(47) 8. Basis probabilistic concepts

8.1. Probability spaces

$(\Omega, \mathcal{A}, \mathbb{P})$ probability space

Ω sample space, space of outcomes

$\omega \in \Omega$ elementary outcome or event

$\mathcal{A}$ σ-algebra or "σ-field"

$A \in \mathcal{A}$ event

$\mathbb{P}$ probability measure defined on $\mathcal{A}$, $\mathbb{P} \geq 0$, $\mathbb{P}(\Omega) = 1$.

Ex. 8.2. $\Omega = \{1, 2, \ldots, 6\}$

$\mathcal{A} = \mathcal{P}(\Omega)$, $\mathbb{P} = \frac{1}{6} \cdot$ counting measure

Pick $\omega \in \Omega$ "at random"

= roll a die

8.3. Random variables

A measurable m.p. $X : \Omega \to \mathbb{R}$ is called a random variable

$(X^{-1}(B) \in \mathcal{A}$ for each Borel set $B \subseteq \mathbb{R})$

$$\mathbb{E}[X] := \int_\Omega X(\omega) \, d\mathbb{P}(\omega)$$

(48) expectation of X.

$$\mathrm{Var}(X) = \mathbb{E}\left[\left(X - \mathbb{E}[X]\right)^2\right]$$
$$= \int \left(X - \mathbb{E}[X]\right)^2 d\mathbb{P}$$
$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2 .$$

"Variance of X"

Lem. 8.4. (Borel-Cantelli I)
A_n, $n \in \mathbb{N}$, events:
If
$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty , \text{ then}$$
$$\mathbb{P}(\underbrace{A_n \text{ i.o.}}_{\text{infinitely often}}) = 0$$
$= \{\omega \in \Omega : \omega \in A_n \text{ for inf. many } n\}$
$$= \bigcap_k \bigcup_{n \geq k} A_n$$

Proof: $\mathbb{P}(A_n \text{ i.o.}) = \lim_{k \to \infty} \mathbb{P}\left(\bigcup_{n \geq k} A_n\right)$
$$\leq \limsup_{k \to \infty} \sum_{n=k}^{\infty} \mathbb{P}(A_n) = 0 . \quad \square$$

Lem. 8.5. (Chebyshev's Ineq.)
If $X \geq 0$, then
$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[X]}{a} , \quad a > 0 .$$

49 Proof:

$$\mathbb{P}(X \geq a) = \int_\Omega \chi_{X \geq a}(\omega)\, d\mathbb{P}(\omega)$$

$$\leq \int_\Omega \frac{1}{a} X \, d\mathbb{P} = \frac{\mathbb{E}[X]}{a}. \quad \square$$

§6. The distribution of a random variable

$X : \Omega \to \mathbb{R}^n$ random variable.

The distribution or law of X is the push-forward measure

$$\mathbb{P}_X := X_* \mathbb{P} \text{ on } \mathbb{R}^n, \text{ i.e.,}$$

$\mathbb{P}_X(B) = \mathbb{P}(X^{-1}(B))$ for each Borel set $B \subseteq \mathbb{R}^n$.

$$\mathbb{E}[X] = \int_{\mathbb{R}} x \, d\mathbb{P}_X(x)$$

Characteristic function of $X : \Omega \to \mathbb{R}^n$

$$f(u) = \mathbb{E}\left[e^{i(u \cdot X)}\right] \text{ for } u \in \mathbb{R}^n$$

$$f(u) = \int_\Omega e^{i(u \cdot X(\omega))} \cdot d\mathbb{P}(\omega)$$

$$= \int_{\mathbb{R}^n} e^{i(u \cdot v)} \, d\mathbb{P}_X(v)$$

= Fourier transform of its distribution.

(50) $X_1, \dots, X_n : \Omega \to \mathbb{R}$ random var.

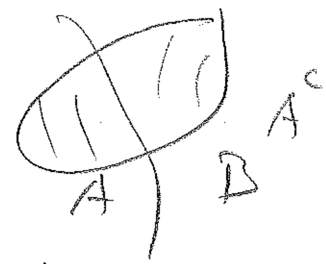
$X := (X_1, \dots, X_n) : \Omega \to \mathbb{R}^n$.

Law of X =: joint law of $X_1, \dots, X_n$.

§7. Independence

$A, B \in \mathcal{A}$ events.

A and B are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$



$A^c := \Omega \setminus A$.

If A, B ind., then A^c, B ind:

$$\mathbb{P}(A^c \cap B) = \mathbb{P}(B) - \mathbb{P}(A \cap B)$$
$$= \mathbb{P}(B)(1 - \mathbb{P}(A)) = \mathbb{P}(A^c) \cdot \mathbb{P}(B)$$

$\mathcal{F}_1, \dots, \mathcal{F}_n \subseteq \mathcal{A}$ σ-algebras are independent if

$$\mathbb{P}(A_1 \cap \dots \cap A_n) = \mathbb{P}(A_1) \cdots \mathbb{P}(A_n)$$

whenever $A_1 \in \mathcal{F}_1, \dots, A_n \in \mathcal{F}_n$.

A, B ind. iff σ-alg. gen. by A and by B are independent.

$X_1, \dots, X_n$ random variables.

(51) They are *independent* if the σ-algebras generated by them are independent.

$$\sigma(X) = \{X^{-1}(B) : B \subseteq \mathbb{R}^n \text{ Borel}\}.$$

If $X_1, \dots, X_n$ are independent, and $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R}$ are Borel, then $f_1(X_1), \dots, f_n(X_n)$ are independent.

Note $\sigma(f(X)) \subseteq \sigma(X)$.

<u>Thm. 8.8.</u> $X_1, \dots, X_n : \Omega \to \mathbb{R}$ random variables, $X = (X_1, \dots, X_n) : \Omega \to \mathbb{R}^n$.

<u>TFAE:</u>

(i) $X_1, \dots, X_n$ are independent,

(ii) $\mathbb{P}(X_1 \in B_1, \dots, X_n \in B_n)$

$$= \mathbb{P}(X_1 \in B_1) \cdots \mathbb{P}(X_n \in B_n)$$

for all Borel sets $B_1, \dots, B_n \subseteq \mathbb{R}$,

(iii) the law of X is a product of the laws of $X_1, \dots, X_n$, i.e.,

$$\mathbb{P}_X = \mathbb{P}_{X_1} \times \cdots \times \mathbb{P}_{X_n},$$

(iv) the characteristic function of X is the product of the characteristic functions of $X_1, \dots, X_n$, that is,

$$\mathbb{E}[e^{i(u \cdot X)}] = \mathbb{E}[e^{iu_1 X_1}] \cdots \mathbb{E}[e^{iu_n X_n}]$$

for $u = (u_1, \dots, u_n) \in \mathbb{R}^n$.

(52) Idea of proof:

(i) $\Longleftrightarrow$ (ii) by definition.

(iii) $\longrightarrow$ (ii) clear.

(ii) $\longrightarrow$ (iii): Follows from fact: if two Borel prob. measures ν, μ on $\mathbb{R}^n$ agree on sets of form $B_1 \times \dots \times B_n$, B_i Borel, then $\nu = \mu$.

(iii) $\not\longrightarrow$ (iv) clear.

(iv) $\longrightarrow$ (iii) follows from fact that a measure is uniquely determined by its Fourier transform.

Cor. 8.9. If X, Y are integrable and independent, then
$$\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y].$$

Proof: $Z = (X, Y): \Omega \to \mathbb{R}^2$.
$$\mathbb{E}[XY] = \int_{\mathbb{R}^2} xy \, d\mathbb{P}_Z(x,y)$$
$$\overset{\text{Thm. 8.8.}}{=} \int_{\mathbb{R}^2} xy \, d\mathbb{P}_X \times \mathbb{P}_Y (x,y)$$
$$= \left(\int_{\mathbb{R}} x \, d\mathbb{P}_X(x) \right) \left(\int y \, d\mathbb{P}_Y(y) \right)$$
$$= \mathbb{E}[X] \cdot \mathbb{E}[Y]. \quad \square$$

Lem. 8.10. (Borel-Cantelli II)
A_n, $n \in \mathbb{N}$, independent events.

(53) If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, then

$\mathbb{P}(A_n \text{ i.o.}) = 1$.

Proof: $e^{-x} \geq 1-x$ for $x \in [0,1]$.

So
$$\mathbb{P}\Big(\bigcup_{n=k}^{N} A_n\Big) = 1 - \mathbb{P}\Big(\bigcap_{n=k}^{N} A_n^c\Big)$$
$$\underset{\text{indep.}}{=} 1 - \prod_{n=k}^{N} (1 - \mathbb{P}(A_n))$$
$$\geq 1 - \prod_{n=k}^{N} e^{-\mathbb{P}(A_n)} = 1 - \exp\Big(-\sum_{n=k}^{N} \mathbb{P}(A_n)\Big)$$
$$\longrightarrow 1 \quad \text{as } N \to \infty.$$

So
$$\mathbb{P}\Big(\bigcup_{n\geq k}^{\infty} A_n\Big) = 1 \quad \text{and}$$
$$\mathbb{P}(A_n \text{ i.o.}) = \mathbb{P}\Big(\bigcap_k \bigcup_{n\geq k} A_n\Big)$$
$$= \lim_{k\to\infty} \mathbb{P}\Big(\bigcup_{n\geq k} A_n\Big) = 1. \quad \square$$

Lem. 8.11. $\underline{X}, \underline{Y} : \Omega \to \mathbb{R}^n$ ind. random var. $Z = \underline{X} + \underline{Y}$.

Then
$$\mathbb{P}_Z = \mathbb{P}_{\underline{X}} * \mathbb{P}_{\underline{Y}} \quad \text{(convolution)},$$
and
$$\phi_Z(u) = \mathbb{E}[e^{i(u \cdot Z)}] = \phi_{\underline{X}}(u) \cdot \phi_{\underline{Y}}(u)$$
for $u \in \mathbb{R}^n$.

Proof: $\pi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\pi(x,y) = x+y$ ..

54 Then $\mathbb{P}_Z = \pi_*(\mathbb{P}_{(X,Y)})$.

Since X, Y and ind.,

$$\mathbb{P}_{(X,Y)} = \mathbb{P}_X \times \mathbb{P}_Y.$$

So if $A \subseteq \mathbb{R}^n$, Borel, then

$$\mathbb{P}_Z(A) = \pi_*(\mathbb{P}_{(X,Y)})$$

$$= \int \chi_A \circ \pi \, d\mathbb{P}_{(X,Y)}$$

$$= \int \chi_A(x+y) \, d\mathbb{P}_X(x) \, d\mathbb{P}_Y(y)$$

$$= \int \chi_A \, d(\mathbb{P}_X * \mathbb{P}_Y).$$

Hence $\mathbb{P}_Z = \mathbb{P}_X * \mathbb{P}_Y$.

$$\phi_Z(u) = \mathbb{E}\left[e^{i u \cdot (X+Y)}\right] = \mathbb{E}\left[e^{i(u \cdot X)} \cdot e^{i(u \cdot Y)}\right]$$

$$\underset{\text{ind.}}{=} \mathbb{E}\left[e^{iuX}\right] \cdot \mathbb{E}\left[e^{iuY}\right] = \phi_X(u) \cdot \phi_Y(u).$$

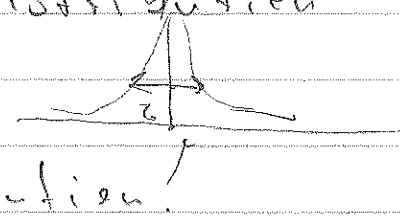
$\square$

§ 12. Gaussian random variables

$X : \Omega \to \mathbb{R}$ real-valued random var.

Then X is Gaussian with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$ if its distribution is given by

$$d\mathbb{P}_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$



Gaussian or normal distribution.

(55) We write $X \sim \mathcal{N}(\mu, \sigma^2)$.

X is <u>standard</u> Gaussian or normal if $X \sim \mathcal{N}(0,1)$, i.e.,

$$\mathbb{P}_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then

$$\mathbb{E}[X] = \mu, \quad \mathrm{Var}[X] = \sigma^2.$$

$\sigma = \mathrm{Var}[X]^{1/2}$ standard deviation.

characteristic function: if $X \sim \mathcal{N}(\mu, \sigma^2)$, then

$$\phi_X(u) = \exp\left(-\frac{1}{2}\sigma^2 u^2 + iu\mu\right).$$

If $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$, and X, Y are independent, then

$$Z = X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2).$$

<u>Proof:</u>

$$\phi_Z(u) = \phi_X(u) \cdot \phi_Y(u)$$
$$= \exp\left(-\frac{1}{2}(\sigma_1^2 + \sigma_2^2)u^2 + iu(\mu_1 + \mu_2)\right). \quad \square$$

It is convenient to consider a random var. X s.t. $X = \mu$ a.s. as a "generalized" Gaussian, where $\sigma^2 = 0$.

Namely,

$$\mathbb{P}_X = \delta_\mu, \quad \phi_X(u) = \exp(iu\mu)$$
$$= \exp\left(-\frac{1}{2} 0 \cdot u^2 + iu\mu\right).$$

56 <u>Def.</u> A random variable $X = (X_1, \dots, X_n) : \Omega \to \mathbb{R}^n$ is a (generalized, vector-valued) <u>Gaussian</u> if

$$\phi_X(u) = \exp\left(-\tfrac{1}{2} u^t C u + i(u \cdot \mu)\right) = \mathbb{E}\left[e^{i(u \cdot X)}\right] \quad \text{for } u \in \mathbb{R}^n,$$

where $\mu \in \mathbb{R}^n$ and C is a pos. semi-def. $n \times n$-matrix

$$\mathrm{Cov}(X, Y) := \mathbb{E}\left[(X - \mathbb{E}[X]) \cdot (Y - \mathbb{E}[Y])\right]$$

covariance of X, Y

Then C is the covariance matrix of X, i.e. $C = (c_{ij})$, where

$$c_{ij} = \mathrm{Cov}(X_i, X_j).$$

X Gaussian iff $X = BY$, where B $n \times n$-matrix, $Y = (Y_1, \dots, Y_n)$ s.t. $Y_1, \dots, Y_n$ real-valued ind. generalized Gaussians iff

$X = DZ + a$, where $a \in \mathbb{R}^n$, D: $n \times k$-matrix, $Z = (Z_1, \dots, Z_k)$, $Z_1, \dots, Z_k$ ind. Gaussians.

$Y = AX$, A $n \times k$ matrix.
$X: \Omega \to \mathbb{R}^k$, $Y: \Omega \to \mathbb{R}^n$.
If X Gaussian, then Y is Gaussian.

(57) <u>Proof:</u> $\phi_{\underline{Y}}(v) = \mathbb{E}[e^{i(v\cdot \underline{Y})}]$

$= \mathbb{E}[e^{i(v\cdot A\underline{X})}] = \mathbb{E}[e^{i(A^t v\cdot \underline{X})}]$

$\phi_{\underline{X}}(\underbrace{A^t v}_{u}) = \exp\left(-\frac{1}{2}(A^t v)^t C A^t v + i(A^t v\cdot \mu)\right)$

$= \exp\left(-\frac{1}{2} v^t (A C A^t) v + i(v\cdot A\mu)\right).$

So $\mu' = A\mu$, $C' = A C A^t$. $\square$

If $\underline{X}: \Omega \to \mathbb{R}^n$ has a <u>multi-normal distribution</u> given by

$$d\mathbb{P}_{\underline{X}}(x) = \frac{|A|^{1/2}}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}(x-\mu)^t A (x-\mu)\right),$$

where $\mu \in \mathbb{R}^n$, and A pos. def. $n\times n$-matrix,

then $\underline{X}$ is Gaussian and

$\phi_{\underline{X}}(u) = \exp\left(-\frac{1}{2} u^t C u + i(u\cdot\mu)\right),$

where $C = A^{-1}$.

§.13. Modes of convergence of random variables

$\underline{X}_n$, $n \in \mathbb{N}\cup\{\infty\}$, real (or vector valued random variables)

$X_n \to X_\infty$ a.s. (almost surely)

iff $\mathbb{P}(X_n \to X_\infty) =$

$\mathbb{P}(\{\omega \in \Omega : X_n(\omega) \to X(\omega)\}) = 1$

iff $X_n(\omega) \to X_\infty(\omega)$ for a.e. $\omega \in \Omega$.

$X_n \to X_\infty$ in probability

iff $\lim_{n \to \infty} \mathbb{P}(|X_n - X_\infty| \geq \varepsilon) = 0$

for all $\varepsilon > 0$.

(equiv. to "convergence in measure").

$X_n \to X_\infty$ in L^p, $p \geq 1$, iff

$\mathbb{E}[|X_n - X_\infty|^p] \to 0$

equiv. $\int_\Omega |X_n(\omega) - X_\infty(\omega)|^p \, d\mathbb{P}(\omega) \to 0$.

$$\begin{tikzcd}
X_n \to X_\infty \text{ a.s.} \arrow[dr, Rightarrow, "1"] & \\
& X_n \to X_\infty \text{ in prob.} \\
X_n \to X_\infty \text{ in } L^p \arrow[uu, Rightarrow, "\exists \text{ subseq.}"] \arrow[ur, Rightarrow, "2"'] &
\end{tikzcd}$$

Proof (easy) e.g. 1: Fix $\varepsilon > 0$

$E_n = \{\omega \in \Omega : |X_n(\omega) - X_\infty(\omega)| \geq \varepsilon\}$.

(59) $X_n \longrightarrow X_\infty$ a.s. implies

$$0 = \mathbb{P}(E_n \text{ i.o.}) = \mathbb{P}\Big(\bigcap_{r} \bigcup_{n \ge k} E_k\Big)$$

$$= \lim_{n\to\infty} \mathbb{P}\Big(\bigcup_{n\ge k} E_k\Big) \ge \limsup_{n\to\infty} \mathbb{P}(E_n),$$

<u>Lem. 8.14.</u> X_n $\mathbb{R}^d$-valued Gaussian random variables, $n \in \mathbb{N}$.

$X_n \longrightarrow X_\infty$ in probability.

Then X_∞ is $\mathbb{R}^d$-valued Gaussian.

<u>Proof</u> (outline): 1. If $X_n \longrightarrow X_\infty$ in prob., then

(*) $\phi_{X_n}(u) \longrightarrow \phi_{X_\infty}(u)$ loc. unif. on $\mathbb{R}^d$.

$$\left| e^{i(u\cdot X_n)} - e^{i(u\cdot X_\infty)} \right| \le |(u\cdot X_n) - (u\cdot X_\infty)|$$

$$\le |u| \cdot |X_n - X_\infty|$$

So $\left|\phi_{X_n}(u) - \phi_{X_\infty}(u)\right|$

$$= \mathbb{E}\left[e^{i(u\cdot X_n)} - e^{i(u\cdot X_\infty)} \right]$$

$$\le |u| \cdot \delta + 2\cdot \mathbb{P}(|X_n - X| \ge \delta) \le \varepsilon$$

for n large.

(*) follows.

2. X_n Gaussian, so

$$\phi_{X_n}(u) = \exp\left(-\tfrac{1}{2} u^t C_n u + i(u\cdot \mu_n)\right)$$

(60) If $\mu_n \in \mathbb{R}^d$ / $\phi_{X_n}(u) \longrightarrow \phi_X(u)$ loc. unif.,

then ϕ_X has the same form, i.e.,

$$\phi_X(u) = \exp\left(-\frac{1}{2}u^t C u + i(u,\mu)\right),$$

where $C \geq 0$, $\mu \in \mathbb{R}^d$. $\square$

<u>Lem. 8.15.</u> $X_1, \ldots, X_n$ real-valued random var. with joint Gaussian distribution (i.e., $X = (X_1, \ldots, X_n)$ is $\mathbb{R}^n$-valued Gaussian random var.)

Then $X_1, \ldots, X_n$ are independent iff they are pairwise uncorrelated, i.e.

$$\mathrm{Cov}(X_i, X_j) = 0 \quad \text{for } i,j = 1, \ldots, n,\ i \neq j.$$

<u>Proof:</u> "$\Rightarrow$" Clear:

$$\mathrm{Cov}(X_i, X_j) = \mathbb{E}[(X_i - \mathbb{E}[X_i])(X_j - \mathbb{E}[X_j])]$$

$$\overset{X_i, X_j \text{ ind.}}{=} \underbrace{\mathbb{E}[X_i - \mathbb{E}[X_i]]}_{0} \underbrace{\cdots}_{0} = 0$$

"$\Leftarrow$" Since X is Gaussian,

$$\phi_X(u) = \exp\left(-\frac{1}{2}u^t C u + i(u,\mu)\right), \quad u \in \mathbb{R}^n,$$

where $C = (c_{ij})$ is the covariance matrix; so $c_{ij} = \mathrm{Cov}(X_i, X_j)$, $i,j = 1, \ldots, n$.

(61) By assumption $c_{ij} = 0$ for $i \neq j$, and so C is a diagonal matrix.
Hence
$$\phi_{\underline{X}}(u) = \phi_{\underline{X}_1}(u_1) \cdots \phi_{\underline{X}_n}(u_n),$$
for $u = (u_1, \dots, u_n) \in \mathbb{R}^n$.
This shows that $\underline{X}_1, \dots, \underline{X}_n$ are independent

by Thm. 8.8. $\square$

§.16. Stochastic processes

A stochastic process in $\mathbb{R}^n$ is a collection $\{\underline{X}_t\}_{t \in T}$ of random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where $T \subseteq \mathbb{R}$ parameter set of "times".
$T = \mathbb{N}_0, \mathbb{N}$ "discrete time stochastic process".
Sequence of random variables:
$\underline{X}_0, \underline{X}_1, \dots$
$T = [0, \infty)$, $[a, b]$, etc.
"continuous time stochastic process".
$t \in T$ fixed: $\omega \longmapsto \underline{X}_t(\omega)$ random var. on Ω.
ω fixed: $t \in T \longmapsto \underline{X}(t, \omega) := \underline{X}_t(\omega)$
sample path of stochastic process.

62 Def. 8.17 (Brownian motion)

A real-valued stochastic process $\{B_t\}_{t \in [0,\infty)}$ is called a (versioned) Brownian motion if the following conditions are true:

(i) the process is a Gaussian process, i.e., for all $n \in \mathbb{N}$, $0 \le t_1 < \dots < t_n$ the random variables $B_{t_1}, \dots, B_{t_n}$ have a joint Gaussian distribution.

(ii) B_t for $t \in [0,\infty)$ is centered, i.e. $\mathbb{E}[B_t] = 0$

(iii) $Cov(B_s, B_t) = \mathbb{E}[B_s \cdot B_t] = s \wedge t$, $s,t \in [0,\infty)$,

(iv) sample paths $t \longmapsto B_t$ are continuous a.s., (i.e., $t \longmapsto B_t(\omega)$ is cont. for a.e. ω).

Rem. 8.18. $\{B_t\}_{t \in [0,\infty)}$ a Brownian motion.

1) $\mathbb{E}[B_t] = 0$, $Var(B_t) = Cov(B_t, B_t) = t$ for $t \ge 0$.

$B_t \sim \mathcal{N}(0, t)$, $t > 0$

$B_0 = 0$ a.s.

Brownian motion starts at 0 for time 0 a.s.

63) 2) Brownian motion has "independent increments":

If $t_1 < t_2 < \dots < t_n$, then

(*) $X_{t_1}, X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \dots, X_{t_n} - X_{t_{n-1}}$

are ind. Gaussian rand. varb.

$X_{t_k} - X_{t_{k-1}} \sim \mathcal{N}(0, t_k - t_{k-1})$

Indeed: r.v.'s in (*) joint Gaussian, centered, and

$k < l$. $\qquad t_{k-1} \le t_k \le t_{l-1} < t_l$

$\mathrm{Cov}(X_{t_k} - X_{t_{k-1}}, X_{t_l} - X_{t_{l-1}})$

$= \mathbb{E}[(X_{t_k} - X_{t_{k-1}})(X_{t_l} - X_{t_{l-1}})$

$= t_k \wedge t_l - t_{k-1} \wedge t_l - t_k \wedge t_{l-1} + t_{k-1} \wedge t_{l-1}$

$= t_k - t_{k-1} - t_k + t_{k-1} = 0.$

So by Lem. 8.15. the r.v.'s in (*) are ind.

8.19 Hilbert space bases

H separable real Hilbert space,

A sequence $\{x_n\}_{n \in \mathbb{N}}$ is called a complete orthonormal system or a Hilbert space basis

if i) the vectors $\{x_n\}_{n \in \mathbb{N}}$ are orthonormal, i.e., $(x_i \cdot x_j) = \delta_{ij}$ $i, j \in \mathbb{N}$,

(64) ii) if $x \in H$ and $(x, x_n) = 0$ for all $n \in \mathbb{N}_0$, then $x = 0$.

In this case,

$$x = \sum_{n=1}^{\infty} (x, x_n)\, x_n .$$

$$\|x\|^2 = \sum_{n=1}^{\infty} |(x, x_n)|^2$$

$$(x, y) = \sum_{n=1}^{\infty} (x, x_n) \cdot (y, y_n)$$

Parseval's Identities.

Equivalent to (ii) is

(ii'): The set S of all (finite) linear combinations of the vectors $x_1, x_2, x_3, \ldots$ is dense in H.

Ex. $H = L^2[0,1]$ $\quad (f, g) = \int_0^1 f(x) g(x)\, dx$

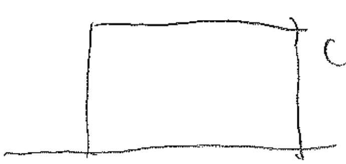
Hilbert space bases: trigonometric functions

$$1, \quad \frac{1}{\sqrt{2}} \cos(2\pi n x), \quad \frac{1}{\sqrt{2}} \sin(2\pi n x), \quad n \in \mathbb{N}$$

2. Haar basis $\quad n \in \mathbb{N}_0$

$$\varphi_{n,k}(x) := \begin{cases} 1 & \left[\frac{k}{2^{n}}, \frac{k+1/2}{2^{n+1}}\right) \\ -1 & \text{for } x \in \left[\frac{k+1/2}{2^{n}}, \frac{k+1}{2^{n}}\right) \\ 0 & \text{else} \end{cases} \quad k = 0, \ldots, 2^n - 1$$

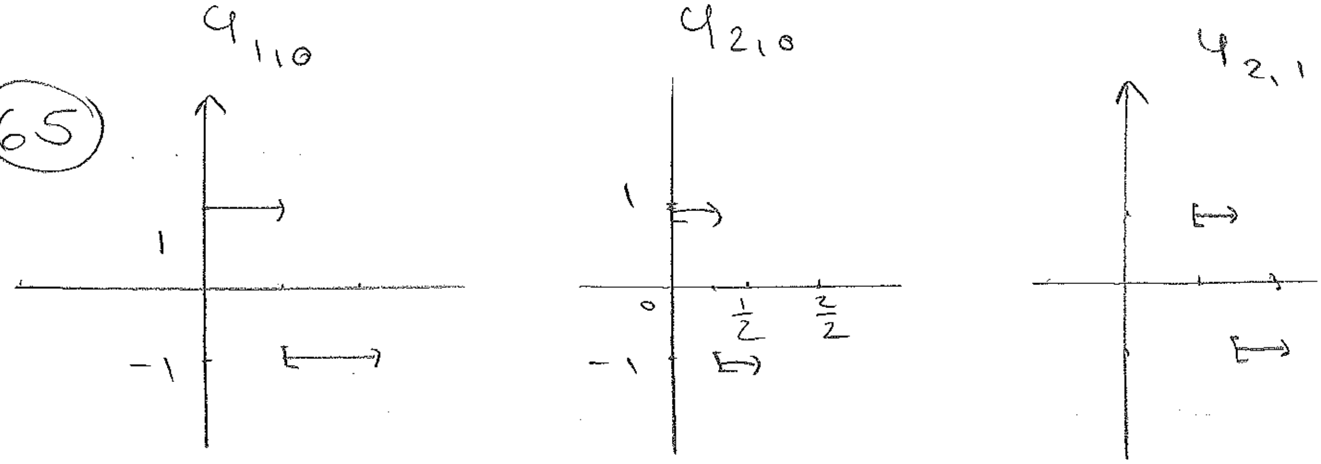
$\varphi_{1,0}$



$\varphi_{-1,0} \equiv 1$

I set of indices

(65)



Obviously, $\varphi_{n,k} \in L^2[0,1]$ pairwise orthogonal

$$\|\varphi_{n,k}\|^2 = \int_0^1 \varphi_{n,k}^2(x)\,dx = \frac{1}{2^n} \qquad n \in \mathbb{N}.$$

$$\psi_{n,k} = 2^{n/2}\varphi_{n,k} \qquad \psi_{-1,0} \equiv 1$$

$\{\psi_{n,k}\}_{(n,k)\in I}$ orthonormal system.

Linear comb. are dense in $L^2[0,1]$:
(because step functions on dyadic intervals are);

So $\{\psi_{n,k}\}_{(n,k)\in I}$ is a Hilbert space basis of $L^2[0,1]$.

If $\{x_n\}_{n\in\mathbb{N}}$ is an orthonormal system, then $\sum_{n=1}^{\infty} a_n x_n$ converges

iff $\sum_{n=1}^{\infty} a_n^2 < \infty$

Follows from Cauchy criterion:

$S_n = \sum_{k=1}^{n} a_k x_k$ partial sum

(66) $\|S_n - S_m\|^2 = \sum_{k=m+1}^{n} a_k^2, \qquad n \geq m.$

§20. Construction of Brownian motion

1. on $T = [0,1]$

Z_n, $n \in \mathbb{N}$, i.i.d. r.v.'s

independent, identically distributed r.v.'s on same prob. space

$(\Omega, \mathcal{A}, \mathbb{P}) \qquad Z_n \sim \mathcal{N}(0,1)$

$\tilde{\Omega} = (\mathbb{R}, \mathcal{B}, \mu)$

$\mathcal{B}$ Borel σ-alg. $\quad d\mu(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$

$\Omega = \tilde{\Omega}^{\mathbb{N}}$, $\quad Z_n$ = projection onto n-th coordinate

Z_n, $n \in \mathbb{N}$, orthonormal system in $L^2(\Omega)$:

$$\int_\Omega Z_n(\omega) \cdot Z_k(\omega)\, d\mathbb{P}(\omega) = \operatorname{cov}(Z_n Z_k) = \delta_{nk}, \quad n,k \in \mathbb{N}.$$

ψ_n, $n \in \mathbb{N}$, a Hilbert space basis of $L^2[0,1]$.

$$f_n(t) = \int_0^t \psi_n(t)\, dt = (\psi_n, \chi_{[0,t]})$$

inner product in $L^2[0,1]$.

$$\boxed{B_t := \sum_{n=1}^{\infty} f_n(t) Z_n \quad \text{for } t \in [0,1]}$$

(67) i) for each $t \in [0,1]$ the sum converges in $L^2(\Omega)$;

equiv.

$$\sum_{n=1}^{\infty} f_n(t)^2 = \sum_{n=1}^{\infty} (\Psi_n, \chi_{[0,t]})^2$$

$$\underset{\text{Parseval}}{=} \|\chi_{[0,t]}\|^2_{L^2[0,1]} = t < \infty.$$

ii) each B_t is a Gaussian; actually, for $t_1 < t_2 < \dots < t_m$, $B_{t_1}, B_{t_2}, \dots, B_{t_m}$ have a joint Gaussian distribution.

$$B^n_t := \sum_{k=1}^{n} f_k(t) Z_k \text{ is Gaussian}$$

(linear comb. of Gaussian).

$B^n_t \longrightarrow B_t$ as $n \longrightarrow \infty$ in $L^2(\Omega)$;

so B_t is Gaussian by Lem. 8.14.

Similarly, $(B^n_{t_1}, \dots, B^n_{t_m})$ have a joint Gaussian distribution, and

$$(B^n_{t_1}, \dots, B^n_{t_m}) \longrightarrow (B_{t_1}, \dots, B_{t_m})$$

as $n \longrightarrow \infty$ in $L^2(\Omega, \mathbb{R}^m)$;

so $(B_{t_1}, \dots, B_{t_m})$ have a joint Gaussian distribution.

68 (iii) B_t is centered:

$$\mathbb{E}[B_t] = \int_\Omega B_t(\omega)\, d\mathbb{P}(\omega)$$

$$= \lim_{n\to\infty} \int_\Omega B_t^n(\omega)\, d\mathbb{P}(\omega)$$

$= 0$, because Z_n, $n \in \mathbb{N}$, is centered.

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f_k(t)\, \mathbb{E}[Z_k] = 0.$$

(iv) $$\mathrm{Cov}(B_s, B_t) = \int_\Omega B_s(\omega) B_t(\omega)\, d\mathbb{P}(\omega)$$

$$= \lim_{n\to\infty} \int_\Omega B_s^n(\omega) B_t^n(\omega)\, d\mathbb{P}(\omega)$$

$$= \lim_{n\to\infty} \sum_{k,l=1}^{n} f_k(s) f_l(t) \underbrace{\mathrm{Cov}(Z_k, Z_l)}_{\delta_{kl}}$$

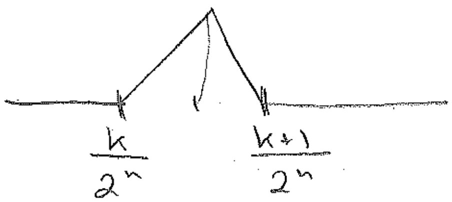
$$= \sum_{k=1}^{\infty} f_k(s) f_k(t)$$

$$= \sum_{k=1}^{\infty} (\varphi_k, \chi_{[0,s]})(\varphi_k, \chi_{[0,t]})$$

$$\overset{\text{Parseval}}{=} (\chi_{[0,s]}, \chi_{[0,t]}) = s \wedge t.$$

To check continuity of $t \longrightarrow B_t(\omega)$ for a.e. $\omega \in \Omega$, we chose the Haar basis for the Hilbert space basis of $L^2[0,1]$.

$$f_{n,k}(t) = \int_0^t \psi_{n,k}(x)\,dx \qquad \mathrm{Lip}(f_{n,k}) = 2^{n/2}.$$



$$\|f_{n,k}\|_\infty \leq \frac{1}{2}\cdot\frac{1}{2^n}\cdot 2^{n/2} = \frac{1}{2^{n/2+1}}.$$
$$\leq 2^{-n/2}$$

<u>Claim</u> $\{Z_{n,k}\}_{(n,k)\in I}$ i.i.d. standard normal r.v.'s.

Then for a.e. $\omega \in \Omega$, the series

$$B_t(\omega) = \underbrace{Z_{-1,0}(\omega)\, f_{-1,0}(t)}_{g_{-1}(t)} + \sum_{n=0}^{\infty} \underbrace{\sum_{k=0}^{2^n-1} Z_{n,k}(\omega)\, f_{n,k}(t)}_{g_n(t)} \quad (*)$$

ω fixed

converges uniformly in t (and hence represents a cont. function in t).

<u>Proof</u>: $\frac{2}{\sqrt{2\pi}} \int_a^\infty e^{-x^2/2}\,dx \leq e^{-a^2/2}$ for $a > 0$.

so $\mathbb{P}(|Z| > a) \leq e^{-a^2/2}$ for $a \geq 0$, if $Z \sim \mathcal{N}(0,1)$.

$$A_{n,k} = \{|Z_{n,k}| > 2\sqrt{\log(2^{n/2} n)}\}.$$

$$\mathbb{P}(A_{n,k}) \leq \exp(-2\log(2^{n/2} n)) = \frac{1}{2^n \cdot n^2}$$

(70) $$\sum_{n=1}^{\infty} \sum_{k=0}^{2^n-1} \mathbb{P}(A_{nk}) \le \sum_{n=1}^{\infty} \frac{2^n}{2^n \cdot n^2}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty .$$

So, by Borel - Cantelli I, we have

$\mathbb{P}(A_{nk} \text{ i.o.}) = 0$; i.e., for a.e. $\omega \in \Omega$ we have

$$|Z_{nk}(\omega)| \le 2 \cdot \sqrt{\log(2^{n/2} \cdot n)} \lesssim \sqrt{n}$$

(**) for all suff. large n (dep. on ω), say for $n \ge N(\omega)$.

For such ω:

$$\sum_{n=N(\omega)}^{\infty} \Big| \underbrace{\sum_{k=0}^{2^n-1} Z_{nk}(\omega) \cdot \|f_{nk}(t)\|_\infty}_{= g_n(t)} \Big|$$

$$\lesssim \sum_{n=N(\omega)}^{\infty} \frac{\sqrt{n}}{2^{n/2}} < \infty \text{ ; so series}$$

(*) represents a cont. function in t by Weierstrass - M-test.

Actually,

g_n is L_n- Lipschitz with $L_n \lesssim \sqrt{n} \cdot 2^{n/2}$

for all $n \ge N(\omega)$; so by adjusting constants wlog for all $n \ge 1$

(71) Moreover,

$$\|g_n\|_\infty \lesssim \sqrt{n} \cdot 2^{-n/2} \quad \text{for all } n \geq N(\omega),$$

w.l.o.g. for all $n \geq 1$.

Suppose ω is "good", i.e. that it satisfies (**).

Let $s,t \in [0,1]$. Pick suitable $N = N(s,t) \in \mathbb{N}$.

Then

$$|B_s(\omega) - B_t(\omega)|$$

$$\leq \sum_{n=-1}^{\infty} |g_n(s) - g_n(t)|$$

$$\leq \sum_{n=-1}^{N(\ldots)} L_n |s-t| + \sum_{n=N+1}^{\infty} 2 \cdot \|g_n\|_\infty$$

$$\underset{\omega}{\lesssim} 1 + \sum_{n=1}^{N} \sqrt{n}\, 2^{n/2} |s-t| + \sum_{n=N}^{\infty} \sqrt{n}\, 2^{-n/2}$$

$$\underset{\omega}{\lesssim} \sqrt{N} \cdot 2^{N/2} |s-t| + \sqrt{N} \cdot 2^{-N/2}$$

Pick $N = N(s,t)$ s.t.

$2^{N/2} |s-t| \approx 2^{-N/2}$ equiv. $|s-t| \sim 2^{-N}$

equiv. $N = \log_2 \frac{1}{|s-t|} \sim \log \frac{1}{|s-t|}$.

Then

$$|B_s(\omega) - B_t(\omega)| \lesssim |s-t|^{1/2} \sqrt{\log \frac{1}{|s-t|}}$$

72 **Conclusion** For a.e. ω there ex. $M(\omega) \geq 0$ s.t.

$$\left| B_s(\omega) - B_t(t) \right| \leq M(\omega) \, |s-t|^{1/2} \sqrt{\log \frac{1}{|s-t|}} .$$

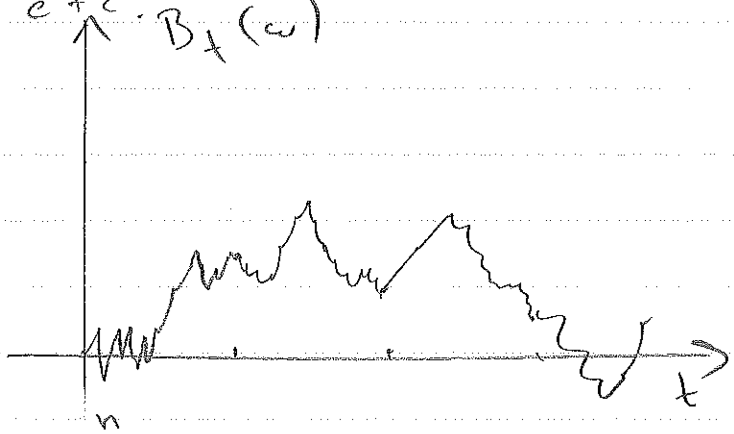
Almost surely, the sample path $t \longmapsto B_t(\omega)$ has modulus of continuity

$$\omega(\delta) = C \, \delta^{1/2} \sqrt{\log\left(\frac{1}{\delta}\right)} ;$$

So for every $\varepsilon > 0$, $t \longmapsto B_t(\omega)$ is Hölder $(1/2 - \varepsilon)$ almost surely.

2. Brownian motion on $[0,\infty)$.

Idea: Let a Brownian motion run until time 1, start a "new" Brownian at endpoint, let it run until time 2, etc.



B^n_t, $n \in \mathbb{N}_0$, independent copies of Brownian motion on $[0,1]$.

Define

$$B_t(\omega) = \sum_{k=0}^{\lfloor t \rfloor -1} B^k_1(\omega) + B^{\lfloor t \rfloor}_{t-\lfloor t \rfloor}(\omega)$$

$$B_{1.5}(\omega) = B^0_1(\omega) + B^1_{0.5}(\omega).$$

(73) Then $\{B_t\}_{t\in[0,\infty)}$ is a Gaussian process.

B_t centered, $s \leq t$

$$\mathrm{Cov}(B_s, B_t) = \mathbb{E}\left[\left(\sum_{k=0}^{\lfloor s\rfloor -1} B_1^k + B^{\lfloor s\rfloor}_{s-\lfloor s\rfloor}\right)\left(\sum_{l=0}^{\lfloor t\rfloor -1} B_1^l + B^{\lfloor t\rfloor}_{t-\lfloor t\rfloor}\right)\right]$$

$$= \sum_{k=0}^{\lfloor s\rfloor -1} 1 + (s-\lfloor s\rfloor) = s = s\wedge t,$$

For each $n\in\mathbb{N}_0$: $t \longmapsto B_t^n(\omega)$ on $[0,1]$ cont. a.s., so for a.e. ω: $t \longmapsto B_t^n(\omega)$ cont. for all $n\in\mathbb{N}_0$.

Hence $t \longmapsto B_t(\omega)$ cont. a.s.

§.21. π-systems

X set, $\mathcal{S}$ family of subsets of X.

$\mathcal{S}$ is a π-system if $A\cap B\in\mathcal{S}$ whenever $A, B\in\mathcal{S}$.
(i.e., a π-system is "stable" under finite intersection).

Facts 1) $\mathcal{S}$ π-system, $\mathcal{A} = \sigma(\mathcal{S})$ σ-algebra generated by $\mathcal{S}$, μ, ν prob. measures on $\mathcal{A}$.
If $\mu(A) = \nu(A)$ for all $A\in\mathcal{S}$,
then $\mu = \nu$ (i.e., $\mu(A) = \nu(A)$ for all $A\in\mathcal{A}$.)

(74) Exercise! $(\Omega, \mathcal{A}, \mathbb{P})$ prob. space

2) $\mathcal{S}, \mathcal{T}$ π-systems in $\mathcal{A}$, $\mathcal{B} = \sigma(\mathcal{S})$, $\mathcal{C} = \sigma(\mathcal{T})$, $\subseteq \mathcal{A}$.

If $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$
whenever $A \in \mathcal{S}$, $B \in \mathcal{T}$, then $\mathcal{B}$ and $\mathcal{C}$ are independent, i.e.
$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$
for all $A \in \mathcal{B}$, $B \in \mathcal{C}$.

Exercise!

8.22. The space $X = C([0,\infty))$.

$X := C([0,\infty)) = \{ f : [0,\infty) \to \mathbb{R} \text{ cont.} \}$
equipped with topology of loc. unif. convergence

$f_n \to f$ iff $f_n \to f$ loc. unif. on $\mathbb{R}$

This is a metrizable topology

$$d_n(f,g) = \sup_{x \in [0,n]} |f(x) - g(x)|$$

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \underbrace{\frac{d_n(f,g)}{1 + d_n(f,g)}}_{\leq 1}$$

d metric on X

$d(f_n, f) \to 0$ iff $f_n \to f$ loc. unif. on $[0,\infty)$.

(X, d) separable metric space.

(75) $\mathcal{B} = \mathcal{B}_{\mathbb{X}}$ Borel σ-algebra on $\mathbb{X}$
(= smallest σ-algebra containing all open sets in $\mathbb{X}$).

Want to find π-system $\mathcal{S}$ s.t. $\mathcal{B} = \sigma(\mathcal{S})$!

$t \in [0,\infty)$

$$\pi_t : \mathbb{X} \longrightarrow \mathbb{R} \quad \text{(evaluation at time } t.)$$
$$f \longmapsto f(t)$$

$$\mathcal{S} = \{ \pi_{t_1}^{-1}(B_1) \cap \dots \cap \pi_{t_k}^{-1}(B_k) : k \in \mathbb{N}, \ t_1 < \dots < t_k \text{ in } [0,\infty), \ B_1, \dots, B_k \in \mathcal{B}_{\mathbb{R}} \}.$$

Obviously, $\mathcal{S}$ is a π-system!

<u>Claim</u> $\sigma(\mathcal{S}) = \mathcal{B}_{\mathbb{X}}$

<u>Proof (Outline)</u>: 1. For $t \in [0,\infty)$ $\pi_t : \mathbb{X} \to \mathbb{R}$ is cont., so $\pi_t^{-1}(B) \in \mathcal{B}_{\mathbb{X}}$ for each $B \in \mathcal{B}_{\mathbb{R}}$ and $\mathcal{S} \subseteq \mathcal{B}_{\mathbb{X}}$.
Hence $\sigma(\mathcal{S}) \subseteq \mathcal{B}_{\mathbb{X}}$.

2. $\mathcal{B}_{\mathbb{X}} \subseteq \sigma(\mathcal{S})$.
Let $f_0 \in \mathbb{X}$ be sch.

$f \longmapsto |f(t) - f_0(t)|$ is $\sigma(\mathcal{S})$-measurable

$$f \longmapsto d_n(f, f_0) = \sup_{t \in [0,n] \cap \mathbb{Q}} |f(t) - f_0(t)| \text{ is } \sigma(\mathcal{S})\text{-measurable}$$

(76) $f \longmapsto d(f, f_0) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(f,f_0)}{1+d_n(f,f_0)}$

is $\sigma(\mathcal{G})$-measurable.

Open balls $B_d(f_0, \varepsilon) = \{ f : d(f,f_0) < \varepsilon \}$
are $\sigma(\mathcal{G})$-measurable.

Since every open set in $\mathbb{X}$ is a countable union of open balls, every open set is in $\sigma(\mathcal{G})$.

Hence $\mathcal{B}_{\mathbb{X}} \subseteq \sigma(\mathcal{G})$.

$(\Omega, \mathcal{A}, \mathbb{P})$ prob. space

$Z : \Omega \longrightarrow \mathbb{X}$ is measurable (w.r.t. $\mathcal{A}$ and $\mathcal{B}_{\mathbb{X}}$)

iff   <u>Claim</u>

$Z_t := \pi_t \circ Z$ is measurable for each $t \in [0, \infty)$.

$$\begin{tikzcd}
\Omega \arrow[r, "Z"] \arrow[dr, "Z_t"'] & \mathbb{X} \arrow[d, "\pi_t"] \\
& \mathbb{R}
\end{tikzcd}$$

<u>Proof:</u> "$\Longrightarrow$" If Z is measurable, then $Z_t := \pi_t \circ Z$ is measurable, because π_t is cont.

"$\Longleftarrow$": $\mathcal{C} = \{ A \subseteq \mathbb{X} : Z^{-1}(A) \in \mathcal{A} \}$.

$\mathcal{C}$ σ-algebra,

$B \subseteq \mathbb{R}$ Borel, $t \in [0, \infty)$:

$Z^{-1}(\pi_t^{-1}(B)) = (\pi_t \circ Z)^{-1}(B) = Z_t^{-1}(B) \in \mathcal{A}$,

so $\pi_t^{-1}(B) \in \mathcal{C}$. Hence $\mathcal{G} \subseteq \mathcal{C}$ and $\sigma(\mathcal{G}) = \mathcal{B}_{\mathbb{X}} \subseteq \mathcal{C}$. $\square$

77 **Thm. 8.23.** (Canonical Brownian motion)
$X = C([0,\infty))$, $\mathcal{B} = \mathcal{B}_X$ Borel σ-algebra on X. There exists a unique probability measure W on $(X, \mathcal{B})$, called Wiener measure with the following property:
if we define $B_t := \pi_t$ then $\{B_t\}_{t\in[0,\infty)}$ is a Brownian motion (on $\mathbb{R}$), more explicitly:

(i) for $t_1 < \dots < t_k < \infty$ the random var. $B_{t_1}, \dots, B_{t_k}$ have joint Gaussian distribution;
equiv. = let $F \subseteq [0,\infty)$ be finite, $\pi_F : X \longrightarrow \mathbb{R}^F = \{\varphi : F \longrightarrow \mathbb{R}\} \cong \mathbb{R}^{|F|}$,
$\mu_F = (\pi_F)_*(W)$ is a Gaussian measure on $\mathbb{R}^F$

$\mu_t := (\pi_t)_*(W)$

(ii) B_t is centered equiv.
$$\int_{\mathbb{R}} x \, d\mu_t(x) = 0 \quad \text{for each } t \in \mathbb{R}$$

(iii) $\mathrm{Cov}(B_s, B_t) = s \wedge t$ equiv.
$$\int_{\mathbb{R}^2} xy \, d\mu_{\{s,t\}}(x,y) = s \wedge t.$$

Proof: 1. Uniqueness: Suppose W, $\tilde{W}$ are two measures with the properties (i) – (iii).
Then $(\pi_F)_*(W) = \mu_F = \tilde{\mu}_F = (\pi_F)_*(\tilde{W})$, for each $F \subseteq [0,\infty)$ finite
because the Fourier transforms of $\mu_F, \tilde{\mu}_F$

(78) and hence $\mu_F, \tilde{\mu}_F$ themselves are uniquely determined by (i) – (iii).
This implies that for $t_1 < \dots < t_n$, $B_1, \dots, B_n \in \mathcal{B}_{\mathbb{R}}$ we have $F = \{t_1, \dots, t_n\}$

$$W(\pi_{t_1}^{-1}(B_1) \cap \dots \cap \pi_{t_k}^{-1}(B_k))$$
$$= W(\pi_F^{-1}(B_1 \times \dots \times B_k)) = \mu_F(B_1 \times \dots \times B_k)$$
$$= \tilde{W}(\pi_{t_1}^{-1}(B_1) \cap \dots \cap \pi_{t_k}^{-1}(B_n)),$$

i.e., $W(S) = \tilde{W}(S)$ for all $S \in \mathcal{S}$.
Since $\sigma(\mathcal{S}) = \mathcal{B}_{X}$, we have $W = \tilde{W}$.

2. Existence: There ex. Brownian motion $\{B_t\}_{t \in [0,\infty)}$ on some prob. space $(\Omega, \mathcal{A}, \mathbb{P})$.
By disregarding a set of measure 0, we may assume that $t \longmapsto B_t(\omega)$ is cont. for every $\omega \in \Omega$.
Define
$$B : \Omega \longrightarrow X = C([0,\infty)) \text{ by}$$
$$\omega \longmapsto (t \in [0,\infty) \longmapsto B_t(\omega))$$
Then for each $t \in [0,\infty)$ we have the commutative diagram:

$$\begin{tikzcd} \Omega \arrow[rr, "B"] \arrow[dr, "B_t"'] & & X \arrow[dl, "\pi_t"] \\ & \mathbb{R} & \end{tikzcd}$$

Since B_t is measurable for each $t \in [0,\infty)$, the map is measurable. Hence $W := B_*(\mathbb{P})$ is a Borel prob. measure on X, and if $F \subseteq [0,\infty)$ is finite, then

(79) $\mu_F = (\pi_F)_*(W) = (\pi_F)_*(B_*(\mathbb{P}))$

$= (\pi_F \circ B)(\mathbb{P}) = (B_F)_*(\mathbb{P})$,

where $B_F(\omega) := (B_{t_1}(\omega), \dots, B_{t_n}(\omega))$.

Hence $\{\pi_t\}_{t\in[0,\infty)}$ is a Brownian motion defined on X. $\square$

§.24 Brownian motion on $\mathbb{R}^n$

An $\mathbb{R}^n$-valued stochastic process $\{B_t\}_{t\in[0,\infty)}$ is called a (version of) <u>Brownian motion on $\mathbb{R}^n$</u> if the following conditions are true:

(i) the process is an $\mathbb{R}^n$-valued Gaussian process, i.e., for all $k \in \mathbb{N}$, $t_1 < \dots < t_k$, the $\mathbb{R}^{nk}$-valued random variable $(B_{t_1}, \dots, B_{t_k})$ has a Gaussian distribution

Let

$B_t = (B^1_t, \dots, B^n_t)$, where

B^i_t is real-valued.

(ii) Then

B^i_t is <u>centered</u> for $i \in \{1, \dots, n\}$,

$\mathbb{E}[B^i_t] = 0 \qquad t \in [0,\infty)$

(iii)

$\mathrm{Cov}(B^i_s, B^j_t) = \delta_{ij}\, s \wedge t, \quad i,j \in \{1,\dots,n\},$
$s,t \in [0,\infty).$

(iv) sample paths $t \longmapsto B_t$ are cont. a.s.

(80) Rem. 8.25 (i) If $B_t = (B_t^1, \dots, B_t^n)$ is a Brownian motion on $\mathbb{R}^n$, then $B_t^1, \dots, B_t^n$ are ind. Brownian motions on $\mathbb{R}$.

Conversely, if $B_t^1, \dots, B_t^n$ are ind. Brownian motions on $\mathbb{R}$, then $B_t = (B_t^1, \dots, B_t^n)$ is a Brownian motion on $\mathbb{R}^n$.

(This proves existence!)

(ii) Uniqueness: One can show (as in Thm. 8.23) that there is a unique Wiener measure W on $X = C([0,\infty), \mathbb{R}^n)$ $= \{ f : [0,\infty) \longrightarrow \mathbb{R}^n \text{ cont.} \}$ s.t. $\{\pi_t\}_{t \in [0,\infty)}$ is a Brownian motion, where $\pi_t : X \longrightarrow \mathbb{R}^n$, $f \longmapsto f(t)$.

Described by "marginals" on $\mathbb{R}^{|F| \times n}$

$\mu_F := (\pi_F)_*(W)$ $\quad \pi_F : X \longrightarrow (\mathbb{R}^n)^F$

$F \subseteq [0,\infty)$ finite. $\quad f \longmapsto f|F$.

(ii) $B_t = (B_t^1, \dots, B_t^n)$, $t \in [0,\infty)$ is an $\mathbb{R}^n$-valued Brownian motion iff

$W_t := \lambda_1 B_t^1 + \dots + \lambda_n B_t^n$ is a 1-dim. Brownian motion for each unit vector $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$:

$\Rightarrow$ Clear. $\Leftarrow$: $\mathrm{Cov}(W_s, W_t)$

$= \lambda_1^2 \, s \wedge t + \dots + \lambda_n^2 \, s \wedge t = s \wedge t$

(81) $\stackrel{n}{\leftarrow}$ " Need fact:

$Z_1, \dots, Z_n$ $\mathbb{R}^k$-valued r.v's.

Then they have a joint Gaussian distribution iff

$\lambda_1 Z_1 + \dots + \lambda_n Z_n$ is Gaussian for all $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

Details left as exercise!

§.26. Basic properties of Brownian motion

Let $\{B_t\}_{t\in[0,\infty)}$ be a Brownian motion in $\mathbb{R}^n$. Then the following processes are also Brownian motions:

(i) $W_t := B_{t+s} - B_s$, $s \in [0,\infty)$ fixed.

(Markov property)

Brownian motion is memoryless!

(ii) if A is an orthogonal transformation, then

$$W_t := A B_t$$

(iii) $W_t = \frac{1}{a} B_{a^2 t}$, $a > 0$ fixed,

(Brownian scaling)

(iv) $W_t = \begin{cases} B_0 & t = 0, \\ t B_{1/t} & t > 0 \end{cases}$

(time inversion.)

(82) Proof: All processes W_t in (i)–(iv) are Gaussian, and W_t is centered.

One checks covariance: for example in (iii) and (iv):

$$\mathrm{Cov}(W_s^k, W_t^l) = \mathrm{Cov}\left(\frac{1}{a} B^k_{a^2 s}, \frac{1}{a} B^l_{a^2 t}\right)$$

$$= \frac{1}{a^2} \delta_{kl} (a^2 s \wedge a^2 t) = \delta_{kl} (s \wedge t)$$

$$\mathrm{Cov}(W_s^k, W_t^l) = st\, \mathrm{Cov}(B^k_{1/s}, B^l_{1/t})$$

$s, t > 0$

$$= st\, \delta_{kl} \frac{1}{s} \wedge \frac{1}{t}$$

$$= \delta_{ks}\, t \wedge s \quad \checkmark$$

Almost sure continuity of sample paths clear for: (i) – (iii), and on $(0, \infty)$ in (iv):

(up to measure 0)

Continuity of W_t at 0 is the following event: $A =$

$$\bigcap_{\substack{\varepsilon > 0 \\ \varepsilon \in \mathbb{Q}}} \bigcup_{\substack{\delta > 0 \\ \delta \in \mathbb{Q}}} \bigcap_{\substack{t \in (0, \delta) \\ t \in \mathbb{Q}}} \left\{ \omega : |W_t(\omega) - W_0(\omega)| < \varepsilon \right\}$$

If we replace W_t by B_t then this is an almost sure event; since W_t and B_t have the same marginals, it follows that A is almost sure!

(Note: this shows that

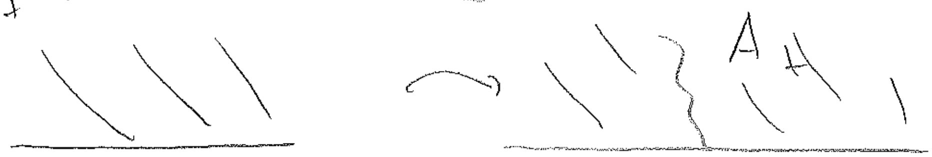
$$\lim_{s \to \infty} \frac{|B_s|}{s} = 0 \quad \text{a.s.})$$

(83) 8.27. The stochastic Loewner equation (SLE)

Chordal Loewner equation:

$\{f_t\}_{t\in[0,\infty)}$ normalized chordal Loewner chain

$$f_t(z) = z - \frac{2t}{z} + \dots \quad \text{near } \infty$$



Loewner-Kufarev eq.

$$\frac{\partial f}{\partial t}(z,t) = V(z,t)\,\frac{\partial f}{\partial z}(z,t)$$

$$V(z,t) = 2\int_{\mathbb{R}} \frac{d\nu_t(u)}{u-z},$$

ν_t prob. measure, $\operatorname{supp}(\nu_t) \subset \mathbb{R}$.

One obtains SLE_κ, $\kappa \geq 0$, if one take a probabilistic driving function

$$\nu_t = \delta_{\sqrt{\kappa} B_t(\omega)}, \quad \text{where } B_t \text{ is 1-dim. Brownian motion.}$$

Then

$$V(z,t) = \frac{2}{\sqrt{\kappa} B_t(\omega) - z}.$$

Depending on ω, one gets a "random"

84 Loewner chains and corr. random hulls $A_t(\omega)$

One is interested in these hulls, because they can be used to study many conformally invariant processes in the plane.

Problems: 1) What are the characterizing properties of SLE?
(i.i.d. increments, Markov (= memoryless) property, conformal invariance)
2) What are the techniques to study SLE?
(Martingale methods, etc.)

9. Survey of martingale theory

9.1. Conditional expectation

Ex. Random exp. in two stages: $\in \{1,\dots,6\}$
roll two dice with outcomes X_1, X_2.
$Z = X_1 + X_2$. $\Omega = \{1,\dots,6\} \times \{1,\dots,6\}$
$\mathbb{E}[Z] = 7$.
Suppose outcome of X_1 is known (partial inform.)
Then we have to adjust
$E[Z]$ depending on X_1

$$E[Z \mid X_1 = x] = x + 3.5 = X_1(\omega) + 3.5.$$

New random vari. $\omega \longmapsto \mathbb{E}[Z \mid X]$.

(85) **Thm. and Def. 9.2.** (Conditional expectation)

$(\Omega, \mathcal{A}, \mathbb{P})$ prob. space, X r.v. with $\mathbb{E}[|X|] < \infty$, $\mathcal{B} \subseteq \mathcal{A}$ σ-algebra. Then there ex. a r.v. Y on Ω s.t.

(i) Y is $\mathcal{B}$-measurable,

(ii) $\mathbb{E}[|Y|] < \infty$,

(iii) for every $B \in \mathcal{B}$ we have

$$\mathbb{E}[Y; B] = \int_B Y(\omega)\, d\mathbb{P}(\omega)$$

$$= \int_B X(\omega)\, d\mathbb{P}(\omega) = \mathbb{E}[X; B].$$

Y is essentially uniquely determined: if $\tilde{Y}$ is another r.v. with prop. (i)–(iii), then $Y = \tilde{Y}$ a.s.

The r.v. Y is called (a version of) conditional expectation of X given $\mathcal{B}$, denoted $\mathbb{E}[X | \mathcal{B}]$.

Idea of proof: Wlog $X \geq 0$.

Define

$$\mu(B) := \int_B X(\omega)\, d\mathbb{P}(\omega) \quad \text{for } B \in \mathcal{B}.$$

Then $\mu \ll \mathbb{P}|\mathcal{B}$, so μ has a Radon-Nikodym derivative Y w.r.t. $\mathbb{P}|\mathcal{B}$.

Then (i) – (iii) are evident.

Uniqueness also clear.

$$\mathbb{E}[X | Z_1, Z_2, \ldots, Z_n] := \mathbb{E}[X | \sigma(Z_1, \ldots, Z_n)]$$

9.3. Properties of conditional expectation

$(\Omega, \mathcal{A}, \mathbb{P})$ prob. space, all r.v.'s X satisfy $\mathbb{E}[|X|] < \infty$,

$\mathcal{B} \subseteq \mathcal{A}$ σ-algebra.

(i) If $Y = \mathbb{E}[X \mid \mathcal{B}]$, then $\mathbb{E}[Y] = \mathbb{E}[X]$.

(ii) If X is $\mathcal{B}$-measurable, then
$\mathbb{E}[X \mid \mathcal{B}] = X$ (a.s.) (iii) linearity

(iv) if $X \geq 0$, then $\mathbb{E}[X \mid \mathcal{B}] \geq 0$

(v) $X_n \geq 0$, $X_n \nearrow X$ (monotone a.s. convergence)
Then
$\mathbb{E}[X_n \mid \mathcal{B}] \nearrow \mathbb{E}[X \mid \mathcal{B}]$ a.s.

(Dominated conv.) + Fatou's lemma
$|X_n(\omega)| \leq V(\omega)$, $\mathbb{E}[V] < \infty$,
$X_n \longrightarrow X$ a.s. Then
$\mathbb{E}[X_n \mid \mathcal{B}] \longrightarrow \mathbb{E}[X \mid \mathcal{B}]$ a.s.

(vi) (Jenson) $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ convex,
$\mathbb{E}[|\varphi(X)|] < \infty$, then

$$\varphi(\mathbb{E}[X \mid \mathcal{B}]) \leq \mathbb{E}[\varphi(X) \mid \mathcal{B}] \quad \text{a.s.}$$

In particular: $|\mathbb{E}[X \mid \mathcal{B}]| \leq \mathbb{E}[|X| \mid \mathcal{B}]$
and

$$\|\mathbb{E}[X \mid \mathcal{B}]\|_p \leq \|X\|_p, \quad p \geq 1.$$

(vii) (Tower Property)
$\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$ σ-algebras. Then

$$\underbrace{\mathbb{E}\big[\mathbb{E}[X \mid \mathcal{B}] \mid \mathcal{C}\big]}_{\mathbb{E}[X \mid \mathcal{B} \mid \mathcal{C}]} = \mathbb{E}[X \mid \mathcal{C}].$$

(87) (ix) If Z is $\mathcal{B}$-measurable, then

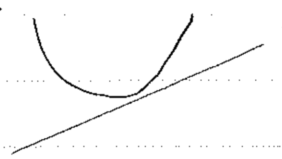
$$\mathbb{E}[ZX \mid \mathcal{B}] = Z\,\mathbb{E}[X \mid \mathcal{B}]$$

(x) If X and $\mathcal{B}$ are ind., then

$$\mathbb{E}[X \mid \mathcal{B}] = \underbrace{\mathbb{E}[X]}_{\text{const. funct.}} \quad \text{a.s.}$$

Proof: Mostly straightforward from the definitions:

(vii) Jensen:



φ convex

$$\varphi(x) = \sup_{L \le \varphi \text{ affine}} L(x)$$

$$L \le \varphi : \quad L(X) = aX + b \le \varphi(X)$$

so

$$L(\mathbb{E}[X \mid \mathcal{B}]) \underset{\text{linearity}}{\le} \mathbb{E}[L(X) \mid \mathcal{B}] \underset{\text{monotonicity}}{\le} \mathbb{E}[\varphi(X) \mid \mathcal{B}]$$

Taking sup over all L gives

$$\varphi(\mathbb{E}[X \mid \mathcal{B}]) \le \mathbb{E}[\varphi(X) \mid \mathcal{B}]$$

Incorrect proof, because we take a sup over an uncountable family; can be corrected if we write

$$\varphi = \sup_{L_n \le \varphi} L_n, \quad \text{for a countable collection } L_n,\ n \in \mathbb{N}.$$

(vi) For dominated convergence we need Fatou Lemma:

$0 \le X_n$. Then:

$$\mathbb{E}[\liminf_{n \to \infty} X_n \mid \mathcal{B}] \le \liminf_{n \to \infty} \mathbb{E}[X_n \mid \mathcal{B}]$$

(88) Ex. 9.4. $(\Omega, \mathcal{A}, \mathbb{P})$ prob. space, $\{A_n\}_{n\in\mathbb{N}}$ countable partition of Ω with $A_n \in \mathcal{A}$, $\mathbb{P}(A_n) > 0$.

$\mathcal{B} = \sigma(\{A_n\}_{n\in\mathbb{N}}) = \{\bigcup_{S\subseteq\mathbb{N}} A_n : S \text{ countable}\}$.

Then

$$\mathbb{E}[X \mid \mathcal{B}] = \sum_{n\in\mathbb{N}} \frac{1}{\mathbb{P}(A_n)} \int_{A_n} X(\omega)\, d\mathbb{P}(\omega) \cdot \chi_{A_n}(\omega)$$

$$= \sum_{n\in\mathbb{N}} \mathbb{E}[X; A_n] \cdot \chi_{A_n}$$

Check definition!

Ex. 9.5. (Fair games and martingales)

Players I + II role dice.

Zero-sum game, at each step PI wins or looses 1 unit.

X_n winnings of P_1 after n rolls.

($-X_n$ " " P_2 after n rolls.)

Game 1: P_1 wins if roll $\in \{1,2\}$ not fair!

(so looses if $\in \{3,4,5,6\}$)

G_2: P_1 wins if roll even fair!

G_3: P_1 wins if roll even; fair!

if one of the players has won ≥ 100 units, then game biased against player as in G_1

G_3 is a fair game ($\mathbb{E}[X_n] = 0$ for all $n\in\mathbb{N}$), but not fair at all times (or all situations).

(89) How to model a game that is "fair at all times":

$\mathbb{E}[X_{n+1} - X_n] = 0$ (true if $\mathbb{E}[X_{n+1}] = \mathbb{E}[X] = 0$)

Better: $\mathbb{E}[X_{n+1} - X_n \mid X_n = x] = 0$ whatever x.

$\mathcal{F}_n$ σ-algebra of events that will be known by time n.

$\mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n] = 0$ equiv.

$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$ $\quad$ ($\mathbb{E}[X_n \mid \mathcal{F}_n] = X_n$)

Def. 9.6. (Martingales; discrete-time case).

$(\Omega, \mathcal{A}, \mathbb{P})$ prob. space with filtration given σ-algebras $\mathcal{F}_n \subseteq \mathcal{A}$ for $n \in \mathbb{N}_0$ ($\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for $n \in \mathbb{N}_0$)

$X = \{X_n\}_{n \in \mathbb{N}_0}$ sequence of r.v's on Ω.

Then X is called a martingale if

(i) X_n is $\mathcal{F}_n$-measurable for $n \in \mathbb{N}_0$, and $X_n \in L^1$, i.e., $\mathbb{E}[|X_n|] < \infty$.

(ii) $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$ (a.s.) for $n \in \mathbb{N}_0$.

If in (ii) we have $\leq$ or $\geq$, then X is called a super-martingale or submartingale, resp.

submartingale: tendency to increase

supermartingale: tendency to decrease

(90) Often: $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$, $n \in \mathbb{N}$.
"natural filtration"

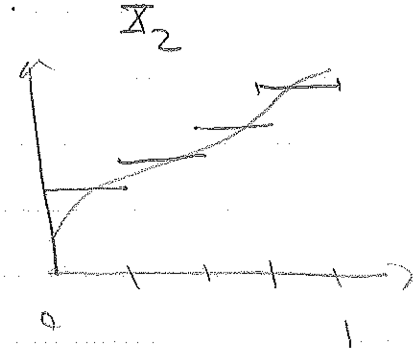
Ex. 9.7 a) Games as in 9.5. with natural filtration, $X = \{X_n\}_{n \in \mathbb{N}}$.
G_1, G_3 not martingales, G_2, G_4 martingales
G_1 supermartingale

b) (Dyadic martingale)
$\Omega = [0,1]$ with Lebesgue measure,
$f \in L^1[0,1]$.

$$D_{n,k} = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right], \quad n \in \mathbb{N}_0,\ k = 1, \dots, 2^n - 1$$

dyadic interval

$\mathcal{F}_n$ σ-algebra generated by dyadic intervals at level $\leq n$.



$$X_n(\omega) = \sum_{k=1}^{2^n-1} \chi_{D_{n,k}}(\omega) \fint_{D_{n,k}} f$$

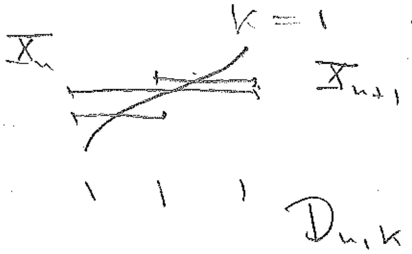
$n \in \mathbb{N}$.

$X = \{X_n\}_{n \in \mathbb{N}_0}$ is martingale.

(i) X_n is $\mathcal{F}_n$-measurable,

(ii) $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n]$

$$= \sum_{k=1}^{2^n-1} \chi_{D_{n,k}}(\omega) \underbrace{\fint_{D_{n,k}} X_{n+1}}_{\fint_{D_{n,k}} f} = X_n.$$



Note: $X_n(\omega) \longrightarrow f(\omega)$ for a.e. ω.
Instance of martingale conv. thm!

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c) (Brownian motion)

$0 \le t_0 < t_1 < \dots$ $\quad B_t$ Brownian motion on $\mathbb{R}$.

$X_n = B_{t_n}$ for $n \in \mathbb{N}_0$.

Then $X = \{X_n\}_{n\in\mathbb{N}_0}$ (with natural filtration)

is a martingale.

$B_{t_{n+1}} - B_{t_n}$ ind. of $B_{t_0}, \dots, B_{t_n}$

$\mathbb{E}[B_t] = 0$.

(i) $X_n = B_{t_n}$ is $\mathcal{F}_n = \sigma(B_{t_0}, \dots, B_{t_n})$ measurable

(ii) $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[B_{t_{n+1}} \mid \mathcal{F}_n]$

$= \mathbb{E}[B_{t_{n+1}} - B_{t_n} \mid \mathcal{F}_n] + B_{t_n}$

$= \mathbb{E}[B_{t_{n+1}} - B_{t_n}] + B_{t_n}$

$= B_{t_n} = X_n$.

Def. 9.8. (Martingales; continuous-time case).
$(\Omega, \mathcal{A}, \mathbb{P})$ prob. space
$\{\mathcal{F}_t\}_{t\ge 0}$ filtration (i.e., $\mathcal{F}_t \subseteq \mathcal{A}$ σ-algebra and $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \le t$)
A stochastic process is called adapted if X_t is $\mathcal{F}_t$-measurable (often extra technical conditions)

(92) for all $t \geq 0$.

$X = \{X_t\}_{t \geq 0}$ is a martingale if

(i) X is adapted and $\mathbb{E}[|X_t|] < \infty$ for $t \geq 0$,

(iii) $\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s$ for all $0 \leq s \leq t$.

Natural filtration $\mathcal{F}_t = \sigma(X_s : 0 \leq s \leq t)$.

Ex 9.9 a) $\{B_t\}$ Brownian motion (with natural filtration is martingale.

b) B_t Brownian motion $\mathcal{F}_t = \sigma(B_s : 0 \leq s \leq t)$.

$X_t = B_t^2 - t$ is a martingale.

(i) $\{X_t\}$ is adapted, and $\mathbb{E}[|X_t|] < \infty$.

(ii) $s \leq t$

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = \mathbb{E}[B_t^2 - t \mid \mathcal{F}_s]$$

$$= \mathbb{E}[(B_t - B_s + B_s)^2 \mid \mathcal{F}_s] - t$$

$$= \mathbb{E}[(B_t - B_s)^2 + 2B_s(B_t - B_s) + B_s^2 \mid \mathcal{F}_s] - t$$

$$= \mathbb{E}[(B_t - B_s)^2 \mid \mathcal{F}_s] + 2B_s\,\mathbb{E}[B_t - B_s \mid \mathcal{F}_s] + B_s^2 - t = \mathbb{E}[(B_t - B_s)^2] + 2B_s\,\mathbb{E}[B_t - B_s] + B_s^2 - t$$

$$= (t-s) + B_s^2 - t = B_s^2 - s = X_s.$$

Thus

(93) Converse:

<u>Thm.</u> (Lévy) $\{X_t\}_{t\ge 0}$ continuous martingale (i.e., martingale with almost surely cont. sample paths).

If $\{X_t^2 - t\}_{t\ge 0}$ is a martingale (w.r.t. $\mathcal{F}_t = \sigma(X_s, s\le t)$),

then $\{X_t\}_{t\ge 0}$ is a Brownian motion.

Important facts about martingales:

martingale convergence thm.,

Doob's L^p-submartingale inequalities,

sub- and supermartingale decompositions,

optional stopping, stochastic integrals.