① Conformally Invariant Processes in the Plane

UCLA, Fall 2012

1. Koebe's Distortion Theorem

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (open) unit disk
$\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ Riemann sphere
$\tilde{\mathbb{D}} := \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} = \{z \in \hat{\mathbb{C}} : |z| > 1\}$
compl. of closed unit disk

Def. 1.1. $\mathcal{S} = \{ f : \mathbb{D} \to \mathbb{C} :$ f holomorphic + injective (= conf. map onto its image),
$f(0) = 0,\ f'(0) = 1\}$
$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$
Taylor series exp.

$\Sigma = \{ g : \tilde{\mathbb{D}} \to \hat{\mathbb{C}} : g$ holom. + injective (= conf. map onto its image) $\}$,

(*) $g(w) = w + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \frac{b_3}{w^3} + \ldots$
Laurent series exp. at ∞.

$g(\infty) = \infty$, $g'(z) = 1 + O(1/z^2)$ near ∞
$g'(\infty) := \lim_{z \to \infty} g'(z) = 1.$

Note: If g is a holomorphic map on $\tilde{\mathbb{D}}$, $g(\infty) = \infty$, g inj., then
$g(\frac{1}{z}) = \frac{1}{z} + b_0 + b_1 z + b_2 z^2 + \ldots$
is holomorphic in $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$, and has 1st order pole

(2) The series in (*) converges uniformly on compact subsets of $\mathbb{C}\setminus\overline{\mathbb{D}}$.

Thm. 1.2. (Area Thm.)

If $g \in \Sigma$, then

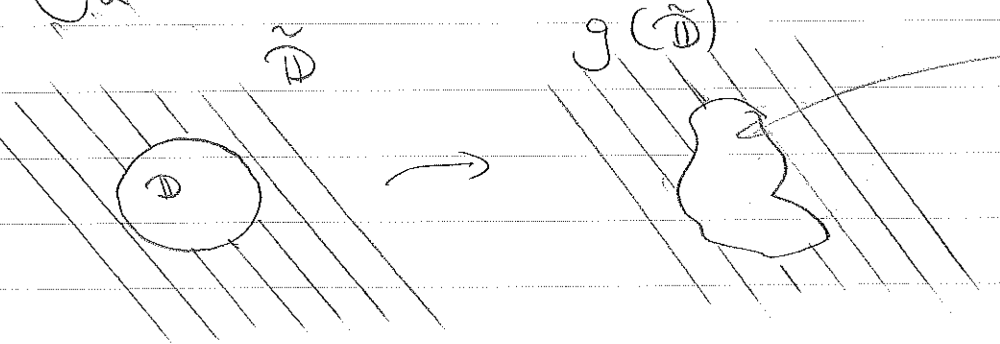
$$1. \quad \underset{\text{Area}}{A}\left(\hat{\mathbb{C}} \setminus g(\tilde{\mathbb{D}})\right) = \pi\left(1 - \sum_{n=1}^{\infty} n|b_n|^2\right) \geq 0$$

In particular,

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1 \quad \text{and} \quad |b_1| \leq 1.$$

Have $|b_1| = 1$ iff

$$g_\alpha(w) = w + b_0 + \frac{e^{2i\alpha}}{w}, \quad \alpha \in \mathbb{R}.$$



Proof: Pick $r > 1$. Define $\gamma_r := g(r\cdot e^{it})$, $t \in [0, 2\pi]$.

γ_r is a (parametrized) Jordan curve.

$$\operatorname{ind}_{\gamma_r}(w) = \begin{cases} 0 & \text{for } w \in O(\gamma_r) \text{ outside of } \gamma_r \\ \pm 1 & w \in I(\gamma_r) \text{ inside of } \gamma_r \end{cases}$$

winding no.

by Jordan Curve Thm.

$$I(\gamma_r) = \hat{\mathbb{C}} \setminus g(\tilde{\mathbb{D}}) \cup g(\{w \in \mathbb{C} : 1 < |w| < r\})$$

Moreover, $\operatorname{ind}_{\gamma_r}(w) = 1$ for $w \in I(\gamma_r)$.

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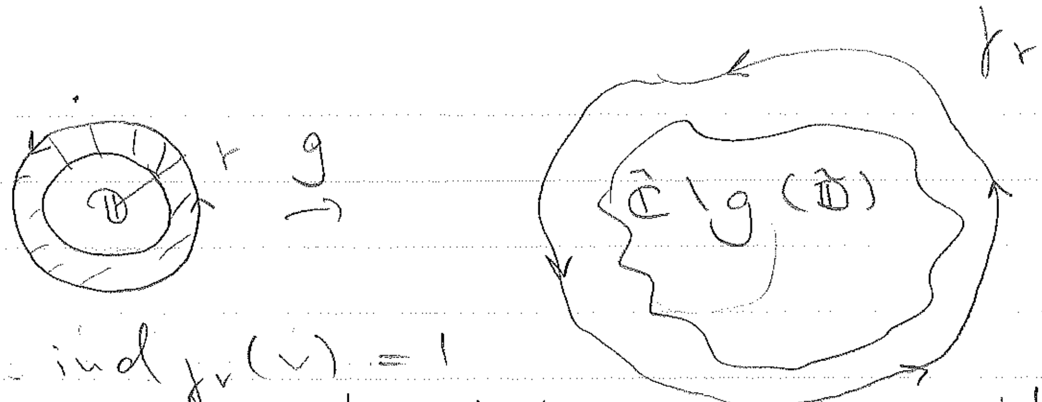


$\supseteq$: $\mathrm{ind}_{\gamma_r}(v) = 1$
follows from homotopy invariance of winding nos. (let $r \to +\infty$)
$\subseteq$ follows because every point not on γ_r or the right hand side lies in the unbdd. comp. of $\mathbb{C} \setminus \gamma_r$

So
$$\hat{\mathbb{C}} \setminus g(\tilde{\mathbb{D}}) = \bigcap_{r>1} I(\gamma_r), \text{ and}$$

$$A(\hat{\mathbb{C}} \setminus g(\hat{\mathbb{D}})) = \lim_{r \to 1^+} A(I(\gamma_r)).$$

By Green's Thm.: (246 B)

$$\frac{1}{2i} \int_{\gamma_r} \bar{u}\, du = \int_{\mathbb{C}} \mathrm{ind}_{\gamma_r}(u)\, dA(u)$$
$$= A(I(\gamma_r)).$$

On the other hand
$$\frac{1}{2i} \int_{\gamma_r} \bar{u}\, du = \frac{1}{2i} \int_0^{2\pi} \overline{g(r \cdot e^{it})} \cdot g'(r \cdot e^{it})\, i\, r \cdot e^{it}\, dt$$

$$\underset{w = r \cdot e^{it}}{=} \frac{1}{2} \int_0^{2\pi} \overline{g(w)} \cdot g'(w) \cdot w\, dt$$

$$g(w) = w + \sum_{n=0}^{\infty} \frac{b_n}{w^n} \qquad g'(w) = 1 - \sum_{n=1}^{\infty} \frac{n\, b_n}{w^{n+1}}$$

④ Note $\displaystyle\int_0^{2\pi} w^k\,dt \overset{w=r\cdot e^{it}}{\underset{\bar{w}=\frac{r^2}{w}}{=}} \begin{cases} 0 & \text{for } k\in\mathbb{Z}\setminus\{0\}, \\ 2\pi r & k=0. \end{cases}$

By unif. convergence, we can integrate "term-by-term" and so,

$$A(I(\gamma_r)) = \frac{1}{2}\int_0^{2\pi} \overline{g(w)}\cdot g'(w)\cdot w\,dt$$

$$= \frac{1}{2}\int_0^{2\pi}\left(\bar{w}+\sum_{n=0}^{\infty}\frac{\bar{b}_n}{\bar{w}^n}\right)\left(w-\sum_{n=1}^{\infty}\frac{n b_n}{w^n}\right)dt$$

$$= \frac{1}{2}\int_0^{2\pi}\left(|w|^2-\sum_{n=1}^{\infty} n\frac{|b_n|^2}{|w|^{2n}}\right)dt$$

$$= \pi\left(r^2-\sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}\right),$$

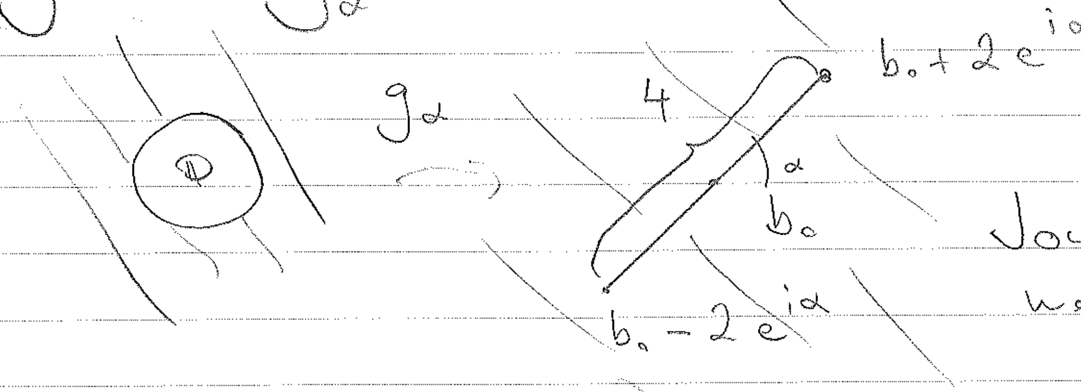
$$\xrightarrow{r\to 1} \pi\left(1-\sum_{n=1}^{\infty} n|b_n|^2\right),$$

has to be justified! The first part follows!

So $|b_1|\le 1$

If $|b_1|=1$, then $b_2=b_3=\dots=0$, and so

$$g(w)=g_\alpha(w)=w+b_0+\frac{e^{2i\alpha}}{w},\quad b_1=e^{2i\alpha},\ \alpha\in\mathbb{R}$$



Joukovsky map

∎

⑤ Cor. 1.3. Let
$g(w) = w + b_0 + \frac{b_1}{w} + \cdots \in \Sigma$.
If $u \in \hat{\mathbb{C}} \setminus g(\tilde{\mathbb{D}})$ (i.e., u is omitted by g), then
$|u - b_0| \leq 2$ and if we have equality then g is a Joukowsky map.

Proof:
Let $h(w) = \sqrt{g(w^2) - u}$ on $\tilde{\mathbb{D}}$

$$= w \cdot \sqrt{\underbrace{\frac{g(w^2)}{w^2} - \frac{u}{w^2}}_{\text{zero-free holomorphic funct. on simply conn. region } \tilde{\mathbb{D}}}} = w$$

$$= w \left(1 + \frac{(b_0 - u)}{w^2} + \cdots\right)^{1/2} \quad \text{odd}$$

$$= w \left(1 + \frac{1}{2}\frac{(b_0 - u)}{w^2} + \cdots\right) = w + \underbrace{\frac{1}{2}(b_0-u)}_{\tilde{b}_1}\frac{1}{w} + \cdots$$

Note h holomorphic and injective on $\tilde{\mathbb{D}}$:
$h(w_1) = h(w_2) \Rightarrow g(w_1^2) - u = g(w_2^2) - u$
$\Rightarrow w_1^2 = w_2^2 \Rightarrow w_1 = \pm w_2$
If $w_2 = -w_1$, then $h(w_1) = h(w_2) = -h(w_1)$, and so $h(w_1) = \infty$ (0 impossible!)
$\Rightarrow w_2 = w_1 = \infty$.
So $h \in \Sigma$. By Thm. 1.2. we have
$\underbrace{\left|\frac{1}{2}(b_0 - u)\right|}_{\tilde{b}_1} \leq 1$ equiv. $|u - b_0| \leq 2$
If $|u - b_0| = 2$, then $|\tilde{b}_1| = 1$, and so h is a Joukowsky map which implies that

⑥ g is a Joukowsky map

$$h(w) = w + \frac{1}{2}\frac{(b_0 - a)}{w}$$

$$g(w^2) = h(w)^2 + a = w^2 + b_0 + \frac{1}{4}\frac{(b_0-a)^2}{w^2}, \text{ so}$$

$$g(w) = w + b_0 + \frac{1}{4}\frac{(b_0-a)^2}{w} = w + b_0 + \frac{\tilde{b}_1^2}{w}.$$

<u>Thm. 1.4.</u> Let $f \in \mathcal{S}$,

$f(z) = z + a_2 z^2 + \cdots$.

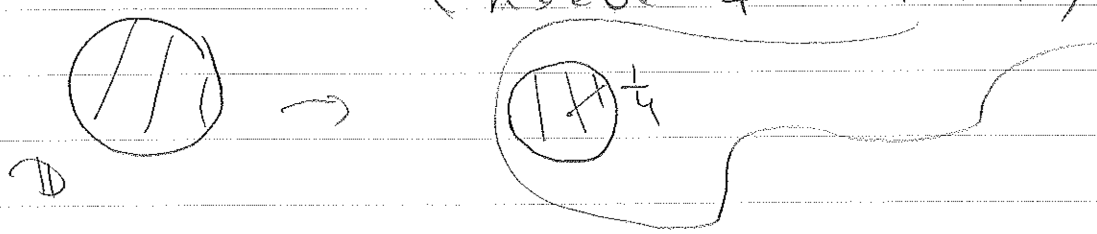
Then:

i) $|a_2| \leq 2$,

ii) if $v \in \mathbb{C} \setminus f(\mathbb{D})$, then $|v| \geq \frac{1}{4}$,

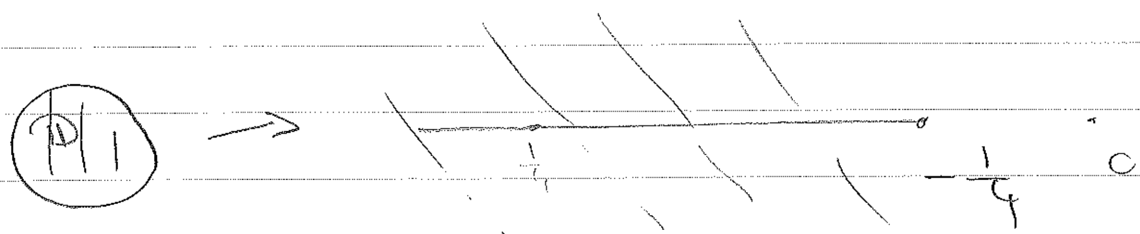
i.e., $B(0, \frac{1}{4}) \subseteq f(\mathbb{D})$

(Koebe $\frac{1}{4}$ - Thm.)



We have equality in (i) or (ii) iff f is a <u>Koebe function</u>,

i.e., $f(z) = e^{-i\alpha} k(e^{i\alpha} z)$, $k(z) = \frac{z}{(1-z)^2}$



$$k(z) = z + 2z^2 + 3z^3 + 4z^4 + \cdots$$

(7) <u>Rem.</u> A long-standing open problem was Bieberbach's Conjecture:
if $f \in \mathcal{S}$, then $|a_n| \leq n$ for $n \geq 2$.
Proved by de Branges (early 1980's).

<u>Proof:</u> If $f \in \mathcal{S}$, then

$$g(w) = \frac{1}{f(1/w)} \in \Sigma \qquad \frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

$$= \frac{1}{\frac{1}{w} + a_2 \frac{1}{w^2} + \cdots} = w \cdot \frac{1}{1 + \frac{a_2}{w} + \frac{a_3}{w^2} + \cdots}$$

$$= w\left(1 - \left(\frac{a_2}{w} + \frac{a_3}{w^2} + \cdots\right) + \left(\frac{a_2}{w} + \frac{a_3}{w^2} + \cdots\right)^2 + \cdots\right)$$

$$= w\left(1 - \frac{a_2}{w} + \frac{(a_2^2 - a_3)}{w^2} + \cdots\right)$$

$$= w - \underbrace{a_2}_{b_0} + \frac{\overbrace{a_2^2 - a_3}^{=b_1}}{w} + \cdots$$

Moreover, $u = 0$ is omitted by g!

i) By Cor. 1.3. $2 \geq |\underbrace{u}_{0} - \underbrace{b_0}_{-a_2}| = |a_2|$.

If equality, then the proof of Cor. 1.3, shows

$$g(w) = w + b_0 + \frac{1}{4}\frac{(b_0 - u)^2}{w^2} = w - a_2 + \frac{1}{4}\frac{a_2^2}{w}$$

$$= w\left(1 - \frac{a_2}{2}\frac{1}{w}\right)^2; \text{ so}$$

$$f(z) = \frac{1}{g(1/z)} = \frac{z}{\left(1 - \frac{a_2}{2} z\right)^2}, \text{ where } |a_2| = 2.$$

(8) ii) If $v \in \mathbb{C}$ is omitted by f, then $u = 1/v$ is omitted by g.

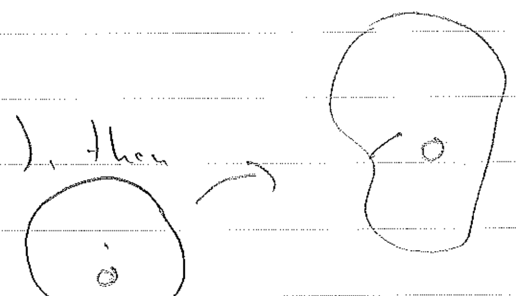
So by Cor. 1.3.

$$2 \geq |u - b_0| = \left|\frac{1}{v} + a_2\right| \text{ ; so}$$

$$\left|\frac{1}{v}\right| \leq |-a_2| + \left|\frac{1}{v} + a_2\right|$$

$$= |a_2| + \left|\frac{1}{v} + a_2\right| \leq 4 \text{ equiv.}$$

$|v| \geq \frac{1}{4}$. If $|v| = \frac{1}{4}$, then $\left|\frac{1}{v}\right| = 4$ and $|a_2| = 2$. Again f is a rotation of the Koebe function. □



Rem 1.5. If $f \in S$ and $\Omega = f(\mathbb{D})$, then

$$\frac{1}{4} \leq \operatorname{dist}(0, \partial\Omega) \leq 1.$$

Proof: First ineq. follows from $\frac{1}{4}$-Thm.

2nd ineq.: $d := \operatorname{dist}(0, \partial\Omega) < \infty$.

Define $g(w) := f^{-1}(dw)$, $w \in \mathbb{D}$.

Then $g(\mathbb{D}) \subseteq \mathbb{D}$, $g(0) = 0$; so by the Schwarz lemma:

$$1 \geq |g'(0)| = \frac{d}{|f'(0)|} = d. \quad \square$$

Lem. 1.6. If $f \in S$, then

$$\left|(1 - |z|^2)\frac{f''(z)}{f'(z)} - 2\bar{z}\right| \leq 4 \text{ for } z \in \mathbb{D}.$$

Proof: Fix $z_0 \in \mathbb{D}$; $\varphi \in \operatorname{Aut}(\mathbb{D})$

$\varphi(0) = z_0$, $\varphi(z) = \frac{z + z_0}{1 + \bar{z}_0 z}$, $\varphi'(z) = \frac{1 - |z_0|^2}{(1 + \bar{z}_0 z)^2}$

$$\varphi''(z) = -2\frac{(1 - |z_0|^2)\bar{z}_0}{(1 + \bar{z}_0 z)^3}$$

(9) $g = f \circ \varphi$ conf. map on $\mathbb{D}$.
Not normalized!

$$h = \frac{g - g(0)}{g'(0)} \in \mathcal{S} \qquad |a_2(h)| \le 2.$$

$$g(0) = f(z_0), \quad g'(0) = f'(z_0) \cdot \varphi'(0)$$
$$= f'(z_0)\,(1 - |z_0|^2).$$

$$a_2(h) = \frac{1}{2} h''(0) = \frac{1}{2} \frac{g''(0)}{g'(0)}.$$

$$g' = (f' \circ \varphi) \cdot \varphi'$$
$$g'' = (f'' \circ \varphi) \cdot \varphi'^2 + (f' \circ \varphi) \cdot \varphi''$$
$$g''(0) = f''(z_0) \cdot (1 - |z_0|^2)^2 + f'(z_0) \cdot (-2\bar{z}_0 (1 - |z_0|^2))$$

So

$$2 \ge |a_2(h)|$$
$$= \frac{1}{2} \left| \frac{g''(0)}{g'(0)} \right| = \frac{1}{2} \left| \frac{f''(z_0)}{f'(z_0)} (1 - |z_0|^2) - 2\bar{z}_0 \right|. \quad \square$$

<u>Thm. 1.7.</u> (Koebe's Distortion Thm.)
Let $f \in \mathcal{S}$. Then:

i) $$\frac{1 - |z|}{(1 + |z|)^3} \le |f'(z)| \le \frac{1 + |z|}{(1 - |z|)^3}$$

ii) $$\frac{|z|}{(1 + |z|)^2} \le |f(z)| \le \frac{|z|}{(1 - |z|)^2}, \quad z \in \mathbb{D}.$$

Estimates are sharp and the Koebe function is the only extremal (up to rotation).

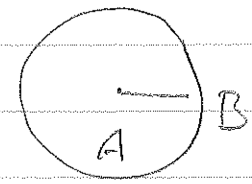
(10) **Proof:** By rotational invariance wlog $x \in [0,1)$.

$$g(z) = \log f'(z) = \log(1 + 2a_2 z + \dots) = 2a_2 z + \dots,$$

$g(0) = 0$

$$g' = \frac{f''}{f'}. \quad \text{By Lem. 1.6:}$$

$$\left| \frac{f''(x)}{f'(x)} - \frac{2x}{1-x^2} \right| \le \frac{4}{1-x^2}$$



By integration

$$\Big| \underbrace{g(x)}_{= \log f'(x)} - \underbrace{\int_0^x \frac{2t}{1-t^2}\,dt}_{= \log \frac{1}{1-x^2} = A} \Big| \le \underbrace{\int_0^x \frac{4\,dt}{1-t^2}}_{= 2\log\frac{1+x}{1-x} = B}, \quad x \in [0,1)$$

$\Rightarrow$

$$\underbrace{A - B}_{= \log \frac{1-x}{(1+x)^3}} \le \underbrace{\log|f'(x)|}_{= \operatorname{Re} g(x)} \le \underbrace{A + B}_{= \log \frac{1+x}{(1-x)^3}}$$

Exponentiating, the first ineq. follows

$$|f(x)| = \left| \int_0^x f'(t)\,dt \right| \le \int_0^x \frac{1+t}{(1-t)^3}\,dt = \frac{x}{(1-x)^2},$$

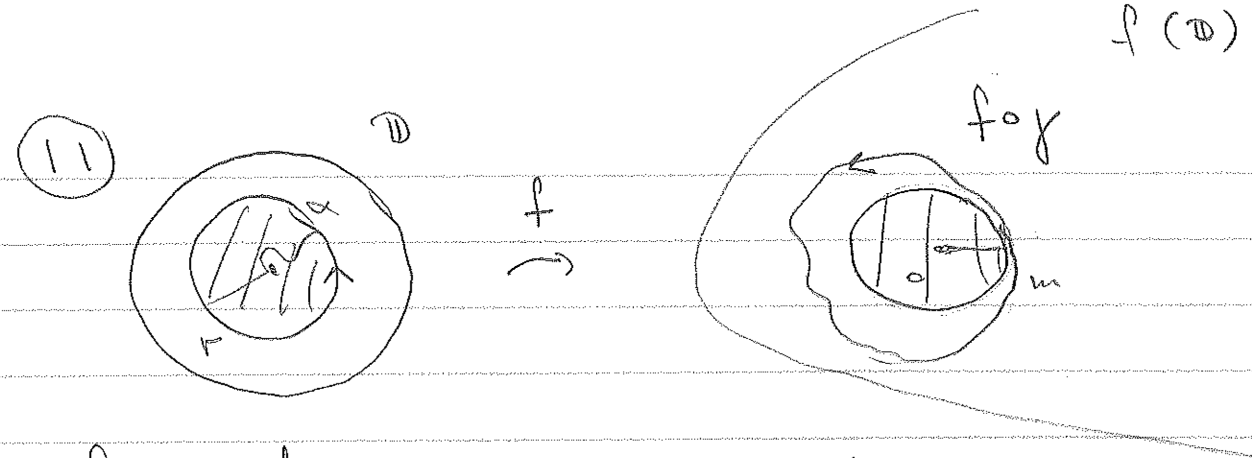
Upper bdd. in (ii)!

Lower bdd.: $r \in (0,1)$, $m = \min_{|z|=r} |f(z)| > 0$

$\gamma(t) := r \cdot e^{it}$, $t \in [0, 2\pi]$. Wlog $f(r \cdot e^{i\theta}) = m$

$f(\mathbb{D}) = \Omega$

(11)



$f\circ\gamma$ does not meet $B(0,m)$, $w \in B(0,m)$

Argument Principle $\Rightarrow$

$\#$ of zeros of $f-w$ in $B(0,r)$

$= \operatorname{ind}_{f\circ\gamma}(w) = \operatorname{ind}_{f\circ\gamma}(0) =$

$\#$ of zeros of $f-0 = f$ in $B(0,r) = 1$.

$\Rightarrow B(0,m) \subseteq f(B(0,r))$

$\Rightarrow \overline{B}(0,m) \subseteq f(\overline{B}(0,r)) \subseteq \Omega = f(\mathbb{D})$

$\Rightarrow [0,m] \subseteq \Omega$

$\alpha(t) := f^{-1}(t)$, $t \in [0,m]$.

α path in $\mathbb{D}$ from $0 = f^{-1}(0)$ to $r \cdot e^{i\theta}$
$= f^{-1}(m)$.

$f(\alpha(t)) = t \Rightarrow f'(\alpha(t)) \cdot \alpha'(t) \equiv 1$;

so

$$m = \int_0^m dt = \int_0^m |f'(\alpha(t))| \cdot |\alpha'(t)|\, dt$$

$$= \int_\alpha |f'(z)|\,|dz| = \int_0^L |f'(\tilde\alpha(s))|\, ds$$

[arc-length reparametrization $\tilde\alpha : [0,L] \to \mathbb{C}$ of α:

$L = \ell(\alpha) = \text{length}(\alpha) \geq r$.

$\tilde\alpha(\ell(\alpha|[0,t])) = \alpha(t)$

$$\int_\alpha g(z)|dz| = \int_0^L g(\tilde\alpha(s))\, ds.$$

(12) Since $\alpha(0) = \tilde{\alpha}(0) = 0$,
$|\tilde{\alpha}(s)| \le s$;

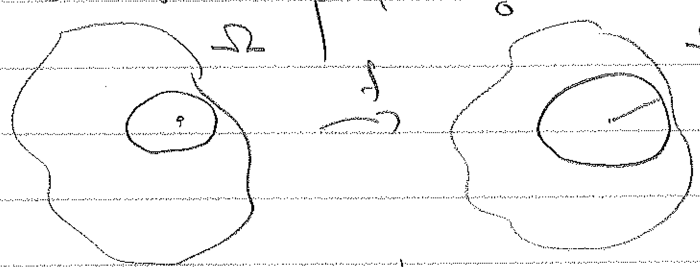
So
$$m = \int_{\alpha}^{L} |f'(\tilde{\alpha}(s))|\, ds \ge \int_{0}^{L} \frac{1 - |\tilde{\alpha}(s)|}{(1 + |\tilde{\alpha}(s)|)^3}\, ds$$
$$\ge \int_0^r \frac{1-s}{(1+s)^3}\, ds = \frac{r}{(1+r)^2}. \quad \square$$

<u>Cor. 1.8.</u> $\mathcal{S}$ is a normal family, i.e., every seq. $\{f_n\}$ in $\mathcal{S}$ has a subsequence $\{f_{n_k}\}$ that converges loc. unif. in $\mathbb{D}$;
moreover every loc. unif. limit of a seq. in $\mathcal{S}$ also lies in $\mathcal{S}$.
(So $\mathcal{S}$ is compact w.r.t. topology of loc. unif. conv.)

<u>Proof</u> By Koebe's Dist. Thm. (upper bdd. in (ii)), $\mathcal{S}$ is locally unif. bdd.
Hence $\mathcal{S}$ is a normal family by Montel's Little Thm. (Thm. 18.10).
If $\{f_n\}$ is a seq. in $\mathcal{S}$, and $f_n \to f$ loc. unif. on $\mathbb{D}$, then f is holomorphic (Weierstrass), and const. or injective (Hurwitz). Moreover,
$\underset{0}{\underbrace{f_n(0)}} \to f(0)$, and $\underset{1}{\underbrace{f_n'(0)}} \to f'(0)$;
so $f(0)=0$, $f'(0)=1$;
so f is non-const. hence injective.
So $f \in \mathcal{S}$. $\square$

(13) <u>Rem. 1.9</u> Koebe's Dist. Thm. often gives useful (non-sharp) quantitative information:

i) $\Omega, \Omega' \subsetneq \mathbb{C}$ regions, $f: \Omega \to \Omega'$ conf. map, $z_0 \in \Omega$.



Then

$$|f'(z_0)| \simeq \frac{\operatorname{dist}(f(z_0), \partial\Omega')}{\operatorname{dist}(z_0, \partial\Omega)}$$

$\simeq$: univ. const.

$A \simeq B$ iff $\frac{1}{C} A \le B \le C A$

<u>Proof</u>: $d' = \operatorname{dist}(\overset{w_0}{f(z_0)}, \partial\Omega')$, $d = \operatorname{dist}(z_0, \partial\Omega)$

$B(z_0, d) \subseteq \Omega$.

By 1/4-Thm. (applied to

$$u \longmapsto \frac{f(z_0 + ud) - f(z_0)}{f'(z_0) d}$$

), we have $B(w_0, \frac{1}{4}|f'(z_0)| d) \subseteq \Omega'$;

so

$d' \ge \frac{1}{4} |f'(z_0)| d$, and $|f'(z_0)| \lesssim \frac{d'}{d}$.

For lower bdd. consider f^{-1}.

ii) $\Omega, \Omega' \subsetneq \mathbb{C}$ regions, $f: \Omega \to \Omega'$ conf. map, $K \subseteq \Omega$ comp.

Then $|f'(z)| \simeq |f'(w)|$ for $z, w \in K$ with impl. const. only depending on Ω, K (and not on f!).

<u>Idea of proof</u>: If $\Omega = \mathbb{D}$, then $|f'(z)| \simeq |f'(0)| \simeq |f'(w)|$ by Koebe. Generalizes to $\Omega =$ disk. General case follows from Harnack chain argument.

2. Boundary extensions of conformal maps

$\Omega \subseteq \mathbb{C}$ bdd. region

TFAE (Thm. 2.2.2):

i) Ω is simply conn.

ii) $\hat{\mathbb{C}} \setminus \Omega$ is conn. ($\Leftrightarrow$ $\mathbb{C} \setminus \Omega$ conn.)

iii) $\partial \Omega \subseteq \mathbb{C}$ is conn.

iv) Ω is conf. equiv. to $\mathbb{D}$, i.e., there ex. conf. map $f : \mathbb{D} \leftrightarrow \Omega$.

<u>Thm. 2.1.</u> $f: \mathbb{D} \to \Omega$ conf. map onto bdd. (simply conn.) region.

<u>TFAE:</u>

i) f has a cont. extension to $\overline{\mathbb{D}}$,

ii) $\partial \Omega$ can be parametrized as a loop, i.e., there ex. a cont. map $\varphi : \partial \mathbb{D} \to \mathbb{C}$ s.t. $\varphi(\partial \mathbb{D}) = \partial \Omega$

iii) $\partial \Omega$ is locally connected,

iv) $\mathbb{C} \setminus \Omega$ is locally connected.

We'll prove this in the following!

<u>2.2. Locally connected sets</u>

$A \subseteq \mathbb{C}$ closed.

A <u>loc. conn.</u> $:\Leftrightarrow$ for all $a \in A$ and $\varepsilon > 0$ there ex. $\delta > 0$ s.t. if $b \in A$ is arb. with $|a-b| < \delta$ then there ex. continuum $E \subseteq A$ with $a, b \in E$ and $\mathrm{diam}(E) < \varepsilon$.

(15)

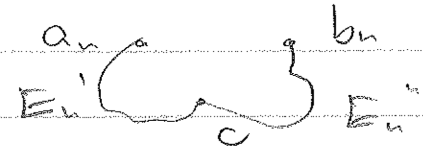
"small connection between a, b"

$A \subseteq \mathbb{C}$ compact is loc. conn. iff for all $\varepsilon > 0$ there ex. $\delta > 0$ s.t. for all $a, b \in A$ with $|a-b| < \delta$ there ex. cont. $E \subseteq A$ with $a, b \in E$ and $\operatorname{diam}(E) < \varepsilon$.

Proof: $\Leftarrow$ trivial

$\Rightarrow$ By contradiction! If not, there ex. $\varepsilon_0 > 0$ ("bad ε") and sequences $\{a_n\}$, $\{b_n\}$ in A s.t. $|a_n - b_n| \to 0$, but no cont. E s.t. $a_n, b_n \in E$ and $\operatorname{diam} E < \varepsilon_0$.

Wlog $a_n, b_n \to c$



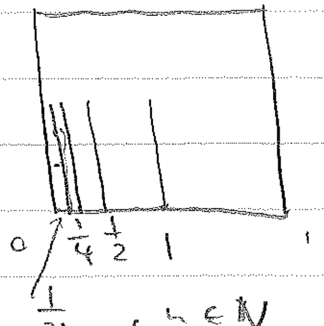
Since A is loc. conn. for sufft. large n there ex. cont. E_n', E_n'' s.t. $a_n, c \in E_n'$, $b_n, c \in E_n''$, $\operatorname{diam}(E_n') < \varepsilon/2$, $\operatorname{diam}(E_n'') < \varepsilon_0/2$.

Then $E_n := E_n' \cup E_n''$ is cont. with $a_n, b_n \in E_n$ and $\operatorname{diam}(E_n) < 2 \cdot \varepsilon_0/2 = \varepsilon_0$, contradiction!

A comp. is loc. conn. iff pts. that are close have a "small connection"

iff there ex. $\omega : (0, \infty) \to (0, \infty)$ s.t. $\lim_{\delta \to 0^+} \omega(\delta) = 0$ s.t. $\forall a, b \in A$ $\exists$ cont. $E \subseteq A$ with $a, b \in E$ and $\operatorname{diam} E \leq \omega(|a-b|)$.

(16) Boundary of comb domain is connected but not loc. conn.



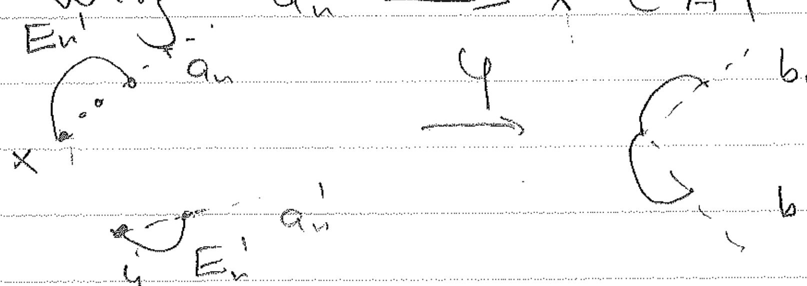
$A \subseteq \mathbb{C}$ comp. and loc. conn., $\varphi : A \to \mathbb{C}$ cont., $B := \varphi(A)$. Then B is loc. conn.

(Cont. images of comp. and loc. conn sets are loc. conn.).

<u>Proof:</u> By contradiction! If not, then there ex. $\varepsilon_0 > 0$ (bad ε) and sequences $\{b_n\}$, $\{b_n'\}$ s.t. $|b_n - b_n'| \to 0$ s.t. there ex. <u>no</u> cont. $E \subseteq B$ with $b_n, b_n' \in E$, $\mathrm{diam}(E) < \varepsilon_0$.

$b_n = \varphi(a_n)$, $b_n' = \varphi(a_n')$.

Wlog $a_n \to x \in A$, $a_n' \to y$



Then $b_n, b_n' \to z = \varphi(x) = \varphi(y)$

Can find small connections E_n' and E_n'' between x, a_n and y, a_n' resp., for n large. Then $E_n := \varphi(E_n') \cup \varphi(E_n'')$ is a small conn. between b_n, b_n' for n large, by unif. continuity of φ. Contradiction! □

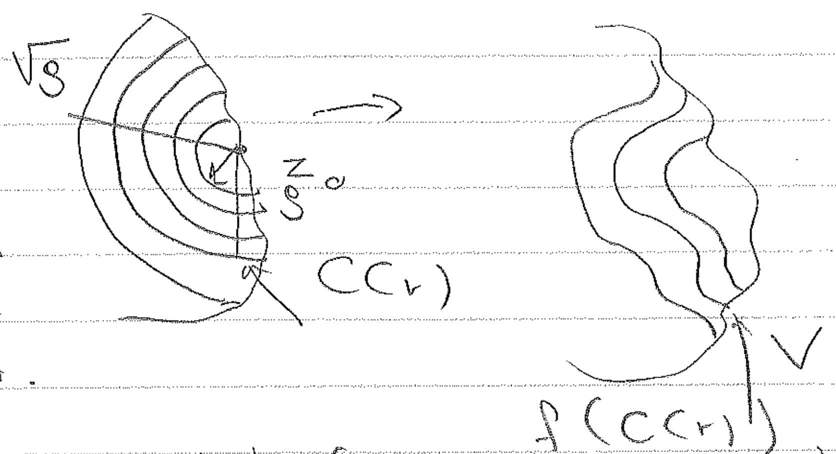
In particular, if $\varphi : \partial\mathbb{D} \to \mathbb{C}$ is cont., then $\varphi(\partial\mathbb{D})$ is loc. conn. (Loops or paths are loc. conn.) (ii) $\Rightarrow$ iii) in Thm. 2.1!)

(17) Lem. 2.3. (Wolff's Lemma)
$U \subseteq \mathbb{C}$ open, $\varphi : U \longrightarrow V \subseteq B(0, R_0)$
conf. map, $z_0 \in \overline{U}$,
$C(r) := U \cap \{ z \in \mathbb{C} : |z - z_0| = r \}$.
Then

$$\inf_{\rho < r < \sqrt{\rho}} \ell(f(C(r))) \le \frac{2\pi R_0}{\sqrt{\log(1/\rho)}} \quad \text{for } 0 < \rho < 1.$$

In particular, there ex. seq. $r_n \longrightarrow 0$
s.t. $\ell(f(C(r_n))) \longrightarrow 0$ as $n \longrightarrow \infty$.

If a "thick" family of curves is confined to a set of controlled area, then one of the curves has to be short.



Proof: $L(r) := \ell(f(C(r)))$. (lower semicont.)
Then

$$L(r)^2 = \left(\int_{C(r)} |f'(z)| |dz| \right)^2$$

$$\underset{\text{Schwarz}}{\le} \left(\int_{C(r)} |dz| \right) \left(\int_{C(r)} |f'(z)|^2 |dz| \right)$$

$$\le 2\pi r \int_{\{t \in [0, 2\pi] : z_0 + re^{it} \in U\}} |f'(z_0 + re^{it})|^2 \, r \, dt.$$

So

$$\int_0^\infty \frac{L(r)^2}{r} dr \le 2\pi \int_U |f'(z)|^2 dA(z) = 2\pi A(V) \le 2\pi^2 R_0^2$$

(18) So

$$\frac{1}{2}\log(1/\delta)\inf_{\delta<r<\sqrt{\delta}} l(r)^2 \le \int_{\delta}^{\sqrt{\delta}} l(r)^2 \frac{dr}{r} \le 2\pi^2 R_0^2.$$

The claim follows. □

<u>Lem 2.4.</u> $\gamma : [0,1) \to \mathbb{C}$ path with
$$\ell(\gamma) = \sup_{0 \le t_0 < \dots < t_n < 1} \sum_{k=0}^{n} |\gamma(t_k) - \gamma(t_{k-1})| < \infty.$$
Then $\lim_{t\to 1^-}\gamma(t)$ exists.

If a path has finite length, then it ends somewhere!

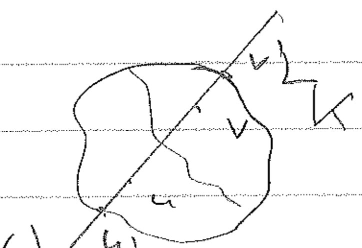
<u>Proof</u>: $L := \ell(\gamma) < \infty$.
$L(t) = \ell(\gamma|[0,t]) \nearrow L$ as $t \to 1^-$,
so $\ell(\gamma|(t,1)) = L - L(t) \to 0$ as $t \to 1^-$.
So for $s, s' \in (t,1)$:
$|\gamma(s) - \gamma(s')| \le \ell(\gamma|(t,1)) \to 0$ as $t \to 1^-$,
this implies that for every sequence $\{s_n\}$ in $[0,1)$ with $s_n \to 1$, $\{\gamma(s_n)\}$ is a Cauchy seq.
The claim follows. □

$A \subseteq \mathbb{C}$ closed, $x, y \in \mathbb{C}$.
A <u>separates</u> x and y if x, y do not lie in one component of $\mathbb{C}\setminus A$ (true if $x \in A$ or $y \in A$)
iff every path joining x, y meets A.

(19) Janiszewski's Thm. (Thm. 27.19):
$K, L \subseteq \mathbb{C}$ compact, $K \cap L$ connected.
If $K \cup L$ separates the pts. $x, y \in \mathbb{C}$, then they are separated by K or by L.

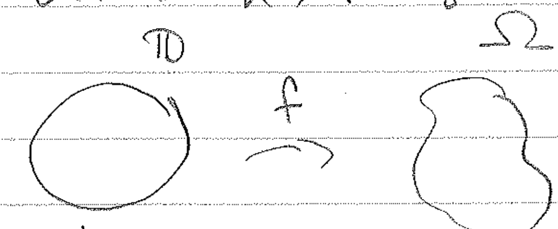
<u>Lem. 2.5.</u> $K \subseteq \mathbb{C}$ compact, $x_0 \in \mathbb{C}$ s.t. $\operatorname{dist}(x_0, K) > \operatorname{diam}(K)$, $u, v \in \mathbb{C}$.
If K separates x_0 and u, and separates x_0 and v, then $|u-v| \leq \operatorname{diam}(K)$.

<u>Proof:</u>



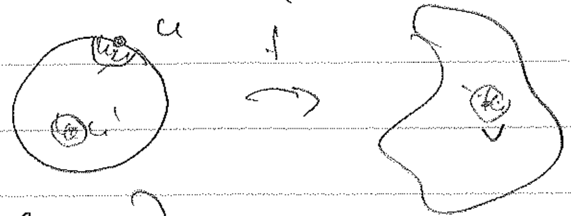
$u \neq v$

Pick $a \in K$, x_0
and let $R = \operatorname{diam}(K)$
Then $K \subseteq \bar{B}(a, R)$ and $|x_0 - a| > R$
So $x_0 \in \mathbb{C} \setminus \bar{B}(a,R) \subseteq \mathbb{C} \setminus K$;
this shows that x_0 lies in the unbdd. comp. of $\mathbb{C} \setminus K$.
So both u, v do <u>not</u> lie in the unbdd. comp. of $\mathbb{C} \setminus K$. This implies that if $L \subseteq \mathbb{C}$ is the line with $u, v \in L$, then there ex. $u', v' \in K$ s.t. $[u, v] \subseteq [u', v']$.
Hence $|u-v| \leq |u'-v'| \leq \operatorname{diam}(K)$. ▯



<u>Proof of Thm. 2.1.</u>:
i) $\Rightarrow$ ii).
Suppose f has a cont. extension $f: \bar{\mathbb{D}} \to \mathbb{C}$. By continuity: $f(\bar{\mathbb{D}}) \subseteq \overline{f(\mathbb{D})} \subseteq \bar{\Omega}$,
by compactness of $\bar{\mathbb{D}}$: $\bar{\Omega} \subseteq \overline{f(\mathbb{D})} \subseteq f(\bar{\mathbb{D}})$,
and so $\bar{\Omega} = f(\bar{\mathbb{D}})$

(20) Since $f(\mathbb{D}) = \Omega$ is open,
$\partial\Omega = \bar{\Omega}\setminus\Omega \subseteq f(\partial\mathbb{D})$.
Moreover, conformality implies $f(\partial\mathbb{D}) \subseteq \bar{\Omega}\setminus\Omega = \partial\Omega$

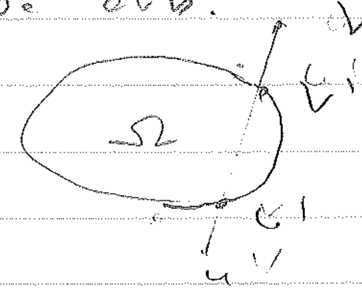


So $f(\partial\mathbb{D}) = \partial\Omega$.

So $\partial\Omega$ has a parametrization as a loop!

(ii) $\rightarrow$ (iii) Cont. images of comp. + loc. conn. sets are loc. connected! (See 2.2.)
Since $\partial\mathbb{D}$ is comp + loc. conn., $\partial\Omega = f(\partial\mathbb{D})$ also has these properties.

(iii) $\rightarrow$ (iv): Let $u, v \in \mathbb{C}\setminus\Omega$ be arb.
Run along $[u,v]$:



1) If $[u,v] \cap \partial\Omega = \emptyset$, then $E = [u,v]$ is a cont. in $\mathbb{C}\setminus\Omega$ joining u,v with $\operatorname{diam}(E) = |u-v|$

2) If $[u,v] \cap \partial\Omega \neq \emptyset$, then we can find $u', v' \in \partial\Omega$ s.t. $u', v' \in \partial\Omega$, [s.t.] $[u,u'] \subseteq \mathbb{C}\setminus\Omega$, $[v',v] \subseteq \partial\Omega$!
By assumption there ex. a cont. $E' \subseteq \partial\Omega$ with $u', v' \in E'$, $\operatorname{diam}(E') \leq \omega(|u'-v'|)$ where $\omega(d) \rightarrow 0$ as $d \rightarrow 0^+$.
Then $E := [u,u'] \cup E' \cup [v',v]$ is a cont. with $E \subseteq \mathbb{C}\setminus\Omega$, $u,v \in E$,

$$\begin{aligned} \operatorname{diam}(E) &\leq |u-v| + \omega(|u'-v'|) \\ &\leq |u-v| + \omega(|u-v|) \\ &= \hat{\omega}(d), \text{ where } \hat{\omega}(d) = d + \omega(d), \end{aligned}$$

$d = |u-v|$. Since $\hat{\omega}(d) \rightarrow 0$ as $d \rightarrow 0^+$,

(21) the claim follows.

iv) $\Longrightarrow$ i):

It suffices to show that f is uniformly continuous on $\mathbb{D}$, i.e., there ex. $\omega : (0,\infty) \to (0,\infty)$ with $\omega(\delta) \to 0$ as $\delta \to 0$ s.t.

$|f(x) - f(y)| \leq \omega(|x-y|)$ for $x, y \in \mathbb{D}$

(then the image of every Cauchy seq. is Cauchy, bla, bla, ...)

equivalently,

$\operatorname{diam}(f(B(z_0,\delta) \cap \mathbb{D}))$

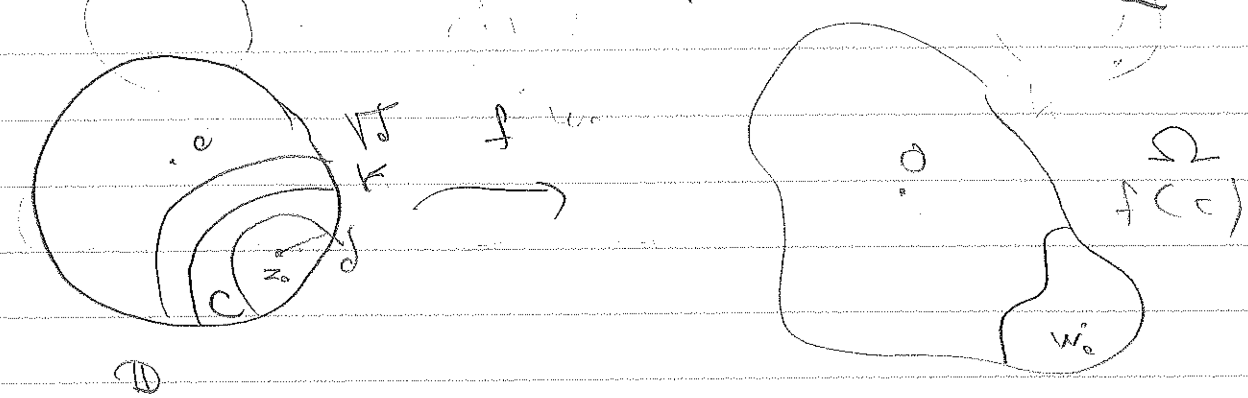
$\leq \omega(\delta)$ for $z_0 \in \mathbb{D}, \delta > 0$.

Here wlog $\delta > 0$ small, and z_0 close to $\partial\mathbb{D}$.

Wlog (by translation and scaling of Ω)

$f(0) = 0$, $z_0 \in \mathbb{D}$, $w_0 = f(z_0)$,

$|z_0|, |w_0| \geq \frac{1}{2}$



By Wolff Lem. 2.3, there ex. $r \in (\delta, \sqrt{\delta})$ s.t.

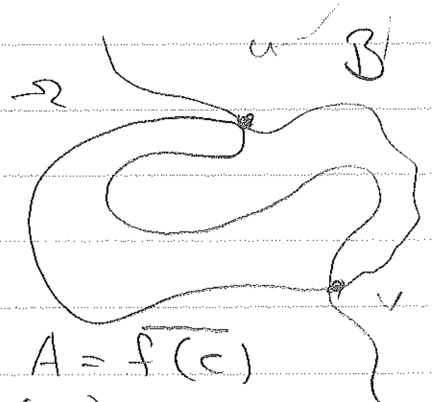
$\ell(f(C)) \leq \omega_1(\delta)$,

where $C = \mathbb{D} \cap \{z \in \mathbb{C} : |z - z_0| = r\}$

Here $\omega_1(\delta) = \dfrac{C_0}{\sqrt{\log(1/\delta)}} \to 0$ as $\delta \to 0$.

(22) Let's assume C is not the whole circle $|z-z_0|=r$, but an open subarc.
Then Lem 2.4 implies that $f(C)$ has two endpts. $u, v \in \partial\Omega$; so
$A := \overline{f(C)} = f(C) \cup \{u,v\}$. (possibly $u=v$!)
Then $|u-v| \le \ell(f(C)) \le \omega_1(\delta)$.
Since $\mathbb{C}\setminus\Omega \supseteq \partial\Omega$ is loc. conn., there ex. a continuum $B \subseteq \mathbb{C}\setminus\Omega$ s.t. $u, v \in B$ and

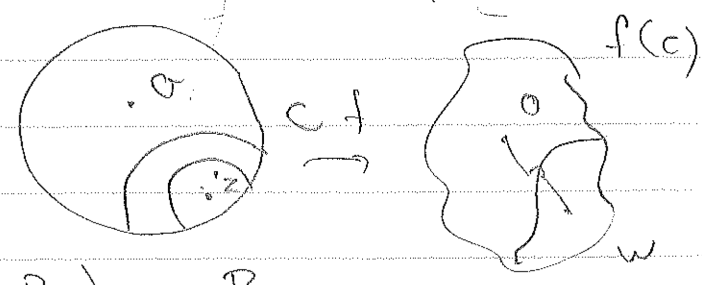
$$\operatorname{diam}(B) \le \omega_2(|u-v|) \to \le \omega_3(\delta)$$



Let $K := A \cup B$.
Then

$$\operatorname{diam}(K) \le \ell(f(C)) + \operatorname{diam}(B) \le \omega_1(\delta) + \omega_3(\delta) = \omega_4(\delta),$$

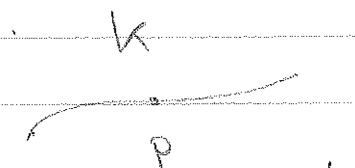
and $K \cap \partial\Omega \ne \emptyset$; so
$\operatorname{dist}(0, K) > \operatorname{diam}(K)$ if δ is small enough.
Now let $z \in B(z_0, \delta) \cap \mathbb{D}$ be arb.,
and $w = f(z)$
Then C separates 0 and z in $\mathbb{D}$,
i.e., $\mathbb{C}\setminus\mathbb{D} \cup C$ separates 0 and z.
This implies that
$\mathbb{C}\setminus\Omega \cup \underbrace{(f(C) \cup B)}_{=K}$
separates $0 = f(0)$ and $w = f(z)$.



Since $\mathbb{C}\setminus\Omega \cap (f(C) \cup B) = B$ is connected, and $\mathbb{C}\setminus\Omega$ does not

(23) separate 0 and w, $K = f(C) \cup B$ separates 0 and w.
If $z' \in B(z_0, \delta) \cap \mathbb{D}$ is another point and $w' = f(z')$, then K separates 0 and w' by the same argument.
Lem. 2.5. implies:
$$|w - w'| \le \operatorname{diam}(K) \le \omega_4(\delta),$$
and so
$$\operatorname{diam}(f(B(z_0,\delta) \cap \mathbb{D})) \le \omega_4(\delta)$$
as desired. $\square$

Rem. 2.6. A similar argument shows: if $f: \mathbb{D} \to \Omega \subseteq \hat{\mathbb{C}}$ is conformal, then f has a cont. extension $f: \bar{\mathbb{D}} \to \bar{\Omega} \subseteq \hat{\mathbb{C}}$ if $\partial\Omega$ (or $\hat{\mathbb{C}} \setminus \Omega$) is loc. connected. Here we use spherical or chordal distance in target! (Versions of Wolff's Lem. + Lem. 2.5 still true for spherical metric).



$K \subseteq \mathbb{C}$ continuum
p is a cut point of K if $K \setminus \{p\}$ is not connected

$\partial\Omega$ loc. conn.

Prop. 2.7. $\Omega \subseteq \mathbb{C}$ bdd. simply conn. region, $f: \mathbb{D} \to \Omega$ conf. map with cont. extension $f: \bar{\mathbb{D}} \to \bar{\Omega}$, $p \in \partial\Omega$.

has empty interior

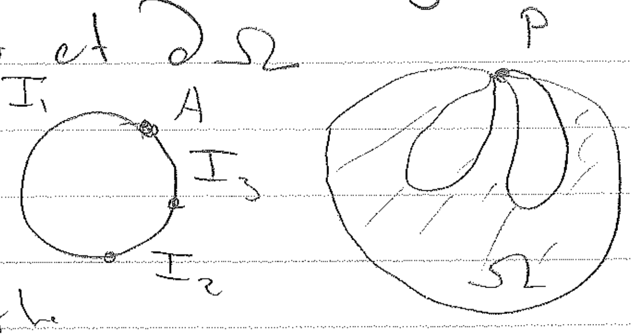
(24) Then $\# f^{-1}(p) \geq 2$ if and only if p is a cut point of $\partial\Omega$

Move precisely,

let $A := f^{-1}(p) \subseteq \partial\mathbb{D}$,

and

$\partial\mathbb{D} \setminus A = \bigcup_{k \in \Lambda} I_k$ be the decomposition into pairwise disjoint open arcs. (Λ countable index set)



Then the sets $f(I_k)$, $k \in \Lambda$, form the pairwise disjoint connected comp. of $\partial\Omega \setminus \{p\}$) $\# \Lambda = \# A$

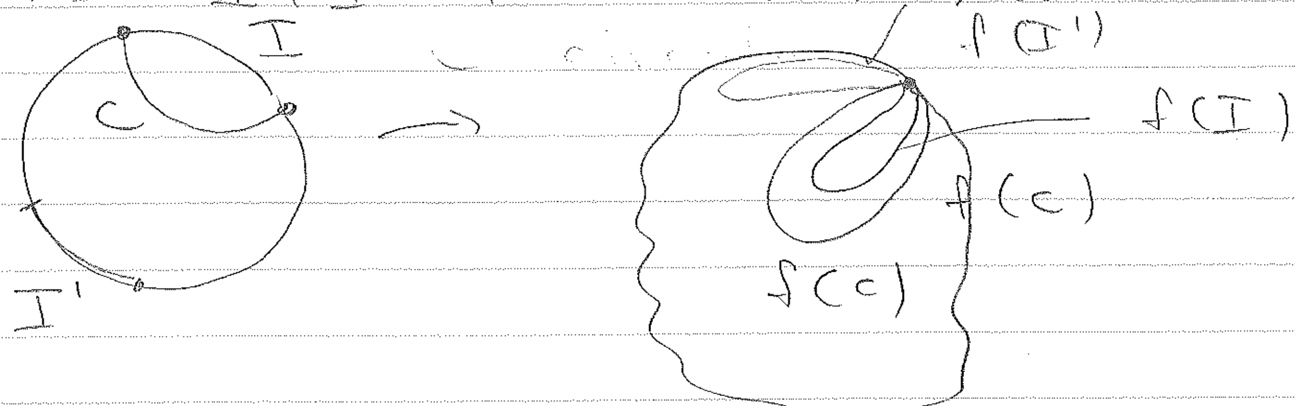
(note $\# \Lambda \geq 2$ iff $\# A \geq 2$).

Proof: Note $\partial\Omega \setminus \{p\} = f(\partial\mathbb{D} \setminus A)$
$= \bigcup_{k \in \Lambda} f(I_k)$

and the sets $f(I_k)$ are connected (cont. image of connected set!)

It suffices to show that $f(I_k)$, $k \in \Lambda$ are pairwise disjoint.

Let I, I' two of these arcs



C circular arc in $\mathbb{D}$ with same end pts. as I.

(25) Then

$\mathbb{D} = D \cup C \cup D'$ disjoint union

$I \subseteq \partial D$, $I' \subseteq \partial D'$

$J = f(C) \cup \{p\}$ Jordan curve

$U := f(D)$, $U' = f(D')$

open, connected sets in $\mathbb{C} \setminus J$.

So $U \subseteq I(J)$ or $U \subseteq O(J)$; and

$U' \subseteq I(J)$ or $U' \subseteq O(J)$.

U, U' cannot lie in the same component

of $\mathbb{C} \setminus J$; say $U, U' \subseteq I(J)$.

By the open mapping theorem

$U \cup C \cup U' \supseteq \Omega$ open neighborhood

of each point on $C \subseteq J$.

On the other hand, $O(J)$ is disjoint

from $U \cup C \cup U'$ and has pts. near each

$u \in J$ ($\partial O(J) = J$!). Contradiction!

So U, U' lie in different comp. of

$\mathbb{C} \setminus J$, say $U \subseteq I(J)$, $U' \subseteq O(J)$.

Then $f(I) \subseteq f(\bar{D}) \subseteq \bar{U} \subseteq J \cup I(J)$,

on the other hand, $f(I) \subseteq \partial \Omega \setminus \{a\}$,

$\partial \Omega \setminus \{a\} \cap J = \emptyset$, and so

$f(I) \subseteq I(J)$

Similarly, $f(I') \subseteq O(J)$.

Hence $f(I) \cap f(I') \subseteq I(J) \cap O(J) = \emptyset$. $\square$

<u>Thm. 2.8. (Carathéodory)</u>

Let $f : \mathbb{D} \to \Omega$ be a conf. map onto

a bdd. simply connected region.

(26) TFAE:

i) f has a homeomorphic extension to $\bar{\mathbb{D}}$ (i.e., continuous and injective)

ii) $\partial\Omega$ is a Jordan curve

iii) $\partial\Omega$ is locally connected and has no cut pts.

Proof: i) $\to$ ii) Obvious, because $\partial\Omega = f(\partial\mathbb{D})$.

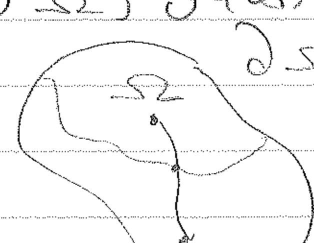
ii) $\to$ iii) Clear.

iii) $\to$ i) By Thm. 2.1. f has a cont. ext. $f : \bar{\mathbb{D}} \to \bar{\Omega}$. By Prop. 2.7 $f|\partial\mathbb{D}$ is injective; since $f(\partial\mathbb{D}) = \partial\Omega$ and $f(\mathbb{D}) = \Omega$ are disjoint, f is injective on $\bar{\mathbb{D}}$. □

A region $\Omega \subseteq \hat{\mathbb{C}}$ is called an (open) Jordan region or domain if $\partial\Omega \subseteq \hat{\mathbb{C}}$ is an Jordan curve.

If $\partial\Omega \subseteq \mathbb{C}$ (i.e., $\infty \notin \partial\Omega$), then $\Omega = I(\partial\Omega)$ or $\Omega = O(\partial\Omega) \cup \{\infty\}$

A closed Jordan region is the closure $\bar{\Omega}$ of an open Jordan region $\Omega \subseteq \hat{\mathbb{C}}$.



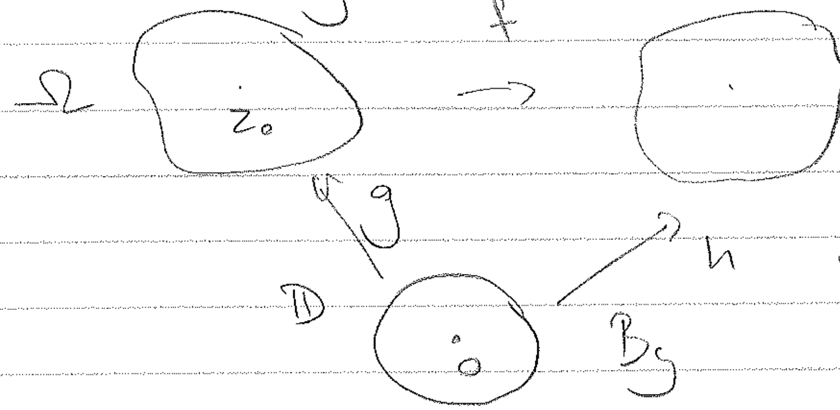
An open Jordan region Ω is simply connected, because $\partial\Omega$ is connected.

Cor. 2.9. Let $\Omega, \Omega' \subseteq \hat{\mathbb{C}}$ be Jordan regions and $f : \Omega \leftrightarrow \Omega'$ be a conf. map.

(unique)

(27) Then f has a homeomorphic extension $\bar f : \bar\Omega \longleftrightarrow \bar\Omega'$ (w.r.t. chordal metric on $\hat{\mathbb{C}}$).

Proof: Wlog $\Omega, \Omega' \subseteq \mathbb{C}$ (use Möbius transf.)



Ω' There ex. a conf. map $g : \mathbb{D} \longrightarrow \Omega$. Then $h := f \circ g$ is a conf. map $h : \mathbb{D} \longrightarrow \Omega'$.

By Thm. 2.8, g, h have homeo. ext. $\bar g : \bar{\mathbb{D}} \longleftrightarrow \bar\Omega$, $\bar h : \bar{\mathbb{D}} \longleftrightarrow \bar\Omega'$, resp.

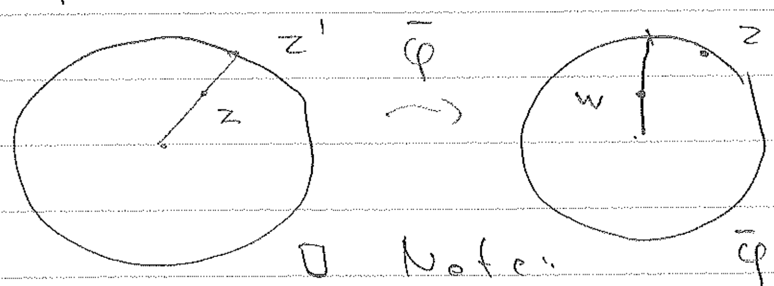
Then $\bar f := \bar h \circ \bar g^{-1} : \bar\Omega \longleftrightarrow \bar\Omega'$ is a homeo. ext. of f. □

Lem. 2.10 Let $\varphi : \partial\mathbb{D} \longleftrightarrow \partial\mathbb{D}$ be a homeo. Then φ can be extended to a homeo. $\bar\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

Proof: Use "radial extension":

$\bar\varphi(r \cdot \xi) = r \cdot \varphi(\xi)$, $0 \le r < \infty$, $\xi \in \partial\mathbb{D}$,

$\bar\varphi(\infty) = \infty$

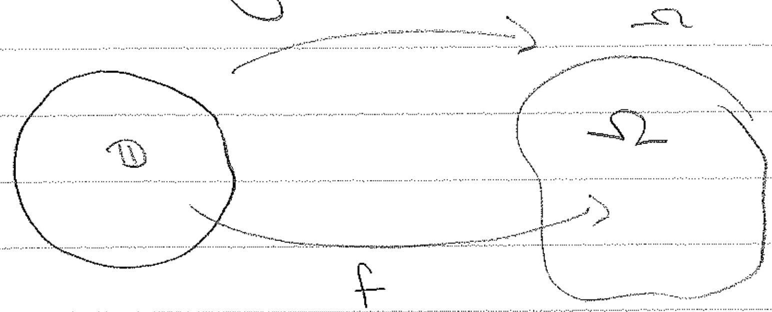


This is a cont. bijection with cont. inverse (rad. ext. of φ^{-1}).

□ Note: $\bar\varphi | \bar{\mathbb{D}}$ is a homeo. ext. $\bar{\mathbb{D}} \longleftrightarrow \bar{\mathbb{D}}$ of φ.

(28) <u>Thm. 2.11.</u> Let $f:\mathbb{D}\to\Omega$ be a conf. map onto a Jordan region $\Omega\subseteq\hat{\mathbb{C}}$. Then f has a homeomorphic ext. $\bar{f}:\hat{\mathbb{C}}\leftrightarrow\hat{\mathbb{C}}$.

<u>Proof:</u> wlog $\partial\Omega = J \subseteq\mathbb{C}$, $\Omega = I(J)$.



Then f has a homeo. ext. $f:\bar{\mathbb{D}}\leftrightarrow\bar{\Omega}$. Note that $\hat{\mathbb{D}} = \hat{\mathbb{C}}\setminus\bar{\mathbb{D}}$ also and $\tilde{\Omega} = \hat{\mathbb{C}}\setminus\bar{\Omega}$ are Jordan regions; so there ex. a conf. map $\tilde{f}:\hat{\mathbb{D}}\to\tilde{\Omega}$ with homeo. ext. $\tilde{f}:\bar{\hat{\mathbb{D}}}\leftrightarrow\bar{\tilde{\Omega}}$.

If $f|\partial\mathbb{D} = \tilde{f}|\partial\mathbb{D}$, then $f,\tilde{f}$ would paste together to homeo. ext. of f.

Not true in general!

Let $\varphi := \tilde{f}^{-1}\circ f|\partial\mathbb{D}$ ("conformal welding map induced by J")

Then φ is a homeo. on $\partial\mathbb{D}$.

By Lem. 2.10. it has a homeo. ext. $\bar{\varphi}:\hat{\mathbb{C}}\leftrightarrow\hat{\mathbb{C}}$. Define

$$\bar{f}(z) = \begin{cases} f(z) & \text{for } z\in\bar{\mathbb{D}} \\ \tilde{f}(\bar{\varphi}(z)) & z\in\hat{\mathbb{C}}\setminus\mathbb{D}. \end{cases}$$

This is well-def. and a homeo. $\hat{\mathbb{C}}\leftrightarrow\hat{\mathbb{C}}$ that extends f. $\square$

 Thm. 2.12 (Schönflies)

Every homeo. $\varphi: J \longleftrightarrow J'$ between Jordan curves can be extended to a homeo. $\bar{\varphi}: \hat{\mathbb{C}} \longleftrightarrow \hat{\mathbb{C}}$.

In particular, every Jordan curve $J \subseteq \hat{\mathbb{C}}$ is "tame", i.e., the image of $\partial\mathbb{D}$ under a homeo. $\bar{\varphi}: \hat{\mathbb{C}} \longleftrightarrow \hat{\mathbb{C}}$.

Not true in $\mathbb{R}^3$ (Knots!)

Proof: WLOG $J = \partial\mathbb{D}$, $J' \subseteq \mathbb{C}$.

$\Omega = I(J')$.

Ex. conf. map $f: \mathbb{D} \longleftrightarrow \Omega$ with homeo. ext. $\bar{f}: \hat{\mathbb{C}} \longleftrightarrow \hat{\mathbb{C}}$ (Thm. 2.11.)

Let $\psi := (f|\partial\mathbb{D})^{-1} \circ \varphi$.

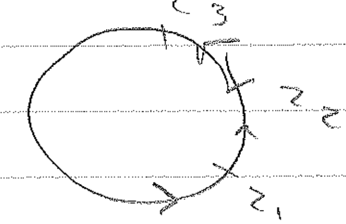
This is a homeo. $\psi: \partial\mathbb{D} \longleftrightarrow \partial\mathbb{D}$, and so has a homeo. ext. $\bar{\psi}: \hat{\mathbb{C}} \longleftrightarrow \hat{\mathbb{C}}$.

Then $\bar{f} \circ \bar{\psi}$ is a homeo. $\hat{\mathbb{C}} \longleftrightarrow \hat{\mathbb{C}}$ with
$\bar{f} \circ \bar{\psi} | \partial\mathbb{D} = f \circ \psi = f \circ (f|\partial\mathbb{D})^{-1} \circ \varphi = \varphi$. $\square$

2.13. Orientation

$z_1, z_2, z_3 \in \partial\mathbb{D}$ distinct

This triple is in positive cyclic order if in the standard parametrization $\gamma: \mathbb{R} \to \partial\mathbb{D}$, $\gamma(t) = e^{it}$, whenever $\gamma(t_1) = z_1$ and $t_2, t_3 \in [t_1, t_1 + 2\pi)$, with $\gamma(t_2) = z_2$, $\gamma(t_3) = z_3$, we have $t_2 < t_3$.



Note: every $\varphi \in \mathrm{Aut}(\mathbb{D})$ preserves

(30) Pos. cyclic order of pts. on $\partial\mathbb{D}$.

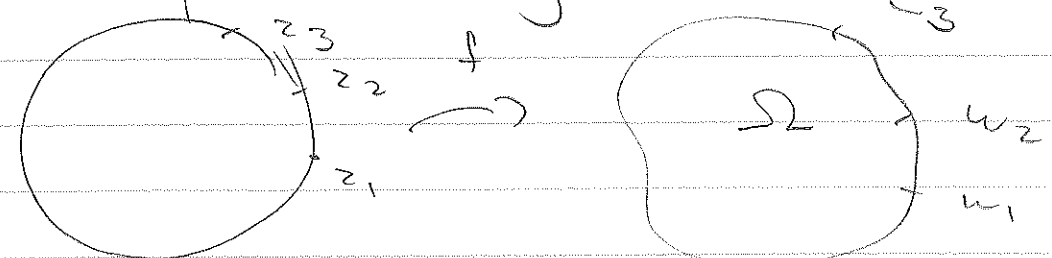
z_1, z_2, z_3 pos. oriented iff

$\operatorname{Im}(a, z_1, z_2, z_3) < 0$ for $a \in \mathbb{D}$

($\mathbb{D}$ lies to the left of $\partial\mathbb{D}$).

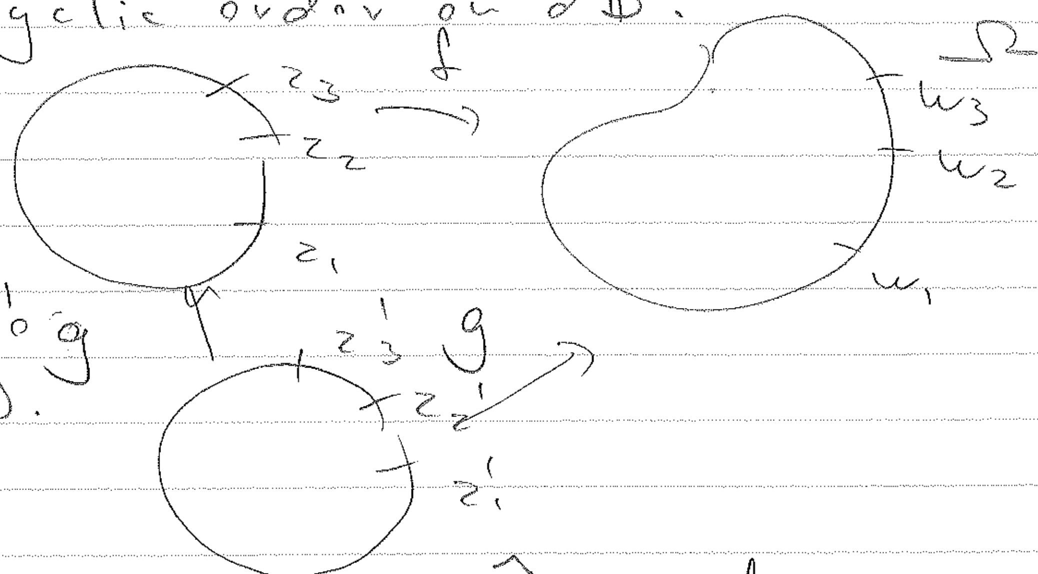
Positive cyclic order on boundary of Jordan region:

$\Omega \subseteq \hat{\mathbb{C}}$ Jordan region, $w_1, w_2, w_3 \in \partial\Omega$ distinct are in <u>positive cyclic order</u> if the following is true:



f conf. map $f : \mathbb{D} \leftrightarrow \Omega$ with homeo. ext. $f : \bar{\mathbb{D}} \leftrightarrow \bar{\Omega}$. Let $z_k = f^{-1}(w_k)$, $k = 1, 2, 3$. The requirement is that z_1, z_2, z_3 are in pos. cyclic order on $\partial\mathbb{D}$.

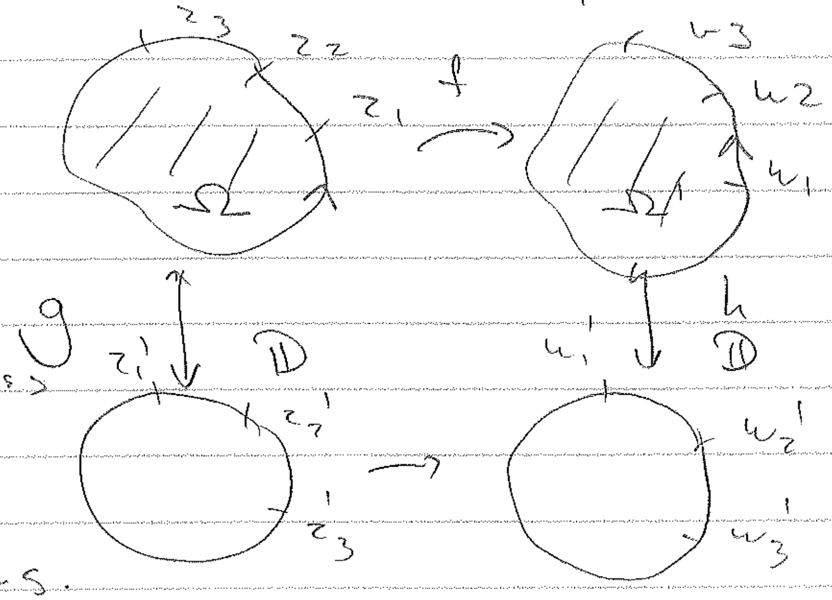
Ind. of f



$\varphi = f^{-1} \circ g \in \operatorname{Aut}(\mathbb{D})$.

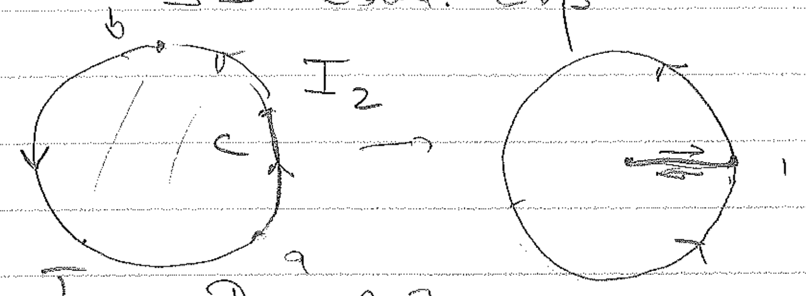
<u>Thm. 2.14.</u> $\Omega, \Omega' \subseteq \hat{\mathbb{C}}$ Jordan regions, z_1, z_2, z_3 in pos. cyc. order on $\partial\Omega$, w_1, w_2, w_3 in pos. cyc. order on $\partial\Omega'$.

(31) Then there ex. a unique conf. map $f: \Omega \longleftrightarrow \Omega'$ whose homeo. ext. $f: \bar{\Omega} \longleftrightarrow \bar{\Omega}'$ satisfies $f(z_k) = w_k$, $k=1,2,3$.



Proof: Pulling back by aux. conf. maps we may assume that $\Omega = \mathbb{D}$, $\Omega' = \mathbb{D}'$. Then ex. a uniqueness follows from that fact that there ex. a unique Möb. trans. $\varphi \in \mathrm{Aut}(\mathbb{D})$ with $w_k' = \varphi(z_k')$. □

Ex. 2.15. $f: \mathbb{D} \to \Omega$ conf. map onto "slit disk" $\Omega = \mathbb{D} \setminus [0,1)$.



$\partial\Omega$ loc. conn. so ex. cont. ext. $f: \bar{\mathbb{D}} \to \bar{\Omega}$ Prop. 2.7.

$\partial\Omega \setminus \{1\}$ has two comp., so $\# f^{-1}(1) = 2$

$f^{-1}(1) = \{a, b\}$

$\partial\mathbb{D} = \partial\mathbb{D} \setminus \{a,b\} = I_1 \cup I_2$

$f(I_1) = \partial\mathbb{D} \setminus \{1\}$ $f(I_2) = [0,1)$.

Since $\partial\mathbb{D} \setminus \{1\}$ has not cut pts,

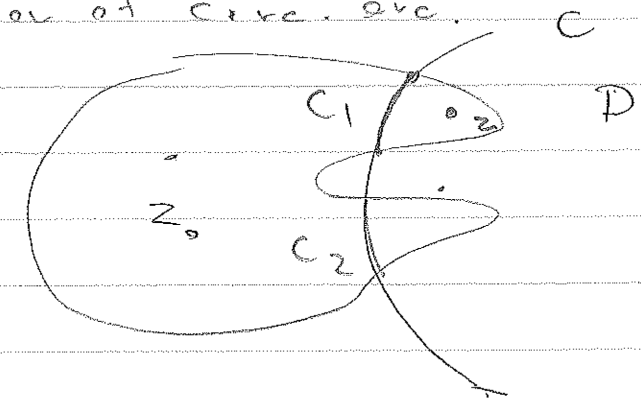
$\# f^{-1}(p) = 1$ for $p \in \partial\mathbb{D} \setminus \{1\}$

So $f|I_1$ homeo.

$\# f^{-1}(0) = 1$, ex. $c \in I_2$ s.t. $f(c_2) = 0$

(32) **Lem. 2.16.** $\Omega \subseteq \mathbb{C}$ simply conn. region, $z_0 \in \Omega$ base point, $D \subseteq \mathbb{C}$ disk, $C = \partial D$, $z_0 \notin D$

$C \cap \Omega = \bigcup_{k \in \{1,2,3,\dots\}} C_k$ pairwise disj. union of circ. arc.



If $z \in \Omega \cap D$, then one of the arcs C_k separates z_0 and z in Ω (i.e. every path in Ω joining z_0 and z meets C_k)

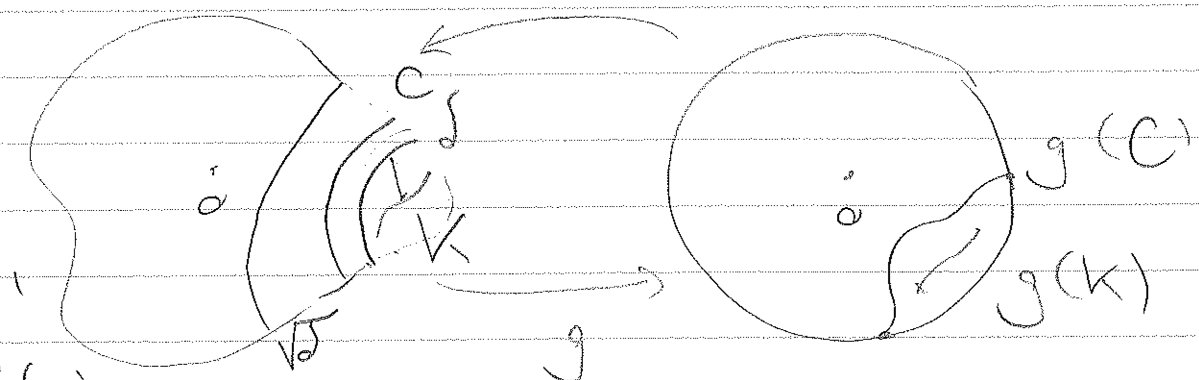
Proof: Suppose not!
Then none of the compact sets
$A_k := \hat{\mathbb{C}} \setminus \Omega \cup C_k$, $k = 1, 2, 3, \dots$
separates z_0 and z.
There ex. a path γ in Ω joining z_0 and z. It has positive distance to $\partial\Omega$; so it can only meets finitely many arcs C_k ($\bar{C}_k \cap \partial\Omega \neq \emptyset$ and $\operatorname{diam}(C_k) \to 0$ as $k \to \infty$ if there are infinitely many).
So there ex. $N \in \mathbb{N}$ s.t.
$B := A_N \cup A_{N+1} \cup \dots$ does not meet γ; so B does not separate z_0 and z.
Since $A_1 \cap B = \hat{\mathbb{C}} \setminus \Omega$ is connected, and neither A_1, B sep. z_0 and z, $A_1 \cup B$ does not sep. z, z_0 either. (Janiszewski!). Repeating this argument, we see that $A_1 \cup A_2 \cup B$ does not sep. z, z_0, etc.

33 So $A_1 \cup \dots \cup A_{N-1} \cup B$
$= \bigcup_{k \in \{1,2,3,\dots\}} A_k \cup \hat{\mathbb{C}} \setminus \Omega = C \cup \mathbb{C} \setminus \Omega$ does not sep. z and z_0.
But C separates z, z_0. Contradiction! $\square$

Thm. 2.17. (Fundamental dist. estimate for cont. maps into $\mathbb{D}$).
There ex. a function ("universal distortion function") $\omega : (0,\infty) \to (0,\infty)$, $\omega(\delta) \to 0$ as $\delta \to 0^+$ with the following property:
Let $\Omega \subseteq \mathbb{C}$ be a simply connected region, $g : \Omega \to \mathbb{D}$ a conformal map, and $K \subseteq \Omega$ a continuum.
Then
$$\operatorname{diam}(g(K)) \le \omega\left(\frac{\operatorname{diam}(K)}{|f'(0)|}\right) \quad (*)$$

$f = g^{-1} : \mathbb{D} \to \Omega$; one can take
$$\omega(\delta) = \frac{C_0}{\sqrt{\lg\left(\frac{1}{\delta}\right)}}.$$

Proof:
wlog
$g(0) = 0$
$= f(0)$,
$f'(0)$
$= 1 = g'(0)$



The proof is similar to the proof of Thm. 2.1. using Wolff's Lem. applied to g.
Wlog $\operatorname{diam}(K)$ very small

(34) Note $f(\frac{1}{2}\overline{\mathbb{D}}) \supseteq \overline{B}(0, \frac{2}{9})$ (follows from lower bdd. in Thm. 1.7. and its proof;
so $g(\overline{B}(0, \frac{2}{9})) \subseteq \frac{1}{2}\overline{\mathbb{D}}$.
By Koebe's Dist. Thm. it follows that $|g'| \le C_0$ on $\overline{B}(0, \frac{2}{9})$ with C_0 ind. of g. So g is uniformly Lipschitz on $\overline{B}(0, \frac{2}{9})$.
$(*)$ follows if K is close to 0.
Pick $z_0 \in K$. Let $\delta := \operatorname{diam}(K)$.
Then $K \subseteq \overline{B}(z_0, \delta)$.
By Wolff's Lem. there ex. $r \in (\delta, \sqrt{\delta})$ s.t. for $C_0 = \{|z - z_0| = r\}$ we have

$$\ell(g(C_0 \cap \Omega)) \le \omega(\delta).$$

We may assume that 0 lies outside C_0.
By Lem. 2.16. there ex. a circular arc $C \subseteq \Omega \cap C_0$ s.t. C separates 0 and z_0 in Ω; then C actually separates 0 and every point in K in Ω (K is connected).
Then
$$\ell(g(C)) \le \ell(g(C_0 \cap \Omega)) \le \omega(\delta) << 1,$$
and $g(C)$ separates 0 and $g(K)$ in $\mathbb{D}$.
Hence
$$\operatorname{diam} g(K) \le 2 \operatorname{diam} g(C) \le 2\,\omega(\delta).$$
(Note: if $\delta = \operatorname{diam}(K)$, $w_0 \in g(K)$ and δ small, then $g(K) \subseteq \overline{B}(w_0, \delta)$). $\square$

(35) **Def. 2.18.** $\Omega \subseteq \mathbb{C}$ region

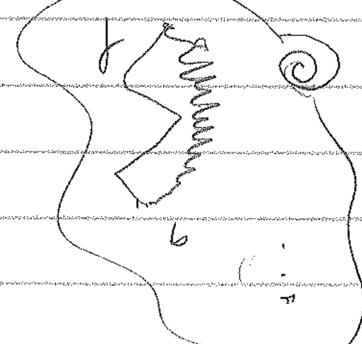
If $a, b \in \Omega$ we define

$$\lambda_\Omega(a,b) = \inf_\gamma \ell(\gamma)$$

where inf is taken over all paths in Ω joining a, b, and

$$\delta_\Omega(a,b) = \inf_K \operatorname{diam}(K),$$

where the inf is taken over all continua $K \subseteq \Omega$ with $a, b \in K$.



Both λ_Ω and δ_Ω are metrics on Ω called the <u>inner length metric</u> on Ω and the <u>diameter metric</u> on Ω.

Note $\delta_\Omega \le \lambda_\Omega$, and $\delta_\Omega, \lambda_\Omega$ induce the Euclidean topology on Ω (if $a \in \Omega$, and b is close to a, then $\delta_\Omega(a,b) = \lambda_\Omega(a,b) = |a-b|$).

Note: On convex regions Ω both δ_Ω and λ_Ω agree with the Eucl. metric.

Cor. 2.19 Let $\Omega \subseteq \mathbb{C}$ be simply connected region, and $g : \Omega \to \mathbb{D}$ be a conformal map. Then $g : (\Omega, \delta_\Omega) \to \mathbb{D}$ and $g : (\Omega, \lambda_\Omega) \to \mathbb{D}$ are unif. cont. (where $\mathbb{D}$ is equipped with the Eucl. metric)

(36) <u>Proof:</u> Let $w_1, w_2 \in \Omega$ be arb.,
$K \subseteq \Omega$ cont. with $w_1, w_2 \in K$
with $\operatorname{diam}(K)$ close to $g_\Omega(w_1, w_2)$.
Let $z_1 = g(w_1)$, $z_2 = g(w_2)$.
By Thm. 2.17.

$$|z_1 - z_2| \leq \operatorname{diam} g(K)$$
$$\leq \tilde{\omega}(\operatorname{diam}(K)) \longrightarrow \tilde{\omega}(g_\Omega(w_1, w_2))$$

as $\operatorname{diam}(K) \longrightarrow g_\Omega(w_1, w_2)$
So

$$|z_1 - z_2| \leq \tilde{\omega}(g_\Omega(w_1, w_2))$$
$$\leq \tilde{\omega}(\lambda_\Omega(w_1, w_2))$$ (if $\tilde{\omega}$ is increasing as we may assume. $\square$

<u>Cor. 2.10.</u> Let $\Omega \subseteq \mathbb{C}$ be a simply connected region, $g : \Omega \longrightarrow \mathbb{D}$ conformal map.
Suppose $\gamma : [0,1) \to \Omega$ is a path with $\lim_{t \to 1^-} \gamma(t) = w_0 \in \partial\Omega$.
Then $\lim_{t \to 1^-} g(\gamma(t)) = z_0 \in \partial\mathbb{D}$ exists.

<u>Proof:</u> Our hypotheses imply
$\operatorname{diam}(\gamma|[t,1)) \to 0$ as $t \to 1^-$.
By Thm. 2.17.
$\operatorname{diam}((g \circ \gamma)|[t,1)) \longrightarrow 0$ as $t \longrightarrow 1^-$.
Hence
$\lim_{t \to 1^-} (g \circ \gamma)(t) = z_0 \in \overline{\mathbb{D}}$ exists.

(37) Then $z_0 \in \partial\mathbb{D}$, because otherwise $z_0 \in \mathbb{D}$, and $\gamma(t) = f(g(\gamma(t))) = \gamma(t)$
$$\underset{g^{-1}}{\overset{\shortparallel}{f}} \qquad \longrightarrow w_0 = f(z_0) \in \Omega,$$
Contradiction!

<u>Rem. 2.20.</u> For every simply connected region $\Omega \subseteq \hat{\mathbb{C}}$ one can introduce a suitable compactification $\hat{\Omega}$ ("prime end compactification") s.t. every conf. map $f: \Omega_1 \hookrightarrow \Omega_2$ between simply connected regions extends to a homeo. $\hat{f}: \hat{\Omega}_1 \hookrightarrow \hat{\Omega}_2$ (Carathéodory 1913)

<u>3. Kernel convergence</u>

$f_n: \mathbb{D} \to \Omega_n$, $n \in \mathbb{N}$, conf. maps, with suitable normalization.

Can we characterize when $\{f_n\}$ converges loc. unif. on $\mathbb{D}$ in terms on the regions Ω_n.

Yes! Answer related to kernel convergence of the seq. $\{\Omega_n\}$

<u>Def. 3.1. (Kernel convergence)</u>
$\{\Omega_n\}$ sequence of regions in $\mathbb{C}$.
$w_0 \in \mathbb{C}$. "basepoint"
Suppose $w_0 \in \Omega_n$ for all $n \in \mathbb{N}$.

(38) The <u>Kernel</u> Kern_{w_0} w.r.t. w_0 of $\{\Omega_n\}$ consists of

i) the point w_0

ii) every point $w \in \mathbb{C}$ with the following property: there ex. a region $\mathcal{U}$ with $w_0, w \in \mathcal{U}$ s.t. $\mathcal{U} \subseteq \Omega_n$ for all suff. large n.

So always $w_0 \in \mathrm{Kern}_{w_0}$, and $\mathrm{Kern}_{w_0} = \{w_0\}$ is possible.

If $\mathrm{Kern}_{w_0} \neq \{w_0\}$, then Kern_{w_0} is a region. ($=$ union of sets $\mathcal{U}$ in ii))

Let $\Omega = \{w_0\}$ or $\Omega \subseteq \mathbb{C}$ be a region with $w_0 \in \Omega$.

We say that $\{\Omega_n\}$ <u>converges to Ω in the sense of kernel convergence</u> (w.r.t. base point w_0), written

$$\Omega_n \longrightarrow \Omega \quad (\text{w.r.t. } w_0)$$

if every subsequence of $\{\Omega_n\}$ has the kernel Ω.

<u>Ex. 3.2.</u> $\Omega_n = \mathbb{C} \setminus (-\infty, -\frac{1}{n}] \cup [\frac{1}{n}, \infty)$

$\mathbb{H}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$

$\mathbb{H}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$

$\bigcap \Omega_n = \mathbb{H}_+ \cup \{0\} \cup \mathbb{H}_-$ $\ni w_0$ <u>base point</u>

$$\mathrm{Kern}_{w_0} = \begin{cases} \mathbb{H}_+ & w_0 \in \mathbb{H}_+ \\ \{0\} & \text{for } w_0 = 0 \\ \mathbb{H}_- & w_0 \in \mathbb{H}_- \end{cases}$$

Moreover

$$\Omega_n \longrightarrow \begin{cases} \mathbb{H}_+ & \\ \{0\} & \text{w.r.t. } w_0 = 0, \\ \mathbb{H}_- & \end{cases} \quad \begin{matrix} \in \mathbb{H}_+, \\ \\ \in \mathbb{H}_- \end{matrix}$$

(39) Lem. 3.3. $w_0 \in \mathbb{C}$, $\{\Omega_n\}$ seq. of regions in $\mathbb{C}$ with $w_0 \in \Omega$ for all $n \in \mathbb{N}$

a) If $\{\Omega_n\}$ is increasing, i.e., $\Omega_n \subseteq \Omega_{n+1}$ for $n \in \mathbb{N}$,
then $\mathrm{Ker}_{w_0} = \Omega_\infty := \bigcup_{n\in\mathbb{N}} \Omega_n$,
and $\Omega_n \longrightarrow \Omega_\infty$ w.r.t. w_0.

b) If $\{\Omega_n\}$ is decreasing, i.e., $\Omega_n \supseteq \Omega_{n+1}$ for $n \in \mathbb{N}$,
let Ω_∞ be the connected component of the interior of $\bigcap_{n\in\mathbb{N}} \Omega_n$ containing w_0 if $w_0 \in \mathrm{int}\left(\bigcap_{n\in\mathbb{N}} \Omega_n\right)$
and $\Omega_\infty = \{w_0\}$ if not.
Then
$\mathrm{Ker}_{w_0} = \Omega_\infty$ and $\Omega_n \longrightarrow \Omega_\infty$ w.r.t. w_0.

Proof: a) $\mathrm{Ker}_{w_0} \subseteq \Omega_\infty$ clear.
$\Omega_\infty \subseteq \mathrm{Ker}\, w_0$: if $w \in \Omega_\infty$, then $w \in \Omega_n$ for some $n \in \mathbb{N}$. Take $U = \Omega_n$ in Def 3.1.; so $w \in \mathrm{Ker}_{w_0}$.
$\Omega_n \longrightarrow \Omega_\infty$ because Kernel (= union) does not change by passing to subseq.
b) $\mathrm{Ker}\, w_0 \subseteq \bigcap \Omega_n$ is $\{w_0\}$ or region containing w_0; so $\mathrm{Ker}\, w_0 \subseteq \Omega_\infty$.
$\Omega_\infty \subseteq \mathrm{Ker}\, w_0$: Clear if $\Omega_\infty = \{w_0\}$; otherwise take $U = \Omega_\infty$ in Def. 3.1.
so $\Omega_\infty = U \subseteq \mathrm{Ker}\, w_0$.
$\Omega_n \longrightarrow \Omega_\infty$ clear, because $\bigcap_{n\in\mathbb{N}} \Omega_n$ does not

40 change by passing to subsequences. □

Prop. 3.4. $f_n : \mathbb{D} \longleftrightarrow \Omega_n$ conf. maps s.t. $f_n(0) = w_0$, $f_n'(0) > 0$.
Suppose $f_n \longrightarrow f$ loc. unif. on $\mathbb{D}$.
Then for the kernel of $\{\Omega_n\}$ w.r.t. w_0 we have
$$\mathrm{Ker}_{w_0} = f(\mathbb{D}).$$

Proof: Note: f const. $\equiv w_0$ or f conf. map onto $(\Omega = f(\mathbb{D})$. (Hurwitz). $f(0) = w_0$

I. $f(\mathbb{D}) \subseteq \mathrm{Ker}_{w_0}$: Obvious if f const.
Assume $f \not\equiv$ const. Let $w \in f(\mathbb{D})$ arb.
Ex. $r \in (0,1)$ s.t. $w \in U := f(B(0,r))$
U region s.t. $w_0, w \in U$. (and so $w \in \mathrm{Ker}_{w_0}$)
Claim: $U \subseteq f_n(\mathbb{D}) = \Omega_n$ for large n.
Otherwise, there ex. seq. $\{n_k\}$ in $\mathbb{N}$ with $n_k \to \infty$ and pts. $w_k \in U$ s.t. $w_k \notin f_{n_k}(\mathbb{D})$. $\overline{U} \subseteq f(\overline{B}(0,r))$ compact; so
w.l.o.g. $w_k \to v \in \overline{U} \subseteq f(\mathbb{D})$. Then
$h_k := f_{n_k} - w_k$ is zero-free on $\mathbb{D}$,
and $h_k \longrightarrow f - v$ loc. unif. on $\mathbb{D}$.
$v \in \overline{U} \subseteq f(\mathbb{D})$, so $f - v$ is not zero-free,
so $f - v \equiv 0$ equiv. $f \equiv v$ by Hurwitz.
Contradiction!

II $\mathrm{Ker}_{w_0} \subseteq f(\mathbb{D})$: $w_0 \in f(\mathbb{D})$.
Let $w \in \mathrm{Ker}_{w_0}$, $w \neq w_0$, be arb.
Then there ex. region U s.t. $w_0, w \in U$.

(41) and $U \subseteq \Omega_n$ for all larger n, wlog for all n.

$g_n := f_n^{-1}|U : U \to \mathbb{D}$ holomorphic.

By Montel there ex. subsequence that converges loc. unif. to $g : U \to \overline{\mathbb{D}}$ holomorphic. Wlog $g_n \to g$ loc. unif. on $\mathbb{D}$. Note $g_n(w_0) = 0 \Rightarrow g(w_0) = 0$, and $g(U) \subseteq \overline{\mathbb{D}}$. So $g(U) \subseteq \mathbb{D}$ by max. principle.

Let $z := g(w) \in \mathbb{D}$. Then $f_n \to f$ loc. unif. near z, and so

$$w = \lim_{n \to \infty} f_n(\underbrace{g_n(w)}_{\to\, g(w) = z}) \overset{\text{unif. conv. of } f_n \text{ near } z}{=} f(z),$$

and so $w \in f(\mathbb{D})$.

I + II $\Rightarrow$ Prop. $\square$

Thm. 3.5. (Main Theorem about Kernel convergence)

$f_n : \mathbb{D} \leftrightarrow \Omega_n$ conf. maps s.t. $f_n(0) = w_0$, $f_n'(0) > 0$ for $n \in \mathbb{N}$.

Then:

i) $\Omega_n \to \{w_0\}$ (w.r.t. w_0)
   iff $f_n \to \text{const.} = w_0$ loc. unif. on $\mathbb{D}$
   iff $f_n'(0) \to 0$.

ii) $\Omega_n \to \Omega$, where $\Omega \subseteq \mathbb{C}$ region with $w_0 \in \Omega$, $\Omega \neq \{w_0\}, \mathbb{C}$
   iff $f_n \to f \not\equiv \text{const.}$ loc. unif. on $\mathbb{D}$

iii) $\Omega_n \to \mathbb{C}$ iff $f_n \to \infty$ loc. unif. on $\mathbb{D}\setminus\{0\}$ iff $f_n'(0) \to \infty$.

(42) In particular,

$\Omega_n \to \Omega \neq \mathbb{C}$ iff $\{f_n\}$ conv. loc. unif. on $\mathbb{D}$.

Proof: By Koebe:

$$|f_n'(0)| \frac{|z|}{(1+|z|)^2} \leq |f_n(z) - w_0| \leq |f_n'(0)| \frac{|z|}{(1-|z|)^2},$$

(*) and

(**) $B(w_0, \frac{1}{4}|f_n'(0)|) \subseteq \Omega_n = f_n(\mathbb{D})$

iii) $\Omega_n \to \mathbb{C} \Rightarrow f_n'(0) \to \infty$:

if not, then $\{f_n'(0)\}$ has a bdd. subseq. w.l.o.g. $\{f_n'(0)\}$ is bdd.

By (*) $\{f_n\}$ is loc. unif. bdd. on $\mathbb{D}$; by Montel a subseq. of $\{f_n\}$ conv. loc. unif. on $\mathbb{D}$; w.l.o.g. $f_n \to f$ loc. unif. By Prop. 3.4 $\Omega_n = f_n(\mathbb{D}) \to f(\mathbb{D})$ w.r.t. w_0, but $f(\mathbb{D}) \neq \mathbb{C}$ (by Liouville). Contradiction!

$f_n'(0) \to \infty \Leftrightarrow f_n \to \infty$ loc. unif. on $\mathbb{D} \setminus \{0\}$ by (*).

$f_n'(0) \to \infty \Rightarrow \Omega_n \to \mathbb{C}$ by (**).

i)+ii) Suppose $\Omega_n \to \Omega \neq \mathbb{C}$ (possibly $\Omega = \{w_0\}$). Then by (iii) $\{f_n'(0)\}$ has no subsequence with $f_{n_k}'(0) \to \infty$, and so $\{f_n'(0)\}$ is bdd.

By (*) $\{f_n\}$ is loc. unif. bdd., and so a normal family by Montel.

To show that $\{f_n\}$ converges loc. unif. on $\mathbb{D}$ it suffices that any two

(43) subsequential limits g, h of $\{f_n\}$ agree. By Prop. 3.4.

$g(\mathbb{D}) = \text{Kern}_{w_0} = \Omega = h(\mathbb{D})$.

So if $\Omega = \{w_0\}$, then $g = h \equiv w_0$, and $f_n \longrightarrow w_0$ loc. unif.

(so $\Omega_n \longrightarrow \{w_0\} \Rightarrow f_n \longrightarrow w_0$ loc. unif.)

If $\Omega \neq \{w_0\}$, then g, h are conf. maps onto Ω by Hurwitz.

We have $g(0) = h(0) = w_0$,

g', h' are subsequential limits of $\{f_n'\}$ (Weierstrass).

So $g'(0), h'(0) > 0$. By uniqueness part of Riemann Mapping Thm. $g = h$.

(so $\Omega_n \longrightarrow \Omega \neq \{w_0\}, \mathbb{C} \Rightarrow$ $f_n \longrightarrow f$ loc. unif., where f is the unique conf. map with $\Omega = f(\mathbb{D})$, $f(0) = w_0$, $f'(0) > 0$).

<u>Converse:</u>

i) $f_n \longrightarrow w_0$ loc. unif. $\Longleftrightarrow f_n'(0) \longrightarrow 0$ by $\dagger$

$\Rightarrow \Omega_n \longrightarrow \{w_0\}$ by Prop. 3.4

ii) $f_n \longrightarrow f \not\equiv$ const. $\Rightarrow$

$\Omega_n \longrightarrow f(\mathbb{D}) = \Omega$ by Prop. 3.4.

f conf. map onto $f(\mathbb{D}) = \Omega \neq \mathbb{C}$. $\square$

(44) # 4. Loewner chains and the Loewner-Kufarev equation

4.1. Loewner chains (whole plane version)

$I = [a, \infty]$; $w_0 \in \mathbb{C}$ basepoint

$\Omega_t \subseteq \mathbb{C}$ simply connected region with $w_0 \in \Omega$ for $t \in I$ s.t.

i) $\Omega_a = \mathbb{C}$ ($\Omega_a = \{w_0\}$ allowed as degenerate case)

ii) $\Omega_s \subsetneq \Omega_t$ for $s, t \in I$, $s < t$.

We say that the family $\{\Omega_t\}$ is a (geometric) Loewner chain if Ω_t is cont. in t in the sense of Kernel convergence w.r.t. w_0, i.e.,

$\Omega_{t_n} \to \Omega_t$ whenever $t_n \in I \to t \in I$.

For $t \in I$ let $f_t : \mathbb{D} \leftrightarrow \Omega_t$ be the unique conf. map with

$f_t(0) = w_0$, $f_t'(0) > 0$

(f_∞ is left undefined and $f_a \equiv w_0$ if $\Omega_a = \{a\}$).

Then $\{f_t\}$ is called an (analytic) Loewner chain if f_t is cont. in t w.r.t. locally unif. conv. on $\mathbb{D}$, i.e.,

$f_{t_n} \to f_t$ loc. unif. on $\mathbb{D}$

whenever $t_n \to t$.

(it is understood that this means:

$f_{t_n}'(0) \to \infty$ if $t_n \to \infty$; no problem if $\Omega_a = \{w_0\}$ and $f_a \equiv w_0$!)

(45) The Loewner chain $\{f_t\}$ is normalized if $f_t'(0) = e^t$, for $t \in I$.

Rem. 4.2. a) $\{\Omega_t\}$ cont. in t if and only if $\{f_t\}$ cont. in t (by Thm. 3.5.)

b) For continuity of $\{f_t\}$ it is enough to check "left" and "right" continuity, i.e., that $f_{t_n} \longrightarrow f_t$ loc. unif. on $\mathbb{D}$ whenever $\{t_n\}$ is a monotone sequence in I (decreasing or increasing) with $t_n \longrightarrow t$ (true essentially because every seq. $\{t_n\}$ has a monotone subsequence)

c) By a) + b) for continuity of $\{\Omega_t\}$ one only has to check that $\Omega_{t_n} \longrightarrow \Omega_t$ whenever $\{t_n\}$ is a monotone seq. in I with $t_n \longrightarrow t$.

By Lem. 3.3 this is equiv. with the following two conditions:

(i) $\Omega_t = \bigcup_{s<t} \Omega_s$ for $t \in I$, and

(ii) $\Omega_t = \{w_0\} \cup$ connected comp. of interior of $\bigcap_{t<r} \Omega_r$, $t \in I$. that cont. w_0

Note: if (ii') $\Omega_t = \text{interior of } \bigcap_{t<r} \Omega_r$ then ii) is true

(i) + (ii') $\longrightarrow$ (i) + (ii) $\longleftrightarrow$ continuity

(46) d) Continuity of $\{\Omega_t\}$ independent of $w_0 \in \bigcap \Omega_t =: \Omega_a$.

Indeed, (i) in c) ind. of w_0.

Let $w_0, w_1 \in \bigcap \Omega_t$; then

$$w_0, w_1 \in \Omega_t \subseteq \operatorname{int}\Big(\bigcap_{t<r} \Omega_r\Big) =: \tilde{\Omega}_t$$

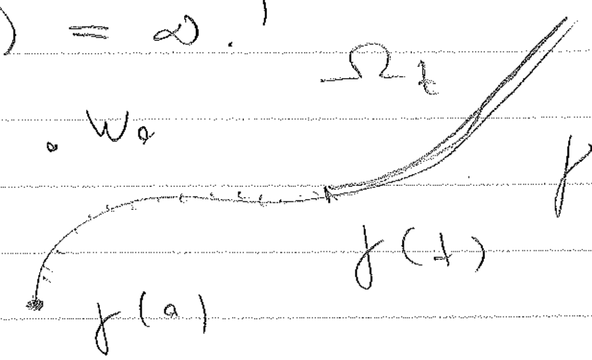
So w_0, w_1 lie in the same connected comp of $\hat{\Omega}_t$. This shows that (ii) true for w_0 iff true for w_1.

Ex. 4.3. (Loewner chains generated by slits).

$\gamma : [a,\infty] \longrightarrow \hat{\mathbb{C}}$ simple path "slit" ending at ∞, i.e.,

$\gamma : [a,\infty) \longrightarrow \mathbb{C}$ cont., injective, $\gamma(\infty) = \infty$.

$\Omega_t := \mathbb{C} \setminus \gamma([t,\infty])$ for $t \in [a,\infty]$



$w_0 \in \mathbb{C} \setminus \gamma([a,\infty])$

(or $w_0 = \gamma(a)$; then $\Omega_a := \{w_0\}$)

Ω_t simply connected regions (the complement of an arc in $\hat{\mathbb{C}}$ has only one component!)

$\Omega_s \subsetneqq \Omega_t$, because $\gamma([s,\infty]) \supsetneqq \gamma([t,\infty])$,

Continuity:

i) $$\bigcup_{s<t} \mathbb{C} \setminus \gamma([s,\infty]) = \mathbb{C} \setminus \bigcap_{s<t} \gamma([s,\infty]) \underset{\gamma \text{ cont.}}{=} \mathbb{C} \setminus \gamma\Big(\bigcap_{s<t} [s,\infty]\Big) = \mathbb{C} \setminus \gamma([t,\infty]) = \Omega_t.$$

(47) (iii) $\bigcap_{t<r} \mathbb{C} \setminus \gamma([r,\infty])$

$= \mathbb{C} \setminus \bigcup_{t<r} \gamma([r,\infty]) = \mathbb{C} \setminus \gamma\left(\bigcup_{t<r} [r,\infty]\right)$

$= \mathbb{C} \setminus \gamma([t,\infty]) = \Omega_t \cup \{\gamma(t)\}$.

$\operatorname{int}(\Omega_t \cup \{\gamma(t)\}) = \Omega_t$.

(if $t = a$, $w_0 = \gamma(a)$, $\Omega_a = \{w_0\}$, then

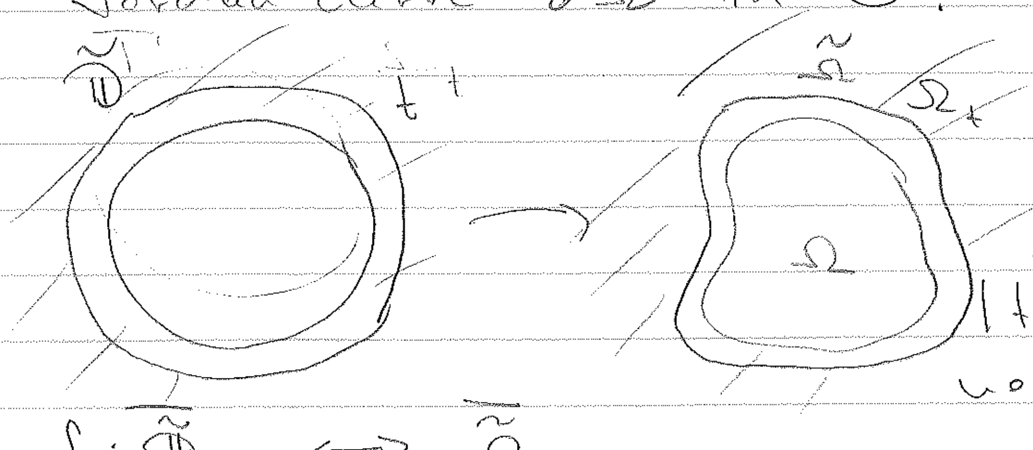
$\tilde{\Omega} := \operatorname{int}\left(\bigcap_{a<r} \mathbb{C} \setminus \gamma([r,\infty])\right) = \mathbb{C} \setminus \gamma([a,\infty))$,

so comp. of $\tilde{\Omega}$ cont. $w_0 = \phi$

and (ii) true for $t \neq a$).

<u>Ex. 4.4.</u> Let $\Omega \subseteq \mathbb{C}$ be bdd. Jordan region. Then there ex. a Loewner chain $\{\Omega_t\}_{t \in [1,\infty]}$ s.t. $\Omega_1 = \Omega$ ($w_0 \in \Omega$).

<u>Proof:</u> Let $\tilde{\Omega}$ be the exterior of the Jordan curve $\partial\Omega$ in $\hat{\mathbb{C}}$. Then there ex. a conf. map $f: \tilde{\mathbb{D}} \to \tilde{\Omega}$ with $f(\infty) = \infty$. It has a homeomorphic extension $f: \overline{\tilde{\mathbb{D}}} \longleftrightarrow \overline{\tilde{\Omega}}$.



For $t \in [1,\infty)$ let Ω_t be the inside of the Jordan curve $f(\{z \in \mathbb{C} : |z| = t\})$ and $\Omega_\infty := \mathbb{C}$.

(48) Thm: $\{\Omega_t\}_{t\in[1,\infty]}$ is a Loewner chain with $\Omega_1 = \Omega$. Ω_t strictly increasing

Indeed: $\Omega_1 = \Omega$ clear.

$\Omega_t = \hat{\mathbb{C}} \setminus f(\tilde{\mathbb{D}}) \cup f(\{z \in \mathbb{C} : 1 < |z| < t\})$ for $1 < t < \infty$

(shown as in the proof of Area Thm.)

Continuity:

$1 \leq t < \infty$:

i) $\bigcup_{s<t} \Omega_s = \hat{\mathbb{C}} \setminus f(\hat{\mathbb{D}}) \cup f(\{z \in \mathbb{C} : 1 < |z| < t\}) = \Omega_t$.

(ii') $\bigcap_{t<r} \Omega_r = \hat{\mathbb{C}} \setminus f(\tilde{\mathbb{D}}) \cup f(\{z \in \mathbb{C} : 1 < |z| \leq t\})$ f homeom.

$= \Omega_t \cup \partial\Omega_t = \overline{\Omega}_t$.

interior $(\overline{\Omega}_t) = \Omega_t$ (Ω_t Jordan region!)

$t = \infty$:

$\bigcup_{s<\infty} \Omega_s = \hat{\mathbb{C}} \setminus f(\hat{\mathbb{D}}) \cup f(\{z \in \mathbb{C} : 1 < |z| < \infty\})$
$= \overline{\Omega}_t \cup \tilde{\Omega}_t \setminus \{\infty\} = \mathbb{C}$.

4.5. The associated semi-group

$f, g : \mathbb{D} \to \mathbb{C}$ holomorphic

f is subordinate to g, written $f \prec g$, if there ex. a holomorphic map $\varphi : \mathbb{D} \to \mathbb{D}$ with $\varphi(0) = 0$ and $f = g \circ \varphi$

(then $f(0) = g(0)$, and $|f'(0)| \leq |g'(0)|$, because $|\varphi'(0)| \leq 1$ (Schwarz's Lemma))

Let $\{f_t\}_{t\in[a,\infty]}$ be a Loewner chain.

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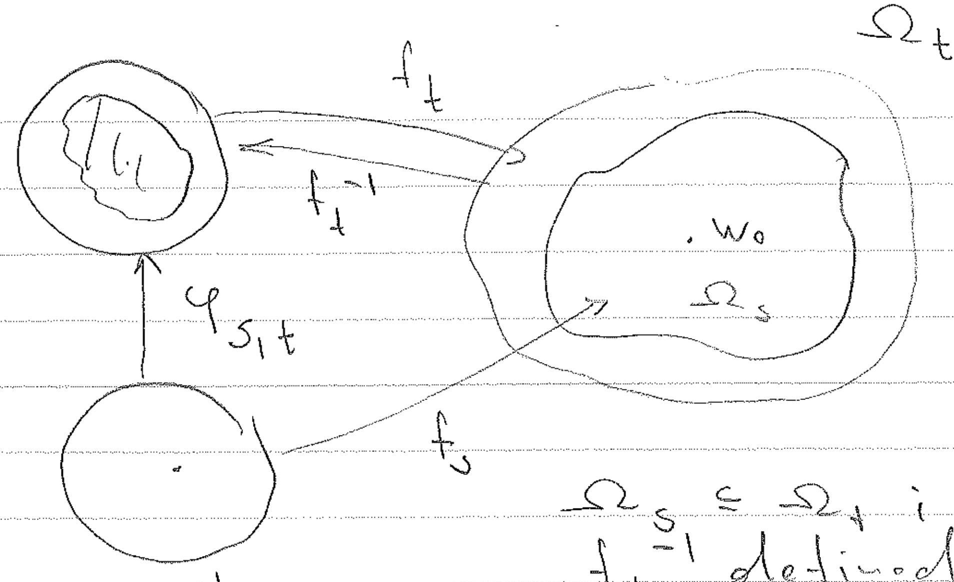


$a \leq s \leq t < \infty$

$\Omega_s \subseteq \Omega_t$; so f_t^{-1} defined on Ω_s

$\varphi_{s,t} := f_t^{-1} \circ f_s$ on $\mathbb{D}$, holomorphic, conf. map onto its image

$\varphi_{s,t}(\mathbb{D}) \subseteq \mathbb{D}$, $\varphi_{s,t}(0) = 0$

$$\boxed{\begin{aligned} & f_s = f_t \circ \varphi_{s,t}, \quad a \leq s \leq t < \infty \\ & \varphi_{t,u} \circ \varphi_{s,t} = \varphi_{s,u} \quad a \leq s \leq t \leq u < \infty \\ & \text{(semigroup property)} \\ & \varphi_{t,t} = \mathrm{id}_{\mathbb{D}} \quad a \leq t < \infty. \end{aligned}}$$

f_s subordinate to f_t; so
$f_s'(0) \leq f_t'(0)$, $s < t$.

Actually, we have strict inequality
$f_s'(0) < f_t'(0)$ here.

Otherwise, $f_t'(0) = f_s'(0) = f_t'(0) \cdot \varphi_{s,t}'(0)$;
so $\varphi_{s,t}'(0) = 1$.

By Schwarz's Lemma: $\varphi_{s,t} = \mathrm{id}_{\mathbb{D}}$,
and $f_t = f_s$, and $\Omega_t = f_t(\mathbb{D}) = f_s(\mathbb{D}) = \Omega_s$,
contradiction.

50 4.6. Heuristics for the Loewner equation

A family of maps $\varphi_{s,t}$ with the semigroup property is generated by a time-dependent vector field.

Assume $\varphi_{s,t}(z)$ is smooth in s,t, holomorphic in z.

Define
$$V(z,s) = \frac{\partial \varphi_{s,t}}{\partial t}(z,s) = \lim_{\delta \to 0^+} \frac{\varphi_{s,s+\delta}(z) - z}{\delta}$$

$\mathbb{C} \cong \mathbb{R}^2$ time-dep vector field $\qquad \varphi_{s,s} = id_{\mathbb{D}}$.

$\varphi_{s,s+\epsilon}$

$\varphi_{s,s+\delta}(z) = z + \delta V(z,s)$

z

$$\frac{\partial}{\partial t}\varphi_{s,t}(z,t) = \lim_{\delta \to 0^+} \frac{\varphi_{s,t+\delta}(z) - \varphi_{s,t}(z)}{\delta}$$

$$= \lim_{\delta \to 0^+} \frac{\varphi_{t,t+\epsilon}(\overbrace{\varphi_{s,t}(z)}^{w}) - \overbrace{\varphi_{s,t}(z)}^{w}}{\delta} = V(w,t)$$

$V(\varphi_{s,t}(z), t)$

So
$$\boxed{\begin{aligned} &\frac{\partial \varphi_{s,t}}{\partial t}(s,z) = V(z,s) \\ &\frac{\partial \varphi_{s,t}}{\partial t}(t,z) = V(\varphi_{s,t}(z), t), \quad t > s \end{aligned}}$$

$\gamma : [s,u] \to \mathbb{C}$ $\quad C^1$-smooth

$\gamma(s) = z, \quad \dot\gamma(t) = V(\gamma(t), t)$

γ integral curve of vector field V.

$t \longmapsto \varphi_{s,t}(z)$ integral curve.

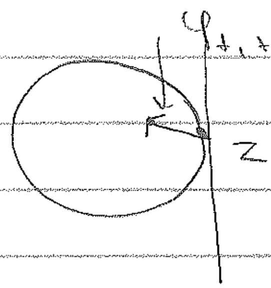
(51) z at time $t \longmapsto \gamma(t)$ at time t

$\varphi_{s,t}(z) = \gamma(t)$ is a map $\varphi_{s,t}$ (map from time s to time t)

z time s $\quad$ time t

What can we say about $V(z,t)$ if $\varphi_{s,t}$ comes from a Loewner chain?

By Schwarz's Lemma: $\varphi_{t,t+\delta}(z) \in \bar{B}(0,|z|)$



So $\mathrm{Re}\left(\dfrac{\varphi_{t,t+\delta}(z)-z}{z}\right) \leq 0$, and

$$\mathrm{Re}\left(\frac{V(z,t)}{z}\right) = \mathrm{Re}\left(\lim_{\delta\to 0^+} \frac{\varphi_{t,t+\delta}(z)-z}{z\,\delta}\right) \leq 0$$

<u>Conclusion</u> $\mathrm{Re}\left(\dfrac{V(z,t)}{z}\right) \leq 0$,

$V(z,t)$ holomorphic in $\mathbb{D}$.

So

$$V(z,t) = -z\,p(z,t),$$

$p(z,t)$ holomorphic in z, $\mathrm{Re}(p(z,t)) \geq 0$.

$\{f_t\}$ Loewner chain

$f(z,t) = f_t(z)$, $\quad f_t'(z) = \dfrac{\partial f}{\partial z}(z,t)$

assume $f(z,t)$ smooth in t. $\quad \dot{f}_t(z) = \dfrac{\partial f}{\partial t}(z,t)$

$\varepsilon > 0$:

$$f_{t+\varepsilon} \circ \varphi_{t,t+\varepsilon}(z) = f_t(z);$$

$$0 = \left.\frac{\partial f_t(z)}{\partial \varepsilon}\right|_{\varepsilon=0} = \left.\frac{\partial}{\partial \varepsilon} f\left(\varphi_{t,t+\varepsilon}(z),\, t+\varepsilon\right)\right|_{\varepsilon=0}$$

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$$= f_t'(z) \cdot \frac{\partial \varphi_{t,t+\varepsilon}(z)}{\partial \varepsilon}\Big|_{\varepsilon=0} + \dot{f}_t(z)$$

$$= f_t'(z) \underbrace{V(z,t)}_{= z\,p(z,t)} + \dot{f}_t(z).$$

So

$$\boxed{\begin{aligned} &\dot{f}_t(z) = z\,p(z,t)\, f_t'(z) \\ &\frac{\partial f}{\partial t}(z,t) = z\,p(z,t) \frac{\partial f}{\partial z}(z,t) \\ &\operatorname{Re} p(z,t) \geq 0 \qquad \text{Loewner-Kuferev equation} \end{aligned}}$$

Have we accomplished anything? Wlog $f(0,t) = w_0 \equiv 0$

$f(z,t) = a_1(t) z + a_2(t) z^2 + \dots$

$\dot{f}(z,t) = \dot{a}_1(t) z + \dot{a}_2(t) z^2 + \dots$ $\quad f_0 \in \mathcal{S}$

$a_1(0) = 1$

$f'(z,t) = a_1(t) + 2a_2(t) z + \dots$

$p(z,t) = c_0(t) + c_1(t) z + \dots$

$$(\dot{a}_1 z + \dot{a}_2 z^2 + \dots) = z(c_0 + c_1 z + \dots)(a_1 + 2a_2 z + \dots)$$
$$= c_0 a_1 z + (c_1 a_1 + 2a_2 c_0) z^2 + \dots$$

Comparing coefficients:

$\dot{a}_1 = c_0 a_1$,

$\dot{a}_2 = c_1 a_1 + 2a_2 c_0$

Making a change of time parametrization

$\dot{a}_1 \equiv a_1$ so $c_0 \equiv 1$ $\quad a_1(t) = e^t$

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$\dot{a}_2 - 2a_2 = c_1 e^t$

$a_2(t) = e^{2t} C(t)$, $e^{2t} C'(t) = c_1(t) \cdot e^t$

$C(t) = \int_0 c_1(t) \cdot e^{-t} dt$

$e^{-t} f_t \in \mathcal{S}$; so $|a_2(t) \cdot e^{-t}|$ bdd.

$C(\infty) = \lim_{t\to\infty} a_2(t) e^{-2t} = 0$

$-C(t) = C(\infty) - C(t) = \int_t^\infty c_1(t) \cdot e^{-t} dt$

$a_2(t) = -e^{2t} \int_t^\infty c_1(t) e^{-t} dt$

$a_2(0) = -\int_0^\infty c_1(u) \cdot e^{-u} du$

Now: if $f(z) = 1 + c_1 z + c_2 z^2$ and $\operatorname{Re} f \geq 0$, then $|c_1| \leq 2$;

so $|a_2(0)| \leq 2 \int_0^\infty e^{-u} du = 2.$

<u>Lem. 4.7.</u> Let $\{f_t\}_{t\in I}$, $I = [a, \infty]$, be an analytic Loewner chain.
Then there ex. $\tilde{a} \in [-\infty, \infty)$, a strictly increasing homöomorphism $\alpha : \tilde{I} \to I$, and a Loewner chain $\{\tilde{f}_t\}_{t\in\tilde{I}}$ s.t.

i) $\tilde{f}_t'(0) = e^t$ for $t \in \tilde{I} \setminus \{-\infty, \infty\}$,

ii) $\tilde{f}_t = f_{\alpha(t)}$

(so by a homeomorphic change of the time parametrization one can normalize an analytic

54 Loewner chain.

<u>Proof:</u> Define $\beta(t) = \begin{cases} f_t'(0) & \text{for } t \in I \setminus \{\infty\}, \\ \infty & t = \infty. \end{cases}$

Then:

i) β is strictly increasing (see 4.5.)

ii) β is continuous:

$\{t_n\}$ seq. in I s.t. $t_n \longrightarrow t_\infty \in I$.

Then: if $t_\infty = \infty$: $\beta(t_n) = f_{t_n}'(0) \longrightarrow \infty = \beta(\infty)$

by def. of Loewner chain.

if $t_\infty \neq \infty$: $f_{t_n} \longrightarrow f_{t_\infty}$ loc. unif. on $\mathbb{D}$;

so $\beta(t_n) = f_{t_n}'(0) \longrightarrow f_{t_\infty}'(0) = \beta(t_\infty)$

by Weierstrass.

By i)+ii) β is a homeo. onto its image

$\tilde{I} := \beta(I) = [b, \infty] \subseteq [0, \infty]$.

Let: $\tilde{a} := \log b \in [-\infty, \infty)$,

and $\alpha(t) := \beta^{-1}(e^t)$, $t \in [\tilde{a}, \infty]$

$(e^{-\infty} = 0, \; e^{\infty} = \infty)$.

Then α is a strictly increasing homeo.

from $\tilde{I} := [\hat{a}, \infty]$ and $I = [a, \infty]$

$$\tilde{I} \xleftrightarrow{\exp} [b, \infty] \xleftrightarrow{\beta^{-1}} [a, \infty].$$

Define $\tilde{f}_t := f_{\alpha(t)}$. Then $\{\tilde{f}_t\}_{t \in I}$ is a Loewner chain (obvious), and

$\tilde{f}_t'(0) = f_{\alpha(t)}'(0) = \beta(\alpha(t)) = e^t$ for $t \in \hat{I}$. $\square$

(55) From now on:

All analytic Loewner chains $\{f_t\}_{t\in I}$ are normalized, i.e.,

$$\boxed{f_t'(0) = e^t \text{ for } t \in I.}$$

<u>Thm. 4.8.</u> (Vitali's Thm. on induced convergence)

Let $\Omega \subseteq \mathbb{C}$ be a region, $\mathcal{F}$ a normal family of holomorphic functions on Ω, and $\{f_n\}$ a sequence in $\mathcal{F}$. Suppose that there exists a sequence $\{z_k\}$ of pts. in Ω s.t.

i) $\{f_n(z_k)\}$ converges for all $k \in \mathbb{N}$,

ii) $\{z_k\}$ has a limit point in Ω.

Then $\{f_n\}$ converges locally unif. on Ω (to some holomorphic limit function f).

<u>Proof:</u> There ex. a sublimit $f \in H(\Omega)$ of $\{f_n\}$ (w.r.t. loc. unif. conv. on Ω).

<u>Claim:</u> $f_n \to f$ loc. unif. on Ω.

Proof by contradiction:

if not, then there ex. $\varepsilon_0 > 0$ ("bad ε"), a compact set $K \subseteq \Omega$, a sequence $\{n_l\}$ in $\mathbb{N}$ with $n_l \to \infty$ and pts. $u_l \in K$ s.t. $|\underbrace{f_{n_l}}_{=g_l}(u_l) - f(u_l)| \geq \varepsilon_0$.

(56) $\{g_l\}$ is a seq. in $\mathcal{F}$; so it has a convergent subsequence;
wlog $g_l \longrightarrow g \in H(\Omega)$ loc. unif. on Ω,
also: wlog $w_l \longrightarrow w_\infty \in K$.
Since $\{f_n(z_k)\}$ converges for each $k \in \mathbb{N}$, we have
$$g(z_k) = f(z_k) \quad \text{for } k \in \mathbb{N}.$$
Since $\{z_k\}$ has a limit pt. in Ω, $g \equiv f$ by the Uniqueness Thm.
So
$$0 < \varepsilon_0 \le \lim_{l \to \infty} |\underbrace{g_{n_l}(w_l)}_{\to g(w_\infty)} - \underbrace{f(w_l)}_{\to f(w_\infty)}|$$
$$= |g(w_\infty) - f(w_\infty)| = 0 \quad \text{contradiction.} \ \square$$

<u>Thm. 4.9</u> (Holomorphic functions with positive real part)
Let $\mathcal{P} = \{ p \in H(\mathbb{D}) : p(0) = 1, \operatorname{Re} p \ge 0 \text{ on } \mathbb{D} \}$
Then the following statements are true.

i) $|p(z)| \le \dfrac{1+|z|}{1-|z|}$ for $p \in \mathcal{P}$, $z \in \mathbb{D}$.

ii) $\mathcal{P}$ is a normal family, and it is closed w.r.t. loc. unif. conv., i.e. if $\{p_n\}$ is a seq. in $\mathcal{P}$ and $p_n \longrightarrow p$ loc. unif. on $\mathbb{D}$, then $p \in \mathcal{P}$.

iii) if $p \in \mathcal{P}$, then there ex. a unique Borel probability measure μ on $\partial\mathbb{D}$ s.t.

(57)

$$p(z) = \int_{\partial\mathbb{D}} \frac{\xi+z}{\xi-z}\, d\mu(\xi) \quad \text{for } z\in\mathbb{D}$$

(Herglotz representation). Conversely, every function of this type belongs to $\mathcal{P}$.

If $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ is the Taylor expansion of p, then

$$c_n = 2\int_{\partial\mathbb{D}} \xi^{-n}\, d\mu(\xi) = 2\int_0^{2\pi} e^{-in\vartheta}\, d\mu(e^{i\vartheta})$$

for $n\in\mathbb{N}$.

iv) Let $p(z) = 1 + c_1 z + c_2 z^2 + \dots \in \mathcal{P}$.

Then: $|c_n| \leq 2$ for $n\in\mathbb{N}$,

$(\operatorname{Re} c_1)^2 \leq 2 + \operatorname{Re} c_2$.

__Proof:__ Note $\operatorname{Re} p \geq 0$ if $p\in\mathcal{P}$

by min. principle for harm. functions.

i) $H = \{u \in \mathbb{C} : \operatorname{Re} u > 0\}$

$\psi : H \longleftrightarrow \mathbb{D}$ conf. map, $\psi(1) = 0$

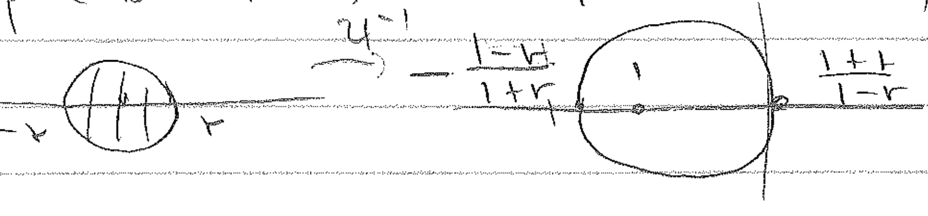
$$\psi(w) = \frac{w-1}{w+1}, \quad \psi^{-1}(u) = \frac{1+u}{1-u}$$

$p\in\mathcal{P}$, $\varphi := \psi\circ p$, $\varphi(\mathbb{D}) \subseteq \mathbb{D}$,
$\varphi(0) = \psi(1) = 0$.

By Schwarz's Lem.:

$\varphi(\bar{B}(0,r)) \subseteq \bar{B}(0,r)$, and so

$$p(\bar{B}(0,r)) \subseteq \psi^{-1}(\bar{B}(0,r)) \subseteq \bar{B}\left(0, \frac{1+r}{1-r}\right).$$



Claim follows

58 ii) By (i) $\mathcal{P}$ is loc. unif. bdd.;
so a normal family by Montel.
if $p_n \in \mathcal{P} \longrightarrow p$ loc. unif. on $\mathbb{D}$,
then $p \in H(\mathbb{D})$ (Weierstrass),
$p(0) = \lim_{n\to\infty} p_n(0) = 1$, and

$\mathrm{Re}\, p(z) = \lim_{n\to\infty} \mathrm{Re}\, p_n(z) \geq 0$ for $z \in \mathbb{D}$.
So $p \in \mathcal{P}$.

iii) Let $p \in \mathcal{P}$. For fixed $r \in (0,1)$
define

$p_r(z) = p(r\cdot z)$. Then $p_r \in H(\mathbb{D})$, and
p_r has a cont. ext. to $\overline{\mathbb{D}}$. Hence (24CC,
Midterm Prob. 2),

$$p_r(z) = i \underbrace{\mathrm{Im}\, p_r(0)}_{0} + \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \mathrm{Re}\, p_r(e^{it})\, dt$$

$$= \int_{\partial\mathbb{D}} \frac{\xi+z}{\xi-z}\, d\mu_r(\xi),$$

where $d\mu_r(\xi) = d\mu_r(e^{it})$
$$= \frac{1}{2\pi}(\mathrm{Re}\, p)(r\cdot e^{it})\, dt$$
$$= \frac{1}{2\pi}(\mathrm{Re}\, p)(r\xi)\, dt.$$

μ_r is a positive Borel measure on $\partial\mathbb{D}$,
and
$$\mu_r(\partial\mathbb{D}) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\, p(r\cdot e^{it})\, dt \overset{\text{mean value prop.}}{=}$$
$$= \mathrm{Re}\, p(0) = 1,$$

59. So μ_r, $r \in (0,1)$ is a pos. Borel prob. measure on $\partial\mathbb{D}$.
By Banach–Alaoglu there ex. a sequence $r_n \longrightarrow 1$ s.t. $\mu_n := \mu_{r_n} \longrightarrow \mu$ w.r.t. the weak-* topology on $C(\partial\mathbb{D})^* = \{\text{complex Borel measures on } \partial\mathbb{D}\}$

i.e., $$\int_{\partial\mathbb{D}} u \, d\mu_n \longrightarrow \int_{\partial\mathbb{D}} u \, d\mu \quad \text{for all } u \in C(\partial\mathbb{D}).$$

μ is also a probability measure.
For fixed $z \in \mathbb{D}$ we have

$$p(z) = \lim_{n\to\infty} p(r_n z) = \lim_{n\to\infty} p_{r_n}(z)$$
$$= \lim_{n\to\infty} \int_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z}\, d\mu_n(\zeta)$$
$$= \int_{\partial\mathbb{D}} \frac{\zeta+z}{\zeta-z}\, d\mu(\zeta)$$

This shows the existence of the Herglotz representation.
Uniqueness + Converse will be HW assignments!
For fixed $z \in \mathbb{D}$, and $\zeta \in \partial\mathbb{D}$ we have
$$\frac{\zeta+z}{\zeta-z} = \frac{1+z/\zeta}{1-z/\zeta} = 1 + 2\sum_{n=1}^{\infty} z^n \zeta^{-n}.$$
Convergence is uniform in ζ.
So we can integrate term-by-term and conclude

(60)

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

$$= \int_{\partial\mathbb{D}} \frac{\xi + z}{\xi - z}\, d\mu(\xi)$$

$$= \int_{\partial\mathbb{D}} \left(1 + 2\sum_{n=1}^{\infty} z^n \cdot \xi^{-n}\right) d\mu(\xi)$$

$$= 1 + 2\sum_{n=1}^{\infty}\left(\int_{\partial\mathbb{D}} \xi^{-n}\, d\mu(\xi)\right) z^n.$$

True for all $z \in \mathbb{D}$. So we can compare coefficients and obtain

$$c_n = 2\int_{\partial\mathbb{D}} \xi^{-n}\, d\mu(\xi) \quad \text{for } n \in \mathbb{N}.$$

In particular: $|c_n| = 2\left|\int_{\partial\mathbb{D}} \xi^{-n}\, d\mu(\xi)\right|$

$$\leq 2\int_{\partial\mathbb{D}} \underbrace{|\xi^{-n}|}_{=1}\, d\mu(\xi) = 2. \qquad \xi = e^{it}$$

$$\operatorname{Re} c_1 = 2\int_{\partial\mathbb{D}} \operatorname{Re}(e^{-it})\, d\mu(\xi)$$

$$= 2\int_{\partial\mathbb{D}} (\cos t)\, d\mu$$

$$\operatorname{Re} c_2 = 2\int_{\partial\mathbb{D}} (\cos 2t)\, d\mu,$$

So $(\operatorname{Re} c_1)^2 = 4\left(\int_{\partial\mathbb{D}} \cos t\, d\mu\right)^2$ $\qquad \cos^2 t = \frac{1+\cos 2t}{2}$

Cauchy-Schwarz
$$\leq 4\int_{\partial\mathbb{D}} \cos^2 t\, d\mu = 4\int \frac{1+\cos 2t}{2}\, dt$$

$$= 2 + 2(\operatorname{Re} c_2). \qquad \square$$

(61) Lem. 4.10 $\{f_t\}$ normalized Loewner chain $\varphi_{s,t} = f_t^{-1} \circ f_s$, $s \le t$, on $I = [a, \infty]$

Then for fixed $z \in \mathbb{D}$:

i) $$|\varphi_{s,t}(z) - z| \le |t-s| \frac{2|z|}{(1-|z|)}$$

$a \le s \le t < \infty$,

ii) $$|f_t(z) - f_s(z)| \le e^t |t-s| \frac{4|z|}{(1-|z|)^4}$$

$a \le s \le t < \infty$

iii) $$|\varphi_{s,u}(z) - \varphi_{t,u}(z)| \le |t-s| \cdot \frac{2}{(1-|z|)^2},$$

$a \le s \le t \le u < \infty$

iv) $$|\varphi_{s,t}(z) - \varphi_{s,u}(z)| \le |u-t| \cdot \frac{2|z|}{1-|z|}$$

$a \le s \le t \le u < \infty$.

So the following functions are Lipschitz:

$t \longmapsto f_t(z)$ on $[a, \infty)$, $z \in \mathbb{D}$ fixed

$t \longmapsto \varphi_{s,t}(z)$ on $[s, \infty)$, $z \in \mathbb{D}$, $s \in [a, \infty)$ fixed

$t \longmapsto \varphi_{t,u}(z)$ on $[a, u]$, $z \in \mathbb{D}$, $u \in [a, \infty)$ fixed.

Moreover, the Lipschitz constants are uniform if the arguments and parameters are restricted to suitable subdomains.

E.g. for each $n \in \mathbb{N}$ there ex. $L = L(n)$ s.t. $t \longmapsto f_t(z)$ is L-Lipschitz on $[a, n]$ for each $z \in \overline{\mathbb{D}}(0, 1-\frac{1}{n})$.