq r < 1$ for $t \in I$,

$p(w_n(u), u) \longrightarrow p(w(u), u)$ for a.e. $u \in I$,

$|p(w_n(u), u)| \leq \frac{2}{1-r}$;

So, $\int_s^t p(w_n(u), u)\, du \longrightarrow \int_s^t p(w(u), u)\, du$ for each $t \in [s, \infty)$

by Lebesgue dom. conv.

For each $t \in I$:

$w(t) = \lim_{n \to \infty} w_{n+1}(t) =$

$\lim_{n \to \infty} z \cdot \exp\left(-\int_s^t p(w_n(u), u)\, du\right)$

$= z \exp\left(-\int_s^t p(w(u), u)\, du\right)$.

Hence:

$w(t) = z \exp\left(-\int_s^t p(w(u), u)\, du\right)$ for $t \in [s, \infty)$.

So

$w(s) = z$, $\dot{w}(t)$ ex. for a.e. $t \in [s, \infty)$,

and

$\dot{w}(t) = -z \cdot \exp(\cdots)\, p(w(t), t)$

(91) $= -w(t)\,p(w(t),t) = V(w(t),t)$.

Existence of w follows.

Uniqueness clear by 5.6. $\square$

<u>Cor. 5.8.</u> For fixed $z \in \mathbb{D}$, $s,t \in I$, $s \le t$ let

$$\varphi_{s,t}(z) = w(t)$$

<u>where</u> w is as in Thm. 5.7.

<u>Then</u>

i) $\varphi_{s,t}(\cdot)$ is holomorphic + injective on $\mathbb{D}$, $\varphi_{s,t}(\mathbb{D}) \subseteq \mathbb{D}$

ii) $\varphi_{s,t}(0) = 0$, $\varphi_{s,t}(0) = e^{s-t}$

iii) $\varphi_{s,u} = \varphi_{t,u} \circ \varphi_{s,t}$

iv) $f_s(z) := \lim\limits_{t \to \infty} e^{t} \varphi_{s,t}(z)$

ex. for $z \in \mathbb{D}$, $s \in I$;

moreover, $e^{t} \varphi_{s,t} \longrightarrow f_s$ loc. unif. on $\mathbb{D}$

v) $\{f_s\}_{s \in I}$ is a normalized Loewner chain

with

$$\dot{f}_s(z) = V(z,s)\, f_s'(z)$$

for $z \in \mathbb{D}$ and a.e. $s \in I$.

<u>Proof:</u> As in the proof of Thm. 5.7. define for $z \in \mathbb{D}$, $t \ge s$

$$w_0(z,t) \equiv 0,$$

$$w_{n+1}(z,t) := z \exp\left(-\int_s^t p(w_n(z,u),u)\,du\right).$$

(92) Using induction and Morera, one shows that $w_n(\cdot,t)$ is holomorphic on $\mathbb{D}$ for each $t \in [s,\infty)$.

$|w_n(z,t)| \leq |z|$; so

$\{w_n(\cdot,t)\}_{n\in\mathbb{N}}$ is a normal family for each $t \in [s,\infty)$;

$w_n((z,t) \longrightarrow w(z,t) = \varphi_{s,t}(z)$ pointwise on $\mathbb{D}$ for each $t \in [s,\infty)$;

so convergence loc. unif. on $\mathbb{D}$ by Vitali.

i) Hence $w(\cdot,t) = \varphi_{s,t}$ holomorphic on $\mathbb{D}$ for $s,t \in I$, $s \leq t$.

Let $s, t_0 \in I$, $s \leq t$, $z_1, z_2 \in \mathbb{D}$, and suppose

$\varphi_{s,t_0}(z_1) = \varphi_{s,t_0}(z_2)$ equiv.

$w(z_1,t) = w(z_2,t)$.

Then by Lemma 5.5. $w(z_1,t) \equiv w(z_2,t)$ for <u>all</u> $t \geq s$; hence

$z_1 = w(z_1,s) = w(z_2,s) = z_2$.

So φ_{s,t_0} is inj. on $\mathbb{D}$.

ii) $w(0,t) \equiv 0$ solves ODE ; so

$\varphi_{s,t}(0) = 0$.

$$\varphi_{s,t}(z) = z \exp\Big(\underbrace{-\int_s^t p(\varphi_{s,u}(z),u)\,du}_{\text{holomorphic in } z}\Big)$$

(93) So

$$\varphi'_{s,t}(0) = \exp\left(-\int_s^t \underbrace{p(\overset{\overset{0}{\shortparallel}}{\varphi_{s,u}(0)},u)}_{=1}\,du\right)$$

$$= \exp(-(t-s)) = e^{s-t}.$$

iii) $v(u) := \varphi_{s,u}(z)$ $\qquad z \in \mathbb{D},\ s \le t$ fixed

$\hat{v}(u) = \varphi_{t,u}(\varphi_{s,t}(z))$ $\qquad u \ge t$

$v(t) = \varphi_{s,t}(z) = ?$

$\hat{v}(t) = \varphi_{t,t}(\varphi_{s,t}(z)) = \varphi_{s,t}(z).$

$\varphi_{t,t}(z) \equiv z$

So $v, \hat{v}$ have same initial value at time $u = t$.

$\dot{v}(u) = V(v(u),u)$ for a.e. u

$\dot{\hat{v}}(u) = V(\hat{v}(u),u)$

So $v(u) \equiv \hat{v}(u)$ for $u \ge t$ by Lem. 5.5,

i.e.,

$\varphi_{s,u}(z) = \varphi_{t,u}(\varphi_{s,t}(z))$ for $z \in \mathbb{D}$, $s \le t \le u$.

iv) $$e^{t-s}\varphi_{s,t}(z) = z \cdot \exp\left(\int_s^t [1 - p(\varphi_{s,u}(z),u)]\,du\right)$$

$e^{t-s}\varphi_{s,t} \in \mathcal{S}$; so

(94) $|\varphi_{s,t}(z)| \le \dfrac{e^{s-t}|z|}{(1-|z|)^2}$; $z \in \mathbb{D}$ by Koebe

So

$\left| 1 - p(\varphi_{s,u}(z), u) \right|$

$= \left| p(0,u) - p(\varphi_{s,u}(z), u) \right|$ $\quad |\varphi_{s,u}(z)| \le |z|$

$\underset{\text{Lem. 5.5.}}{\le} |\varphi_{s,u}(z)| \cdot \dfrac{2}{(1-|z|)^2}$

$\le e^{s-u} \cdot \dfrac{2|z|}{(1-|z|)^4} \le C \cdot e^{-u}$ for fixed s, z.

So $\displaystyle\int_s^\infty \left| 1 - p(\varphi_{s,u}(z), u) \right| du < \infty$

with unif. conv. in z on comp. subsets of $\mathbb{D}$

Hence

$f_s(z) := \lim\limits_{t\to\infty} e^t \cdot \varphi_{s,t}(z) = \lim\limits_{t\to\infty} e^s e^{t-s} \varphi_{s,t}(z)$

$= e^s \cdot z \cdot \exp\left(\displaystyle\int_s^\infty \left[1 - p(\varphi_{s,u}(z), u) \right] du \right)$

ex. with loc. unif. conv. in $z \in \mathbb{D}$.

So $f_s \in H(\mathbb{D})$, $f_s(0) = \lim\limits_{t\to\infty} e^t \varphi_{s,t}(0)$

$= 0$,

$f_s'(0) = \lim\limits_{t\to\infty} e^t \underbrace{\varphi_{s,t}'(0)}_{e^{s-t}} = e^s$;

Since $e^t \cdot \varphi_{s,t}$ is inj. on $\mathbb{D}$, f_s is inj. on $\mathbb{D}$

(95) by Hurwitz.

For $z \in \mathbb{D}$, $s \le t$:

$$f_t(\varphi_{s,t}(z)) = \lim_{u \to \infty} e^u \varphi_{t,u}(\varphi_{s,t}(z))$$

(iii) $\lim_{u \to \infty} e^u \varphi_{s,u}(z) = f_s(z)$.

So $f_t \circ \varphi_{s,t} = f_s$, $s \le t$.

Hence $\Omega_t := f_t(\mathbb{D}) \supseteq f_t(\varphi_{s,t}(\mathbb{D}))$
$= \Omega_s$

[Strict inclusion for $s < t$; for otherwise $\varphi_{s,t}(\mathbb{D}) = \mathbb{D}$, and $\varphi_{s,t} \in \mathrm{Aut}(\mathbb{D})$. Since $\varphi_{s,t}(0) = 0$, $\varphi_{s,t}(z) = e^{i\nu} z$, $\nu \in \mathbb{R}$, and $|\varphi_{s,t}'(0)| = 1$; but $\varphi_{s,t}'(0) = e^{s-t} < 1$, contradiction.]

As in Prop. 5.1 we conclude that $\{f_s\}_{s \in I}$ is a Loewner chain.

Since $\{f_s\}$ is a Loewner chain $(z,t) \longmapsto f(z,t) \in HL(\mathbb{D} \times I)$;

since $f(\varphi_{a,t}(z), t) = f_a(z)$, there ex. $E \subseteq I = [a,b)$, $|E| = 0$, s.t.

(96) $\dot{\phi} = \frac{d}{dt} f_a(z) = \frac{d}{dt} f(\varphi_{a,t}(z), t)$

$= f_t'(\varphi_{a,t}(z)) \cdot \underbrace{\frac{d\varphi_{s,t}}{dt}(z)}_{= V(\varphi_{s,t}(z), t)} + \dot{f}_t(\varphi_{a,t}(z))$;

and

$\dot{f}_t(w) = -V(w,t) \cdot f_t'(w)$
for $t \in I \setminus E$, $w \in \varphi_{a,t}(\mathbb{D}) \subseteq \mathbb{D}$.

We may assume that $f_t(\cdot)$ and $V(\cdot, t)$ are holomorphic for $t \in I \setminus E$.
Then by the Uniqueness Thm.

$\dot{f}_t(z) = -V(z,t) \cdot f_t'(z)$
for $z \in \mathbb{D}$, $t \in I \setminus E$. □

Continuity of $w_n(z,t)$ in z for t fixed:

$w_0(z,t) \equiv 0$

$w_{n+1}(z,t) = z \exp\left(-\int_s^t p(w_n(z,u),u)\,du\right)$

By induction on n! $n \to n+1$:

$z_k \in \mathbb{D} \to z_0 \in \mathbb{D}$ $|z_k| \le r < 1$

$w_n(z_k, u) \to w_n(z_0, u)$ as $k \to \infty$
for each $u \in [s,t]$.

Moreover:

$|w_n(z_k, u)| \le r$ and so

$p(w_n(z_k, u), u) \le \frac{1+r}{1-r}$.

So $\int_s^t p(w_n(z_k,u),u)\,du \to \int_s^t p(w_n(z_0,u),u)\,du$
by Lebesgue Dominated Conv.

(97) In the proof of Thm. 5.7 the following fact was used.

<u>Lem. 5.9.</u> $U \subseteq \mathbb{R}^d$ open, $M \subseteq \mathbb{R}^{d'}$ measurable, $g : U \times M \to \mathbb{C}$ a.e. def. s.t.

i) $g(\cdot, t)$ is cont. on U for a.e. $t \in M$,

ii) $g(z, \cdot)$ is a.e. def. on M and measurable

Let $\psi : M \to U$ be measurable. Then $h : M \to \mathbb{C}$ a.e. def. by
$h(t) := g(\psi(t), t)$ for $t \in M$
is measurable.

<u>Proof</u> (Outline):

I. For each $n \in \mathbb{N}$ pick a "countable" open cover $\mathcal{U}_n = \{U_{n,k} : k \in \mathbb{N}\}$ of U s.t. $U_{n,k} \subset\subset U$ and
$$\operatorname{mesh}(\mathcal{U}_n) = \sup\{\operatorname{diam}(U_{n,k}) : k \in \mathbb{N}\} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Pick $z_{n,k} \in U_{n,k}$ and let $\{\varphi_{n,k} : k \in \mathbb{N}\}$ be a partition of unity subordinate to $\mathcal{U}_n$.

For $f \in C(U)$ define
$$T_n f := \sum_{k \in \mathbb{N}} f(z_{n,k}) \varphi_{n,k} \in C(U).$$

Then $T_n f \longrightarrow f$ loc. unif. on U for <u>all</u> $f \in C(U)$.

For $z \in U$:

(9f) $|f(z) - T_n f(z)|$

$$\le \sum_{k \in \mathbb{N}}' |h(z) - h(z_{n,k})| \, \varphi_{n,k}(z)$$

$$\le \sup\{|h(u) - h(u')| : |u-u'| \le n \cdot \mathrm{sh}(\mathcal{U}_n)\}$$

II. Ex. $E \subseteq M$, $|E| = 0$, s.t.
$g(\cdot, t) \in C(U)$ for $t \in M \setminus E$.
Then
$$T_n g(z,t) = \sum_{k \in \mathbb{N}}' g(z_{n,k}, t) \, \varphi_{n,k}(z)$$

$$\longrightarrow g(z,t) \text{ as } n \to \infty \text{ for}$$
$$z \in U, \; t \in M \setminus E$$

So for a.e. $t \in M$:

$$\underbrace{\sum_{k \in \mathbb{N}}' \overbrace{g(z_{n,k}, t)}^{\text{measurable}} \cdot \overbrace{\varphi_{n,k}(\psi(t))}^{\text{measurable in } t}}_{\text{meas. because finite sum for each } t}$$

$$\longrightarrow g(\psi(t), t) \text{ as } n \to \infty$$
$$= h(t)$$

So h is measurable. $\square$

Lem. 5.10. Let $f \in \mathcal{S}$. Then

$$|f(z) - z| \le \frac{C\,|z|^2}{(1-|z|)^2} \quad \text{for } z \in \mathbb{D},$$

where C is an abs. const. ind. of f.

Proof: Define

$$g(z) = \frac{1}{z^2}(f(z) - z), \quad z \in \mathbb{D}. \qquad g \in H(\mathbb{D})$$
(0 is removable sing.)

(99) Pick $0 < r < 1$. Then by Koebe and the Max. Princip.:

$$|g(z)| \leq \frac{1}{r^2}\left[\frac{r}{(1-r)^2} + r\right] \quad \text{for } |z| \leq r.$$

$$\leq \frac{2}{r(1-r)^2}.$$

$|z| \leq \frac{1}{2}$, $r = \frac{1}{2}$

$$|g(z)| \leq 16 \leq \frac{16}{(1-|z|)^2}$$

$\frac{1}{2} \leq |z| < 1$, $r = |z|$

$$|g(z)| \leq \frac{4}{(1-|z|)^2}.$$

So $C = 16$ works. $\square$

<u>Prop. 5.11.</u> $\{f_t\}$ normalized Loewner chain on $I = [a, \infty)$, $f_t(0) = 0$, $f_t'(0) = e^t$, $t \in I$.
Let $\varphi_{s,t} := f_t^{-1} \circ f_s$ for $a \leq s \leq t$.

$$e^t \varphi_{s,t} \longrightarrow f_s \text{ loc. unif. on } \mathbb{D}$$
as $t \longrightarrow \infty$

(i.e., along any seq. $t_n \longrightarrow \infty$).

<u>Proof:</u> $a \leq s \leq t$, $\varphi_{s,t}(0) = 0$,
$\varphi_{s,t}(\mathbb{D}) \subseteq \mathbb{D}$

So (1) $|\varphi_{s,t}(z)| \leq |z|$ for $z \in \mathbb{D}$ by Schwarz

$\varphi_{s,t}$ inj. on $\mathbb{D}$, $\varphi_{s,t}'(0) = e^{s-t}$.

(100) So

(2) $|\varphi_{s,t}(z)| \le e^{s-t} \dfrac{|z|}{(1-|z|)^2}$ for $z \in \mathbb{D}$

by Koebe.

$f_t \circ \varphi_{s,t} = f_s$, $e^{-t} f_t \in \mathcal{S}$.

So, by Lem. 5.10.

$$|f_t(w) - e^t w| \le C \frac{e^t |w|^2}{(1-|w|)^2}.$$

Using this for $w = \varphi_{s,t}(z) \in \mathbb{D}$ and (1) + (2), we obtain

$$|f_s(z) - e^t \varphi_{s,t}(z)|$$

$$= |f_t(\underbrace{\varphi_{s,t}(z)}_{w}) - e^t \underbrace{\varphi_{s,t}(z)}_{w}| \qquad |w| \le |z|$$

$$\le C \frac{e^t |\varphi_{s,t}(z)|^2}{(1-|z|)^2}$$

$$\le C \frac{e^t \cdot e^{2s-2t} \cdot |z|^2}{(1-|z|)^4}$$

$$\le e^{-t} \frac{C e^{2s} |z|^2}{(1-|z|)^4} \longrightarrow 0 \quad \text{l.c. unif. on } \mathbb{D}$$

as $t \to \infty$. □

$\{f_t\}$ Loewner chain

$\downarrow \varphi_{s,t} = f_t^{-1} \circ f_s \qquad \uparrow f_s = \lim_{t \to \infty} e^t \varphi_{s,t}$

$\varphi_{s,t}$ semigroup

(101) Thm. 5.12. (Existence and uniqueness for solutions of Loewner-Kufarev eqs.)

$I = [a, \infty) \subseteq \mathbb{R}$

Herglotz vector field: $V : \mathbb{D} \times I \to \mathbb{C}$ a.e. def. measurable function s.t.

i) $V(z, \cdot)$ a.e. def. + meas. for each $z \in \mathbb{D}$,

ii) $V(\cdot, t)$ holomorphic for a.e. $t \in \mathbb{D}$,

iii) $V(z,t) = -z\, p(z,t)$ for $z \in \mathbb{D}$, a.e. $t \in I$, where $p(\cdot, t) \in \mathcal{P}$

Then there ex. a unique normalized Loewner chain $\{f_t\}_{t \in I}$ with $f_t(0) = w_0 = 0$ s.t. the Loewner-Kufarev eqn holds:

(*) $\dot{f}_t(z) = -V(z,t)\, f_t'(z)$ for $z \in \mathbb{D}$, a.e. $t \in I$.

Suppose $g : \mathbb{D} \times I \to \mathbb{C}$ is a function s.t.

i) $g(\cdot, t) \in H(\mathbb{D})$, $g(0,t) = 0$, $g'(0,t) = e^t$ for $t \in I$,

ii) $g(z, \cdot)$ is unif. Lipschitz on comp. subsets of $\mathbb{D} \times I$

} $g \in HL(\mathbb{D} \times I)$

iii) g solves (*), i.e.,

$$\frac{\partial g}{\partial t}(z,t) = -V(z,t)\frac{\partial g}{\partial z}(z,t)$$

for each $z \in \mathbb{D}$ and a.e. $t \in I$.

Then there ex. an entire function $h : \mathbb{C} \to \mathbb{C}$ with $h(0) = 0$, $h'(0) = 1$, $g_t = h \circ f_t$ for $t \in I$.

(102) Suppose g satisfies the following additional assumption:

iv) there ex. $r_0 \in (0,1)$ and $C \geq 0$ s.t.
$$|g_t(z)| \leq C e^t \text{ for } t \in I,\ z \in \overline{B}(0, r_0)$$
Then $h = id_{\mathbb{C}}$ and so $g_t = f_t$ for all $t \in I$

<u>Proof</u>: We know that there ex. a norm. Loewner chain $\{f_t\}$ solving (*) (see Cor 5.f.): Find unique $\varphi_{s,t}(z)$ s.t.
$$\varphi_{s,s}(z) \equiv z,\quad z \in \mathbb{D},\quad \frac{\partial \varphi_{s,t}}{\partial t} = -V(\varphi_{s,t}(z), t) \text{ for a.e. } t \geq s,$$
let $f_s := \lim_{t \to \infty} e^t \varphi_{s,t}$. Then $\varphi_{s,t} = f_t^{-1} \circ f_s$.

$\{f_t\}_{t \in I}$ is a Loewner chain solving (*).

Let g be a function as in hypotheses, $g_t := g(\cdot, t)$.

<u>Claim</u> $g_t \circ \varphi_{s,t} = g_s$ for $a \leq s \leq t$.

Fix s: Then for $z \in \mathbb{D}$ and a.e. $t \geq s$
$$\frac{d}{dt} g_t \circ \varphi_{s,t}(z) = \frac{d}{dt} g(\varphi_{s,t}(z), t)$$
g diff. by Prop. 4.12.(iii)
$$= \frac{\partial g}{\partial z}(\varphi_{s,t}(z), t) \cdot \frac{\partial \varphi_{s,t}(z)}{\partial t} + \frac{\partial g}{\partial t}(\varphi_{s,t}(z), t)$$
Chain rule
$$= g_t' \circ \varphi_{s,t}(z) \cdot (+V(\varphi_{s,t}(z), t)) + \dot{g}_t \circ \varphi_{s,t}(z)$$

(103) $w = \varphi_{s,t}(z)$

$$= g_t'(w) \cdot V(w,t) + \dot{g}_t(w)$$

$= 0$ by $(*)$.

Since $t \longmapsto g(\varphi_{s,t}(z), t)$ is loc. Lipschitz, we have

$$g_t \circ \varphi_{s,t}(z) \equiv \text{const. in } t$$

(for fixed s, $z \in \mathbb{D}$).

For $t = s$:

$$g_s \circ \underbrace{\varphi_{s,s}(z)}_{z} = g_s(z)$$

The claim follows.

Lipschitz property:

$J = [c,d] \subseteq I$, comp. $z \in \mathbb{D}$ fixed

$r = |z| < r' < 1$.

g cont. so unif. bdd. on $\overline{B}(0,r') \times J$

$g_t'(z)$ unif. bdd. on $\overline{B}(0,r) \times J$,

so $|g_t'(z)| \le M$

$t_1, t_2 \in J$: $w_1 = \varphi_{s,t_1}(z)$, $w_2 = \varphi_{s,t_2}(z)$

$|w_1|, |w_2| \le r$

$$|g(w_1, t_1) - g(w_2, t_2)|$$

$$\le |g(w_1,t_1) - g(w_2,t_1)| + |g(w_2,t_1) - g(w_2,t_2)|$$

$$\le M|w_1 - w_2| + L|t_1 - t_2|$$

unif. bdd. for g_t' $\quad g(\cdot,t)$ unif. Lip.

$$\le M|\varphi_{s,t_1}(z) - \varphi_{s,t_2}(z)| + L|t_1 - t_2|$$

$\varphi_{s,t}(z)$ unif. Lip. in t (Lem. 4.10)

$$\le M'|t_1 - t_2| + L|t_1 - t_2| \le L'|t_1 - t_2|$$

(104) By Claim:

$$g_t \circ \varphi_{s,t} = g_s \quad \text{equiv.}$$

$$g_t \circ f_t^{-1} \circ f_s = g_s \quad \text{equiv.}$$

$$g_t \circ f_t^{-1} = g_s \circ f_s^{-1} \quad \text{on } \Omega_s := f_s(\mathbb{D})$$

$$t \geq s.$$

Note $\bigcup_{t \geq a} \Omega_t = \mathbb{C}$, because

$\Omega_t \supseteq B(0, \frac{1}{4} e^t)$ by Koebe.

Define

$$h(z) = (g_t \circ f_t^{-1})(z) \quad \text{if } z \in \Omega_t.$$

Then h is well-def. and holomorphic on $\mathbb{C} = \bigcup_{t \in I} \Omega_t$; hence entire.

$g_t = h \circ f_t$ for $t \in I$ by def.

$0 = g_t(0) = h(f_t(0)) = h(0).$

$$h'(0) \cdot \underbrace{f_t'(0)}_{e^t} = \underbrace{g_t'(0)}_{e^t} \Rightarrow h'(0) = 1.$$

Suppose that g sat. (iv) in addition, so

$$|g_t(z)| = |h(f_t(z))|$$
$$\leq Ce^t \quad \text{for } z \in \bar{B}(0, r_0), \; t \in I.$$

By Koebe: $f_t(B(0, r_0)) \supseteq B(0, e^t r_0/4)$, and so

$$|h(w)| \leq Ce^t \quad \text{for } w \in B(0, e^t r_0/4), \; t \in I.$$

(105) So there ex. $C' \geq 0$ s.t.

$$|h(w)| \leq C'(1+|w|), \quad w \in \mathbb{C}.$$

By Cauchy est. $h(w) \equiv aw+b$, $a,b \in \mathbb{C}$.
Since $h(0)=0$, $h'(0)=1$,
$b=0$, $a=1$, and so $h(w) \equiv w$,
i.e., $h = id_{\mathbb{C}}$.
Suppose $\{\hat{f}_t\}$ is another normalized Loewner chain with $\hat{f}_t(0) \equiv 0$, $t \in I$, solving $(*)$.
Then

$$|\hat{f}_t(z)| \leq e^t \frac{|z|}{(1-|z|)^2}, \quad z \in \mathbb{D}, t \in I,$$

by Koebe and so

$$|\hat{f}_t(z)| \leq 2e^t, \quad |z| \leq \tfrac{1}{2}, t \in I,$$

i.e., (iv) is true. Moreover, (i)–(iii) are also true and so $\hat{f}_t = f_t$ for all t,
i.e., there ex. a unique norm. Loewner chain solving $(*)$.

Rem 5.13. It is likely that the second part of Thm. 5.12 can be proved under weaker regularity assumptions on g; namely that $g(\cdot,t) \in H(\mathbb{D})$ for each $t \in I$, and $g(z,\cdot)$ is abs. cont. on comp. $J \subseteq I$ for each $z \in \mathbb{D}$.
It is not clear that under these hypotheses g is diff. for a.e. $(z,t) \in \mathbb{D} \times I$.
Not even local boundedness is clear!

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The Loewner triangle

"Herglotz vector fields"

$V(z,t)$

Loewner-Kufarev eqs.

$\{f_t\}$ norm. Loewner chain

unique Loewner chain solution

unique solution of flow eq.

$V(z,t) = \lim_{\varepsilon \to 0} \frac{\varphi_{t,t+\varepsilon}(z) - z}{\varepsilon}$

$\varphi_{s,t} = f_t^{-1} \circ f_s$

semigroup $\varphi_{s,t}$

$f_s = \lim_{t \to \infty} e^t \varphi_{t,s}$

$$\dot{f}_t(z) = -V(z,t)\, f_t'(z) \quad \text{Loewner-Kufarev eq.}$$

$$\frac{d}{dt} \varphi_{s,t}(z) = V(\varphi_{s,t}(z), t), \quad \varphi_{s,s}(z) = z \in \mathbb{D}.$$

a.e. $t \geq s$

$$V(z,t) = -z\, p(z,t), \quad p(\cdot, t) \in \mathcal{P}$$

for a.e. $t \in I$.

Recent papers by:
Bracci, Contreras, Diaz-Madrigal.

Thm. 5.14 (Becker's univalence criterion)
Let $f \in H(\mathbb{D})$, $f'(z) \neq 0$ for $z \in \mathbb{D}$, and

(*) $$(1-|z|^2) \left| z \frac{f''(z)}{f'(z)} \right| \leq 1 \quad \text{for } z \in \mathbb{D}.$$

Then f is univalent on $\mathbb{D}$ (= injective + holomorphic).

(07) Conversely, if f is univalent on $\mathbb{D}$, then $f'(z) \neq 0$ for $z \in \mathbb{D}$, and

$$(1-|z|^2)\left| z \frac{f''(z)}{f'(z)} \right| < 6 \text{ for } z \in \mathbb{D}.$$

<u>Proof</u>: I. Suppose first that f is univalent on $\mathbb{D}$. Wlog $f(0)=0$, $f'(0)=1$; so $f \in \mathcal{S}$. Then $f'(z) \neq 0$ for $z \in \mathbb{D}$, and by Lem. 1.6:

$$\left| (1-|z|^2) \frac{f''(z)}{f'(z)} - 2\bar{z} \right| \leq 4 \text{ for } z \in \mathbb{D}.$$

Hence

$$(1-|z|^2)\left| z \frac{f''(z)}{f'(z)} \right| \leq 4|z| + 2|z|^2 < 6 \text{ for } z \in \mathbb{D}.$$

II. Suppose now that f satisfies the hypotheses of the first part.

Wlog $f(0)=0$, $f'(0)=1$.

Define

$$f(z,t) := f(e^{-t}z) + (e^{t}-e^{-t})\, z f'(e^{-t}z),$$

$$z \in \mathbb{D},\ t \geq 0.$$

$f_t(\cdot) = f(z,t)$. $\qquad I = [0,\infty)$

$f(\cdot,t) \in \mathbb{H}(\mathbb{D})$, $t \in I$

$f(z,\cdot) \in C^1[0,\infty)$, $z \in \mathbb{D}$

$$\frac{\partial f}{\partial t}(z,t) = -e^{-t} z f'(e^{-t}z) + (e^{t}+e^{-t})\, z f'(e^{-t}z) - (e^{t}-e^{-t})\, z^2 e^{-t} f''(e^{-t}z)$$

$$= e^{t} z f'(e^{-t}z) + (e^{t}-e^{-t})\, z^2 e^{-t} f''(e^{-t}z).$$

$$\left| \frac{\partial f}{\partial t}(z,t) \right| \leq M(r,T) \text{ for } |z| \leq r < 1,\ 0 \leq t \leq T$$

$f \in HL(\mathbb{D} \times I)$.

(108) $\frac{\partial f}{\partial z}(z,t) = e^{-t} f'(e^{-t}z)$

$\qquad + (e^{t}-e^{-t})\left[f'(e^{-t}z) + ze^{-t} f''(e^{-t}z)\right]$

$= e^{t} f'(e^{-t}z) + (e^{t}-e^{-t})\, z e^{-t} f''(e^{-t}z)$

Note: $w = ze^{-t}$

$$\frac{\partial f}{\partial z} = e^{t} f'(w)\left[1 + \underbrace{(1-e^{-2t})\frac{w f''(w)}{f'(w)}}_{A}\right]$$

$$A \le (1-|w|^2)\left|\frac{w f''(w)}{f'(w)}\right| \le 1$$

$|w| < e^{-t}$ so $\frac{\partial f}{\partial z}(z,t) \neq 0$.

$1-|w|^2 > 1-e^{-2t}$

Define

$$V(z,t) = -\frac{\dot{f}_t(z)}{f_t'(z)}$$

$$= -z\underbrace{\left[\frac{1-B(z,t)}{1+B(z,t)}\right]}_{p(z,t)}.$$

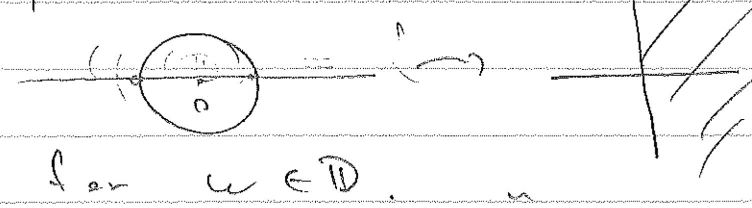
$$B(z,t) = (1-e^{-2t})\frac{e^{-t}z\, f''(e^{-t}z)}{f'(e^{-t}z)}.$$

Then: $B(\cdot,t) \in H(\mathbb{D})$, $t \in I$, $B \in C(\mathbb{D}\times I)$

$|B(z,t)| < 1$ for $(z,t) \in \mathbb{D}\times I$.

$B(0,t) \equiv 0$, $p(0,t) \equiv 1$

$\varphi(w) = \frac{1-w}{1+w}$



$\operatorname{Re}\varphi(w) > 0$ for $w \in \mathbb{D}$.

So $p(\cdot,t) \in \mathcal{P}$, i.e., V is a Herglotz vector field.

$\dot f_t(z,t) = -V(z,t)\, f_t'(z)$;

so f_t solves the Loewner-Kufarev-eq.

Ex. $M \ge 0$ s.t.

$|f(z)| \le M$, $|f'(z)| \le M$ for $|z| \le \frac{1}{2}$.

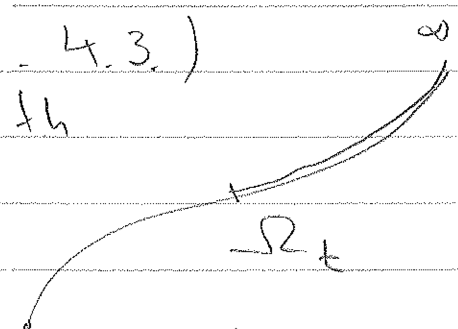
Then

$$|f_t(z)| \le |f(e^{-t}z)| + e^{t}|z|\,|f'(e^{-t}z)|$$
$$\le M(1+e^{t}) \le 2Me^{t} \text{ for } t \ge 0.$$

3, Thm. 5.12: $\{f_t\}_{t\in[0,\infty)}$ is a Loewner chain; so f_t is univalent for $t \ge 0$. In particular, $f_0 = f$ is univalent. □

6. Variants and special cases of the Loewner-Kufarev eq.

6.1. Slit domains (cf. Ex. 4.3)



$\gamma : [a,\infty] \to \hat{\mathbb{C}}$ simple path ending at ∞

$0 \notin \gamma[a,\infty]$, $\gamma(\infty) = \infty$

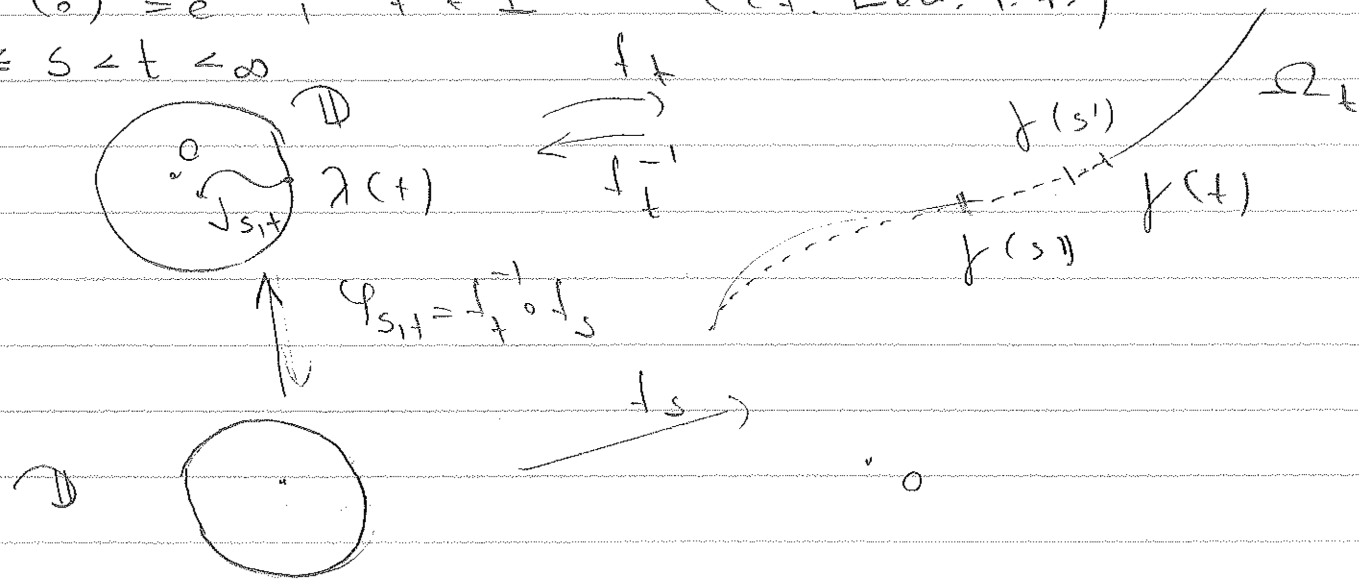
$\Omega_t = \mathbb{C} \setminus \gamma([t,\infty))$. simply connected

$\{\Omega_t\}$ geometric Loewner chain

$f_t : \mathbb{D} \to \Omega_t$ unique conf. map s.t. $f_t(0) = 0$, $f_t'(0) > 0$. $\{f_t\}$ Loewner chain

(110) By homeomorphic reparametrization of time we may wlog assume that $\{f_t\}$ is a normalized, i.e., $f_t'(0) = e^t$, $t \in I$ (cf. Rem. 4.7.)

$a \le s < t < \infty$



$$\gamma([s,t)) \subseteq \Omega_t, \quad \lim_{s' \to t^-} \gamma(s') = \gamma(t) \in \partial\Omega_t.$$

Hence by Cor. 2.10.:

$$\lambda(t) := \lim_{s' \to t^-} f_t^{-1}(\gamma(s')) \in \partial\mathbb{D}$$

exists.

$$J_{s,t} := f_t^{-1}(\gamma([s,t))) \subseteq \mathbb{D}$$

$$\overline{J_{s,t}} = J_{s,t} \cup \{\lambda(t)\}.$$

[Since $\hat{\mathbb{C}} \setminus \Omega_t = \gamma([t,\infty])$ is loc. connected (w.r.t. chordal metric), f_t has a cont. extension $f_t : \overline{\mathbb{D}} \to \hat{\mathbb{C}}$ (c.f. Thm. 2.1. + Rem. 2.6).

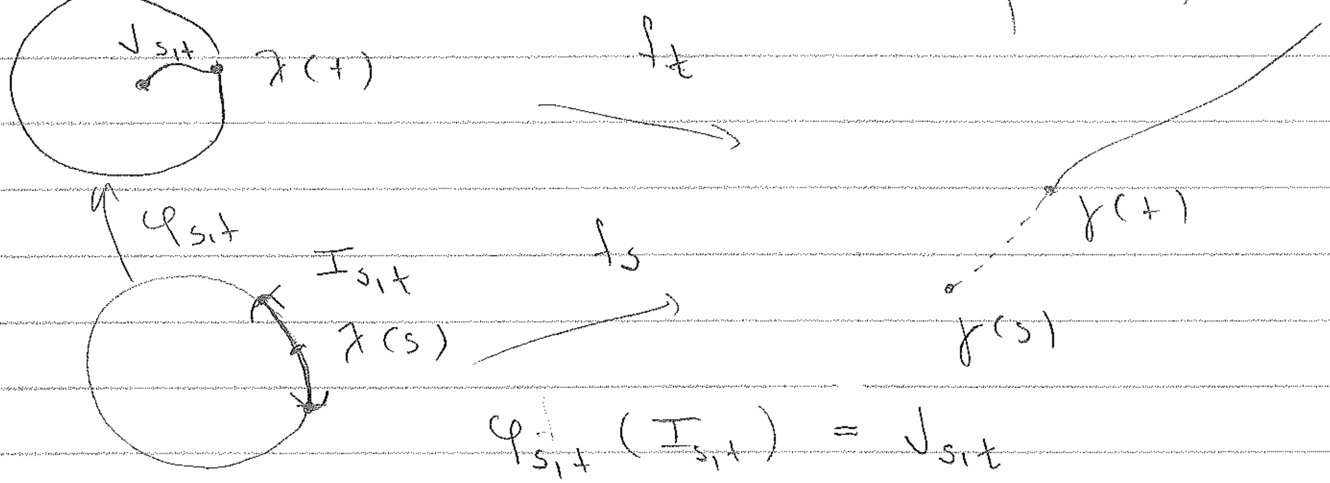
Then $f_t(\lambda(t)) = \lim_{s' \to t^-} f_t(f_t^{-1}(\gamma(s')))$

(iii) $= \lim_{s' \to t} \gamma(s') = \gamma(t)$.

So $f_t(\lambda(t)) = \gamma(t)$, (Prop. 2.7.)
$\lambda(t)$ uniquely determined by this eq.

$\varphi_{s,t}$ is a conf. m.p of $\mathbb{D}$ onto
the slit domain $\mathbb{D} \setminus J_{s,t} = U_{s,t}$.
$\partial U_{s,t} = J_{s,t} \cup \partial\mathbb{D}$ is loc. conn.;
so by Thm. 2.1. $\varphi_{s,t}$ has a cont.
ext. $\varphi_{s,t}: \overline{\mathbb{D}} \to \overline{U_{s,t}}$

As in Ex. 2.15. one shows that
there ex. an open arc $I_{s,t} \subseteq \partial\mathbb{D}$
s.t. $\varphi^{-1}_{s,t}(J_{s,t}) = I_{s,t}$
(cf. Prop. 2.7)



$\varphi_{s,t}(I_{s,t}) = J_{s,t}$

Then $\lambda(s) \in I_{s,t}$. $\varphi_{s,t}(\lambda(s)) \in J_{s,t}$

Lem. 6.2. Fix $T \in I = [a, \alpha)$.
Then there ex. a distortion function
$\omega: (0,\infty) \to (0,\infty)$, $\omega(\delta) \to 0$ as $\delta \to 0^+$
s.t.

i) $\operatorname{diam}(J_{s,t}) \le \omega(|s-t|)$, $\quad a \le s < t \le T$,
ii) $\operatorname{diam}(I_{s,t}) \le \omega(|s-t|)$,

$\omega_1, \omega_2, \dots$ distortion functions $\omega(\delta) \to 0$ as $\delta \to 0$

(112) Proof: By uniform cont. of γ on $[a,T]$ it follows that

$$\operatorname{diam}(\gamma[s,t]) \le \tilde{\omega}_1(|s-t|), \quad a \le s < t \le T$$

for some dist. funct. $\tilde{\omega}_1$.

By Thm. 2.17 $\qquad g_t = f_t^{-1}$

$$\operatorname{diam}(J_{s,t}) = \operatorname{diam}(g_t(\gamma[s,t]))$$

$$\le \omega_2\left(\frac{\operatorname{diam}(\gamma[s,t])}{f_t'(0)}\right)$$

$$\le \omega_2\left(e^{-a}\operatorname{diam}(\gamma([s,t]))\right)$$

$$\le \omega_3(|s-t|).$$

So $\operatorname{diam}(J_{s,t})$ is unif. small if $s<t$ are close in $[a,T]$.

Wlog $s<t$ are so close that $\operatorname{diam}(J_{s,t}) < \frac{1}{2}$.

$z_0 := \lambda(t)$, $r = 2\operatorname{diam}(J_{s,t})$.

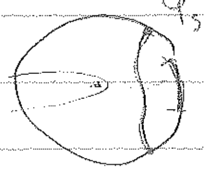
Then $J_{s,t} \subseteq B(z_0,r) =: B$, $0 \notin B(z_0,r)$.

So an arc $C \subseteq \mathbb{D} \cap \partial B$ separates 0 and $J_{s,t}$ in $\mathbb{D}$.

Then $\tilde{C} = \varphi_{s,t}^{-1}(C)$ separates 0 and $I_{s,t}$ in $\mathbb{D}$.

$\varphi_{s,t}$

$\varphi_{s,t}^{-1}(C) = \tilde{C}$



Hence by Thm. 2.17

$$\operatorname{diam}(I_{s,t}) \le \omega_4(\operatorname{diam}(\tilde{C}))$$

$$\le \omega_5\left(\frac{\operatorname{diam}(C)}{\underbrace{\varphi_{s,t}'(0)}_{=e^{s-t}}}\right) \le \omega_5(e^{t-s}\operatorname{diam}(C))$$

wlog ≤ 1

(113)

$\leq \omega_6(\operatorname{diam}(J_{s,t})) \leq \omega_7(|s-t|)$. $\square$

$\Omega \overset{\text{open}}{\subseteq} \hat{\mathbb{C}}$, $f: \Omega \to \mathbb{C}$ holomorphic
(at ∞: $z \mapsto f(1/z)$ holom. near 0)

$Cl(f,\Omega)$
$= \{ w \in \hat{\mathbb{C}} : \text{ex. seq. } \{z_n\} \text{ in } \Omega \text{ s.t. } z_n \to z_0 \in \partial\Omega$
and $f(z_n) \to w \}$ set of cluster values of f on Ω

"set of cluster values"

Prop. 6.3. $\Omega \subsetneq \hat{\mathbb{C}}$ open, $f: \hat{\mathbb{C}} \to \mathbb{C}$ holomorphic. Then:

i) $\sup_{z\in\Omega} |f(z)| = \sup\{|w| : w \in Cl(f,\Omega)\}$ (version of max. principle.
$[0,\infty]$

ii) if $Cl(f,\Omega) \subseteq \mathbb{C}$, then
$$\operatorname{osc}(f,\Omega) := \sup\{|f(z_1)-f(z_2)| : z_1,z_2 \in \Omega\}$$
$$= \sup\{|w_1-w_2| : w_1,w_2 \in Cl(f,\Omega)\}$$
$$= \operatorname{diam}(Cl(f,\Omega)).$$

Proof: i) (standard):

$\geq$ clear.

$\leq$: ex. seq. $\{z_n\}$ in Ω s.t.
$|f(z_n)| \to M := \sup_{z\in\Omega} |f(z)|$. $M = |w|$.

Wlog $z_n \to z_0 \in \overline{\Omega}$, $f(z_n) \to w \in \hat{\mathbb{C}}$.

1. Case: $z_0 \in \partial\Omega$. Then $w \in Cl(f,\Omega)$, and $M = |w|$. Done!

2. Case: $z_0 \in \Omega$: Then $|f|$ attains a max. at z_0. By the max. principle $f \equiv w$ on the comp. U of Ω with

(14) $z_0 \in G$.

Then also $w \in Cl(f, \Omega)$,

and $M = |w|$.

ii) $\geq$ clear

$\leq$: Let $z_1, z_2 \in \Omega$ be arb.

Consider $z \longmapsto f(z) - f(z_2)$.

Holomorphic on Ω so by (i) there ex. $w_1 \in Cl(f, \Omega) \subseteq \mathbb{C}$ s.t.

$$|f(z_1) - f(z_2)| \leq |w_1 - f(z_2)|.$$

Applying the same argument to $z \longmapsto w_1 - f(z)$ we find $w_2 \in Cl(f, \Omega) \subseteq \mathbb{C}$ s.t.

$$|f(z_1) - f(z_2)| \leq |w_1 - f(z_2)| \leq |w_1 - w_2|.$$

$\leq$ follows. $\square$

Lem. 6.4 Setup as in 6.1. $T \in [a, \alpha)$

Then there ex. a dist. function ω s.t.

$$\left| \varphi_{s,t}(z) - e^{t-s} z \right| \leq \omega(|s-t|) \, C(\operatorname{diam}(I_{s,t}) + \operatorname{diam}(J_{s,t}))$$

$z \in \overline{\mathbb{D}}$, $a \leq s < t \leq T$, $|s-t|$ small.

Proof: $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ reflection in $\partial\mathbb{D}$,

$R(z) = 1/\bar{z}$, $z \in \hat{\mathbb{C}}$.

$J^*_{s,t} = R(J_{s,t})$.

By the Schwarz Reflection Principle, $\varphi_{s,t}$ has an extension to a conformal map

$$\varphi_{s,t} : \underbrace{\hat{\mathbb{C}} \setminus I_{s,t}}_{= \Omega} \longleftrightarrow \Omega' = \hat{\mathbb{C}} \setminus (J_{s,t} \cup J^*_{s,t})$$

(115)

$$\varphi_{s,t}(z) = R(\varphi_{s,t}(R(z)))$$
for $|z| > 1$.

Near 0: $\qquad \varphi_{s,t}(0) = 0$

$$\varphi_{s,t}(z) = e^{s-t} z + a_2 z^2 + \cdots \qquad \varphi'_{s,t}(0) = e^{s-t}.$$

Near ∞:

$$\varphi_{s,t}(z) = \frac{1}{\overline{\varphi_{s,t}(1/\bar z)}}$$

$$= \frac{1}{e^{s-t}(1/z) + \bar a_2 (1/z)^2 + \cdots} \qquad w = 1/z$$

$$= \frac{1}{e^{s-t} w + \bar a_2 w^2 + \cdots}$$

$$= \frac{e^{t-s}}{w}\left(\frac{1}{1 + b_1 w + b_2 w^2 + \cdots}\right)$$

$$= \frac{e^{t-s}}{w}\left(1 - b_1 w + \cdots\right)$$

$$= e^{t-s} z + c_0 + \frac{c_1}{z} + \cdots$$

So $\varphi_{s,t}$ has a 1^{st} order pole at ∞.

Let

$$f(z) := \varphi_{s,t}(z) - e^{t-s} z, \quad z \in \Omega.$$

Then $f : \Omega \longmapsto \mathbb{C}$ is holomorphic on Ω (with removable sing. at ∞).

$CL(f, \Omega)$

$$= \{ w \in \hat{\mathbb{C}} : \text{ex. } \{z_n\} \text{ in } \Omega,\ z_n \longrightarrow z_0 \in \partial\Omega = \bar I_{s,t},\ f(z_n) \longrightarrow w \}.$$

$$\subseteq A + B = \{a + b : a \in A,\ b \in B\},$$

where $A = J_{s,t} \cup J^{*}_{s,t}$

$$B = \{-e^{t-s} z_0 : z_0 \in \bar I_{s,t}\}.$$

(116) By Prop. 6.3. : $f(0) = 0$

(1) $\sup_{z \in \mathbb{D}} |\varphi_{s,t}(z) - e^{t-s} z| = \sup_{z \in \mathbb{D}} |f(z)|$

$= \sup_{z \in \mathbb{D}} |f(z) - f(0)| \leq \operatorname{osc}(f, \Omega)$

$\leq \operatorname{diam}(Cl(f, \Omega))$

$\leq \operatorname{diam}(A) + \operatorname{diam}(B)$.

For $|s-t|$ small, $\operatorname{diam}(J_{s,t})$ small,

$\operatorname{diam}(J_{s,t}^{*}) \lesssim \operatorname{diam}(J_{s,t})$

$\leq \omega_1(|s-t|)$,

so $\operatorname{diam}(A) \leq \omega_2(|s-t|)$.

If $|s-t|$ small $e^{t-s} \lesssim 1$, and

$\operatorname{diam}(B) \lesssim \operatorname{diam}(I_{s,t})$

$\lesssim \omega_3(|s-t|)$.

Hence

$\sup_{z \in \mathbb{D}} |f(z)| \lesssim \omega(|s-t|)$ as desired. $\square$

<u>Cor. 6.5.</u> λ (as in 6.1) is a cont. function on $[a, \infty)$.

Proof: $a \leq s < t \leq T$.

Then $\varphi_{s,t}(\lambda(s)) \in J_{s,t}$.

(1) $|\lambda(t) - \varphi_{s,t}(\lambda(s))| \leq \operatorname{diam}(J_{s,t})$

$\leq \omega_1(|s-t|)$

(2) $|\varphi_{s,t}(\lambda(s)) - e^{t-s}\lambda(s)|$

$\leq \omega_2(|s-t|)$

(117)

(3) $\left| e^{t-s}\lambda(s) - \lambda(s) \right| = \left| e^{(t-s)} - 1 \right| \lesssim \omega_3(s-t)$.

By (1) – (3):

$|\lambda(t) - \lambda(s)| \lesssim \omega(|s-t|)$; so λ is cont. on $[a, T]$; T arb. so λ cont. on $[a, \infty)$. $\square$

<u>Thm. 6.6</u> (Loewner eq. (for slit mappings))

$\{f_t\}$ normalized Loewner chain generated by a slit (as in 6.1).

Then

$$\dot{f}_t(z) = -V(z,t) \cdot f_t'(z) \quad \text{for a.e. } t \in [a,\infty),\ z \in \mathbb{D},$$

where

$$V(z,t) = -z\,\frac{\lambda(t)+z}{\lambda(t)-z}, \quad (z,t) \in \mathbb{D} \times I.$$

Have $\lambda : I = [a,\infty) \to \partial\mathbb{D}$ is continuous.

Proof. $\varphi_{s,t} = f_t^{-1} \circ f_s$

We know that $\{f_t\}$ satisfies the Loewner-Kufarev eq. with (cf. 4.13)

$$V(z,t) = \lim_{\varepsilon \to 0} \frac{\varphi_{t,t+\varepsilon}(z) - z}{\varepsilon} \quad z \in \mathbb{D},\ \text{a.e. } t \in I.$$

(118) For $a \le s < t < \infty$ define

$$\Phi_{s,t}(z) := \log\left(\frac{z}{\varphi_{s,t}(z)}\right)$$

$= (t-s) + \cdots$ holomorphic in $\mathbb{D}$ (cf. 4 (a))

Actually, $z \longmapsto \frac{z}{\varphi_{s,t}(z)}$ has a zero-free cont. ext. to $\overline{\mathbb{D}}$;

hence this function has a cont. logarithm on $\overline{\mathbb{D}}$ (uniquely determined by a point normalization).

Hence $\Phi_{s,t}$ has a cont. ext. to $\overline{\mathbb{D}}$.

$R<1$:

$$\Phi_{s,t}(z) = \underbrace{i \operatorname{Im} \Phi_{s,t}(0)}_{=0} + \frac{1}{2\pi}\int_0^{2\pi} \frac{\xi+z}{\xi-z} \operatorname{Re} \Phi_{s,t}(\xi)\,|d\xi|$$

(246c, Midterm) $\xi = e^{it}, |d\xi| = dt$

$$\operatorname{Re} \Phi_{s,t}(\xi) = \log\left|\frac{\xi}{\varphi_{s,t}(\xi)}\right| = \log \frac{1}{|\varphi_{s,t}(\xi)|} \ge 0$$

for $\xi \in \partial\mathbb{D}$. $|\varphi_{s,t}(\xi)| = 1$ for $\xi \in \partial\mathbb{D} \setminus I_{s,t}$;

so $\operatorname{Re} \Phi_{s,t}(\xi)$ supported on $I_{s,t} \ni \lambda(s)$

$$t-s = \Phi_{s,t}(0) = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} \Phi_{s,t}(\xi)\,|d\xi|.$$

Define a prob. measure $\mu_{s,t}$ on $\partial\mathbb{D}$

(119) by

$$d\mu_{s,t}(\xi) = \frac{1}{2\pi(t-s)} \operatorname{Re} \Phi_{s,t}(\xi)\, |d\xi|$$

$\operatorname{Supp}(\mu_{s,t}) = \overline{I}_{s,t} \ni \lambda(s)$

Fix s, $t = s+\varepsilon$, $\varepsilon \longrightarrow 0^+$:

Then $\operatorname{diam} I_{s,s+\varepsilon} \longrightarrow 0$ (Lem. 6.2.)

Hence

$$\mu_{s,s+\varepsilon} \xrightarrow{w^*} \delta_{\lambda(s)} \quad (\text{Dirac mass at } \lambda(s)),$$

as $\varepsilon \longrightarrow 0$

i.e.,

$$\int_{\partial\mathbb{D}} h(\xi)\, d\mu_{s,s+\varepsilon}(\xi) \longrightarrow \int_{\partial\mathbb{D}} h(\xi)\, d\delta_{\lambda(s)}(\xi)$$

$f \in C(\partial\mathbb{D})$ $\qquad = h(\lambda(s))$.

So

$$\lim_{\varepsilon \to 0^+} \frac{\Phi_{s,s+\varepsilon}(z)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0^+} \int_{\partial\mathbb{D}} \frac{\xi+z}{\xi-z}\, d\mu_{s,s+\varepsilon}(\xi)$$

$$= \int_{\partial\mathbb{D}} \frac{\xi+z}{\xi-z}\, d\delta_{\lambda(s)}(\xi) = \frac{\lambda(s)+z}{\lambda(s)-z}.$$

for all $s \in I$, $z \in \mathbb{D}$.

On the other hand,

$$\varphi_{s,t}(z) = z \cdot \exp(-\Phi_{s,t}(z))$$

and

$$V(z,s) = \lim_{\varepsilon \to 0^+} \frac{\varphi_{s,s+\varepsilon}(z) - z}{\varepsilon}$$

(120)

$$= \lim_{\varepsilon \to 0^+} z \cdot \frac{\exp\left(-\Phi_{s,s+\varepsilon}(z)\right) - 1}{\varepsilon}$$

$$= z \cdot \partial_\varepsilon \exp\left(-\Phi_{s,s+\varepsilon}(z)\right) \Big|_{\varepsilon=0}$$

Chain rule

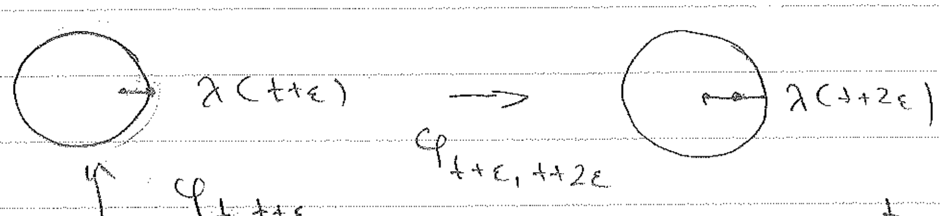
$$= -z \exp(0) \cdot \frac{\partial \Phi_{s,s+\varepsilon}(z)}{\partial \varepsilon}\Big|_{\varepsilon=0}$$

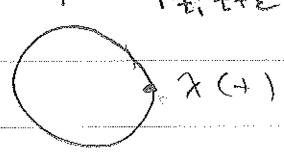
$$= -z \frac{\lambda(s)+z}{\lambda(s)-z}.$$

N.b. $\lim_{\varepsilon \to 0^+} \Phi_{s,s+\varepsilon}(z) = \lim_{\varepsilon \to 0^+} \varepsilon \cdot \frac{\Phi_{s,s+\varepsilon}(z)}{\varepsilon}$

$= 0$, $\square$

Ex. 6.7. $\lambda(t) \equiv 1$



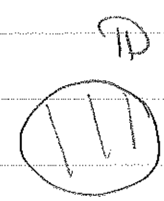


$$f_t(z) = \frac{e^t z}{(1+z)^2}$$

$$\dot f_t = \frac{e^t z}{(1+z)^2}, \quad f_t' = e^t \frac{1-z}{(1+z)^3}$$

$$\dot f_t(z) = + z \underbrace{\frac{1+z}{1-z}}_{= \frac{\lambda(t)+z}{\lambda(t)-z}} f_t'(z)$$

f_t :



$\frac{1}{4} e^t$

(121) Ex. 6.8. Stationary solutions of the Loewner–Kuferev eq.

Suppose $f_t(z) = a(t) \cdot f(z)$ is a normalized Loewner chain.

$f(0) = 0, \; f'(0) = 1$

$f_t'(0) = a(t) \cdot f'(0) = a(t) = e^t$

$f_t(z) = e^t \cdot f(z)$

$\dot{f}_t(z) = e^t f(z), \quad f_t'(z) = e^t f'(z)$

$\dot{f}_t = e^t f = -V(z,t) \cdot e^t \cdot f'$

$= + z\, p(z,t)\, e^t f'$

$p(z,t) = \dfrac{f(z)}{z f'(z)}$ (removable sing. at 0)

$\operatorname{Re} p > 0$ equiv. $\operatorname{Re}\left(\dfrac{f(z)}{z f'(z)}\right) > 0$

equiv. $\operatorname{Re}\left(\dfrac{z f'(z)}{f(z)}\right) > 0.$

Thm. 6.9. $f \in H(\mathbb{D})$, $f(0) = 0$, $f'(0) = 1$.

TFAE:

i) $\operatorname{Re}\left(\dfrac{z f'(z)}{f(z)}\right) > 0$

has removable singularities by assumption.

ii) $f \in \mathcal{S}$ and $\Omega = f(\mathbb{D})$ is starlike w.r.t. 0, i.e., $[0,w] \subseteq \Omega$ for all $w \in \Omega$.

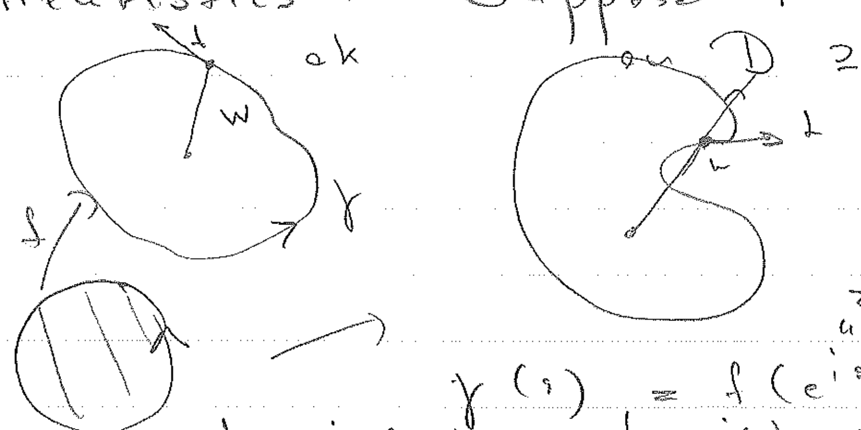
Proof: (i) $\to$ (ii): By Ex. 6.8.

$F(z,t) = f_t(z) := e^t f(z)$ solves Loewner–Kuferev eq.

(122) F is C^∞-smooth on $\mathbb{R}\times\mathbb{D}$ and
$|f_t(z)| \le e^t C$ for $t\in\mathbb{R}$, $z\in\overline{B}(0,\frac{1}{2})$.
$f_t'(0) = e^t$, $t\in\mathbb{R}$.
Hence $\{f_t\}$ is a normalized Loewner chain; so
$f = f_0$ is a conf. map and
$\Omega_t := f_t(\mathbb{D}) = e^t\,\Omega \subseteq \Omega_0 = \Omega$
for all $t<0$.
Hence Ω is starlike w.r.t. 0.

ii) If $f\in\mathcal{S}$ and Ω is starlike,
then $\{\Omega_t\}_{t\in\mathbb{R}}$ with $\Omega_t = e^t\,\Omega$
forms a geometric Loewner chain
corr. to the analytic Loewner chain $\{f_t\}_{t\in\mathbb{R}}$ with $f_t(z) = e^t f(z)$.
Hence
$$\operatorname{Re}\left(\frac{z f'(z)}{f(z)}\right) > 0 \quad \text{by Ex. 6.8.}$$

Heuristics: Suppose f holomorphic on $D \supseteq \overline{\mathbb{D}}$.



starlike $\simeq \operatorname{Im}\left(\frac{t}{w}\right)$

z

$$\gamma(s) = f(e^{is})$$
$$t = \dot\gamma(s) = f'(e^{is})\cdot i e^{is}$$
$$w = \gamma(s)$$
$$0 \le \operatorname{Im}\left(\frac{t}{w}\right) = \operatorname{Im}\left(\frac{\dot\gamma(s)}{\gamma(s)}\right) = \operatorname{Im}\left(i\,\frac{z f'(z)}{f(z)}\right)$$
$$= \operatorname{Re}\left(\frac{z f'(z)}{f(z)}\right)$$