MS5127

(1) Conformally Invariant Processes in the Plane, II

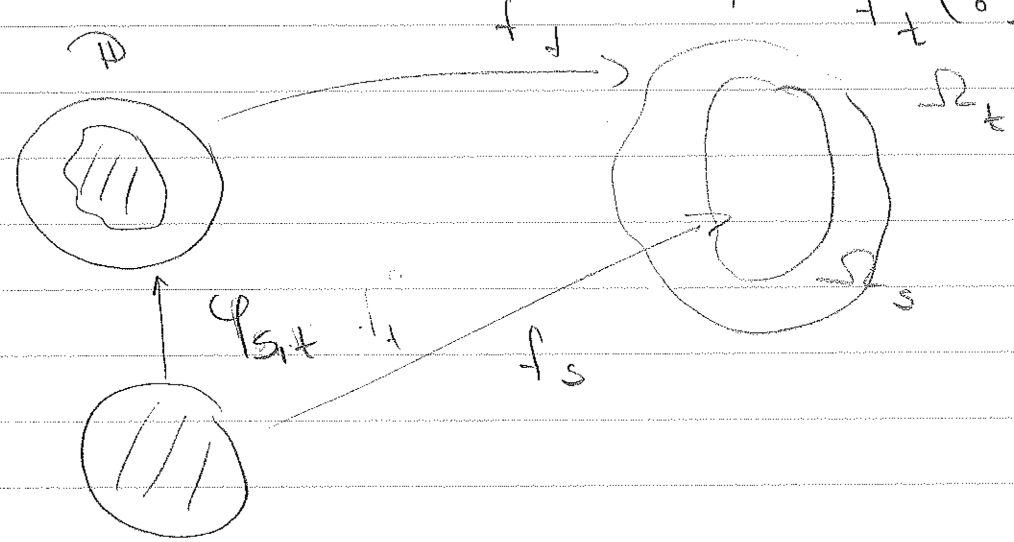
UCLA, Winter 2013

Review from Fall 2012

$\{\Omega_t\}$ geometric Loewner chain

$\{f_t\}_{t\in I}$ normalized analytic Loewner chain

$f_t : \mathbb{D} \longleftrightarrow \Omega_t$ conf. h.p., $f_t(0)=0$, $f_t'(0)=e^t$,



$(z,t) \longrightarrow f(z,t) := f(z,t)$

holo. in z, Lip. in t

(actually unif. Lip. on comp. subset of $\mathbb{D}\times I$).

$\dot{f}_t(z) = -V(z,t)\, f_t'(z)$ Loewner-Kufarev Eq.

V Herglotz vector field

(2) $V(z,t)$ holo. in z, measurable in t

$$V(z,t) = -z\, p(z,t), \text{ where}$$

$p(\cdot,t) \in \mathcal{P}$, i.e., $p(\cdot,t)$ holo.,

$$p(0,t) = 1, \ \operatorname{Re} p(\cdot,t) \geq 0.$$

<u>The Loewner triangle</u>

Loewner-Kufarev eq. → Herglotz vector fields

Loewner chain ← unique

$\{f_t\}$ norm. Loewner chain

Herglotz vector fields $V(z,t)$ → unique sol. of flow eq. → Semigroup $\varphi_{s,t}$

Semigroup $\varphi_{s,t}$ → Herglotz vector fields: $V(z,t) = \lim\limits_{\varepsilon \to 0} \dfrac{\varphi_{t,t+\varepsilon}(z) - z}{\varepsilon}$

Loewner chain → Semigroup: $\varphi_{s,t} = f_t^{-1} \circ f_s$

Semigroup → Loewner chain: $f_s = \lim\limits_{t\to\infty} e^t \varphi_{t,s}$

flow eq. $\dot{w}(t) = V(w(t),t)$

Special case for slit domains

$$\Omega_t = \mathbb{C} \setminus \gamma([t,\infty)) \qquad \gamma : [a,\infty] \to \hat{\mathbb{C}} \text{ simple path}$$

$$\dot{f}_t(z) = -V(z,t)\, f_t'(z),$$

where

$$V(z,t) = -z \frac{\lambda(t)+z}{\lambda(t)-z}, \qquad \lambda \text{ [illegible].}$$

$\lambda : [a,\infty) \longrightarrow \partial\mathbb{D}$ cont.

Note: for every Herglotz vector field:

$$V(z,t) = -z \int_{\partial\mathbb{D}} \frac{\xi+z}{\xi-z}\, d\mu_t(\xi)$$

μ_t prob. measure on $\partial\mathbb{D}$.

(3) 7. The radial and chordal versions of the Loewner - Kufarev equation.

7.1. Radial Loewner chains

(disk version of Loewner chains)

$I = [0, b]$ $\quad b \in (0, \infty]$.

$\{\Omega_t\}_{t \in I}$ (geometric) radial Loewner chain if

i) $\Omega_t \subseteq \mathbb{D}$ simply connected region with $0 \in \Omega_t$ for $t \in I$,

ii) $\Omega_0 = \mathbb{D}$

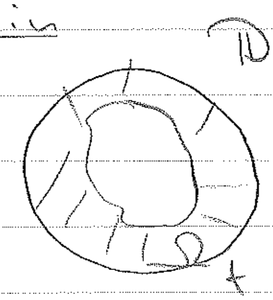
iii) $\Omega_s \supsetneq \Omega_t$ for $s < t$, $s, t \in I$

iv) $\{\Omega_t\}$ is cont. in t w.r.t. kernel conv. w.r.t. $w_0 = 0$.

$f_t : \mathbb{D} \leftrightarrow \Omega_t$ unique conf. map with $f_t(0) = 0$, $f_t'(0) > 0$.

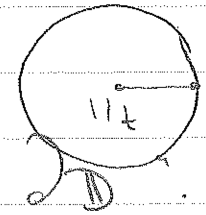
$\{f_t\}_{t \in I}$ is the corr. (analytic) radial Loewner chain

It is normalized if $f_t'(0) = e^{-t}$ for $t \in I$.



Simplest situation:

$\Omega_t = \mathbb{D} \setminus [1/t, 1]$



"a radius grows out of $\partial\mathbb{D}$ towards 0"

(4) Study of radial Loewner chain can be reduced to whole plane version.

If $\{\Omega_t\}_{t\in[0,b]}$ is a radial Loewner chain define

$$\tilde{\Omega}_t := \begin{cases} \Omega_{-t} & \text{for } t\in[-b,0], \\ e^t\mathbb{D} & t \geq 0 \end{cases}$$

(Continuity clear; also at $t=0$).

Then $\{\tilde{\Omega}_t\}_{t\in[-b,\infty)}$ is a (whole plane) Loewner chain.

If then $\{\Omega_t\}$ is normalized (i.e., the corr. analytic Loewner chain is), then $\{\tilde{\Omega}_t\}$ is normalized.

$\{\Omega_t\}$ can be obtained from $\{\tilde{\Omega}_t\}$ by "time reversal" and restriction of time interval;

So the regularity theory for whole plane Loewner chains remains valid in radial case; in particular, if $\{f_t\}_{t\in[0,b]}$ is a normalized radial Loewner chain, then

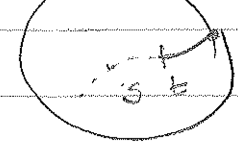
$\dot{f}_t(z) = V(z,t)\cdot f'_t(z)$ for a.e. $t\in I$, all $z\in\mathbb{D}$,

where V is a Herglotz vector field. (radial Loewner-Kufarev equation)

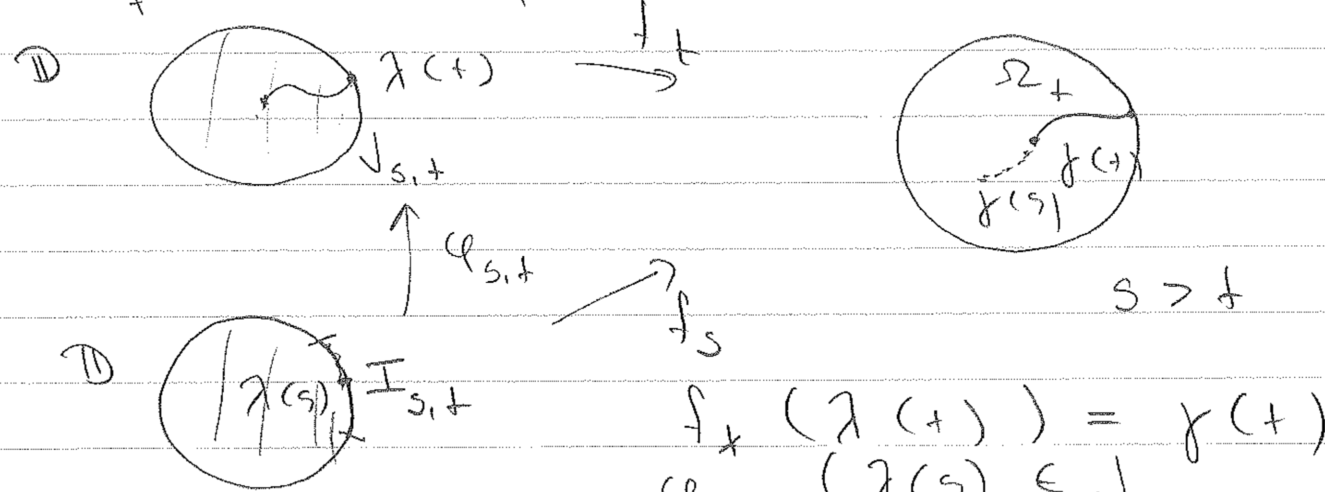
Note sign change in comp. to Loewner-Kuf. Eq. due to time reversal!

⑤ 7.2. Radial Loewner chains $\boxed{\text{generated by slits}}$

$\gamma : [0,b] \to \mathbb{C}$ simple path,
$\gamma(0) = 1$, $\gamma(t) \in \mathbb{D}$, $t \in (0,b]$,
$0 \notin \gamma[0,b]$



$\Omega_t := \mathbb{D} \setminus \gamma([0,t]) \subseteq \mathbb{D}$
simply connected region
$0 \in \Omega_t$ $\quad \Omega_0 = \mathbb{D}$ $\quad$ & $\quad \Omega_t \supseteq \Omega_s$, $t < s$.
$\{\Omega_t\}_{t \in [0,b]}$ geometric radial & Loewner chain.
Can assume that corr. analytic radial Loewner chain $\{f_t\}$ is normalized.
$f_t(0) = 0$, $f_t'(0) = e^{-t}$.



$f_t(\lambda(t)) = \gamma(t)$
$\varphi_{s,t}(\lambda(s)) \in J_{s,t}$

Lem. 6.2. : $J_{s,t}$, $I_{s,t}$ unif. small
if $|s-t|$ small

Lem 6.4: $\varphi_{s,t}$ unif. close to $id_{\mathbb{D}}$
if $|s-t|$ small

Cor. 6.5: $|\lambda(s) - \lambda(t)|$ unif. small
if $|s-t|$ small $\Rightarrow$ λ is cont.

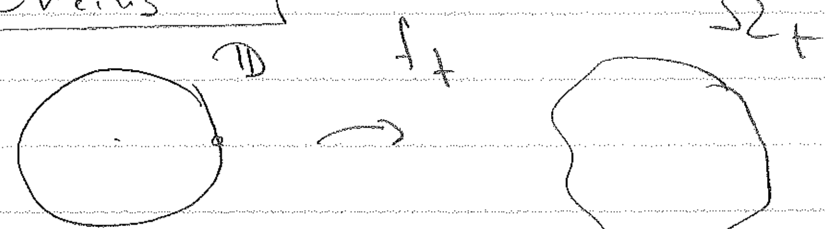
(6) Proof of Thm. 6.6. shows:

$$\dot f_t(z) = V(z,t)\, f_t'(z),$$

where $V(z,t) = -z \dfrac{\lambda(t)+z}{\lambda(t)-z}$,

$(z,t) \in \mathbb{D}\times[0,b]$.

7.3. Idea of chordal Loewner chains



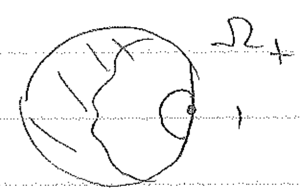
Want to normalize conf. maps at boundary point, say $1 \in \partial\mathbb{D}$.

Meaningless, unless we have additional assumptions:

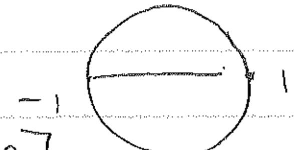
$\Omega_t \subseteq \mathbb{D}$ s.t.

$B(1, r(t)) \cap \mathbb{D} \subseteq \Omega_t$

$\Omega_t \supseteq \Omega_s$ as $t < s$.



Simplest situation:



$\Omega_t = \mathbb{D}\setminus(-1, t-1)$, $t \in [0,2]$

Mostly one switches to upper-half plane

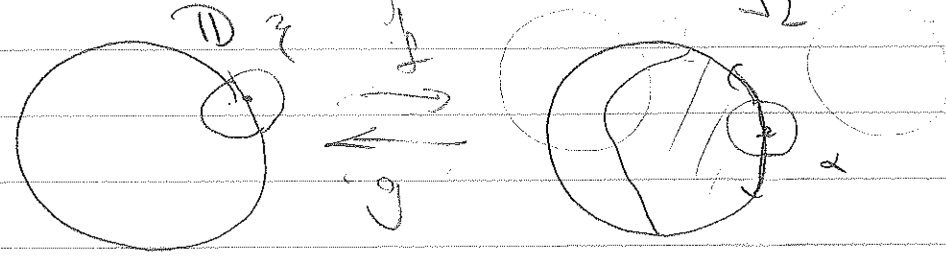
$\mathbb{H} = \{w \in \mathbb{C} : \operatorname{Im} w > 0\}$

$\partial\mathbb{H} = \hat{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$

$\mathbb{D}\cup\{1\} \longleftrightarrow \mathbb{H}\cup\{\infty\}$.

Lem. 7.4 $\Omega \subseteq \mathbb{D}$ simply connected, $f: \Omega \leftrightarrow \mathbb{D}$ conf. map. Suppose $\xi \in \partial\mathbb{D}\cap\partial\Omega$ and there ex. $r > 0$ s.t. $\mathbb{D}\cap B(\xi, r) \subseteq \Omega$.

⑦ Then g has a holomorphic extension to a neighborhood of ξ and $g'(\xi) \neq 0$.



Proof:

Wlog $\xi = 1$

$\alpha \subseteq \partial\mathbb{D} \cap \partial\Omega$ open arc with $1 \in \alpha$

By Wolff's lemma f has a cont. extension to $\Omega \cup \alpha$.

Then $g(\alpha) \subseteq \partial\mathbb{D}$, and g extends to a holomorphic function near ξ.

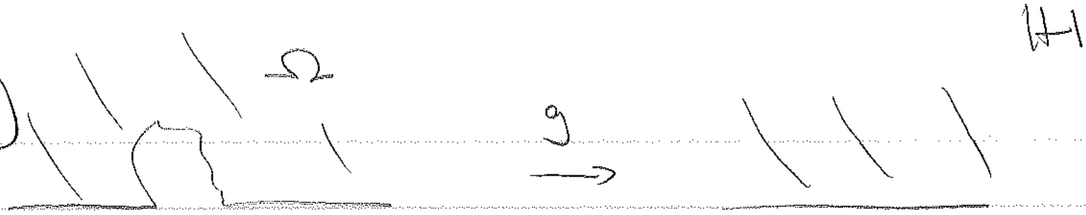
Points in $\mathbb{D}$ near $g(\xi) \in \partial\mathbb{D}$ have precisely one preimage near ξ, so g is loc. inj. near ξ and $g'(\xi) \neq 0$. □

Note that $f = g^{-1}$ has a loc. inj. extension to $\eta = g(\xi) \in \partial\mathbb{D}$

Cor. 7.5. $\Omega \subseteq \mathbb{H}$ simply connected s.t. $\mathbb{H} \setminus B(0,R) \subseteq \Omega$ for some $R > 0$. Then there ex. a unique conformal map $f: \mathbb{H} \longleftrightarrow \Omega$ s.t. f has a holomorphic extension near ∞ and

$$f(z) = z + \frac{a_1}{z} + \cdots, \text{ for } z \text{ near } \infty.$$

Proof: Existence: By Lem. 7.4. there ex. a conf. map $g: \Omega \longleftrightarrow \mathbb{H}$

(f) 

s.t. g has a holomorphic and loc. inj. ext. to ∞.

$g(\infty) \in \hat{\mathbb{R}}$; post-comp. by a Möbius transf. we may assume $g(\infty) = \infty$.

Since g is loc. injective, g has a 1^{st} order pole near ∞ equals:

$$g(z) = b_1 z + b_0 + \frac{b_{-1}}{z} + \frac{b_{-2}}{z^2} + \dots$$

$g(x) \in \mathbb{R}$ for $x \in \mathbb{R}$ near ∞; so

$$b_1 = \lim_{x \in \mathbb{R} \to \infty} \frac{g(x)}{x} \in \mathbb{R};$$

$$b_0 = \lim_{x \in \mathbb{R} \to \infty} g(x) - b_1 x \in \mathbb{R};$$

$\operatorname{Im} g(ix) > 0$ for $x > 0$; so

$$b_1 = \operatorname{Re} b_1 = \lim_{\substack{x \to +\infty \\ \in \mathbb{R}}} \operatorname{Re}\underbrace{\left(\frac{g(ix)}{ix}\right)}_{\operatorname{Im} \frac{g(ix)}{x}} \geq 0;$$

so $b_1 > 0$.

$\varphi(w) = \frac{1}{b_1}(w - b_0)$ preserves $\mathbb{H}$;

$\tilde{g} := \varphi \circ g$ conf. map of Ω onto $\mathbb{H}$ with

$$\tilde{g}(z) = z + \frac{\tilde{b}_{-1}}{z} + \dots, \quad z \text{ near } \infty.$$

$$= z + o(1);$$

(9) Let $f := \tilde{g}^{-1}$.
Then $f : \mathbb{H} \longleftrightarrow \Omega$ conf. map, holo. near ∞, and $f(z) = z + o(1)$;
so
$f(z) = z + \frac{a_{-1}}{z} + \dots$ for z near ∞.

Uniqueness: Suppose $f_1, f_2 : \mathbb{H} \longleftrightarrow \Omega$ conf., holom. near ∞, and
$f_1(z) = z + o(1)$, $f_2(z) = z + o(1)$;
then
$\varphi := f_2 \circ f_1^{-1} : \mathbb{H} \longleftrightarrow \mathbb{H}$ conf.;
hence Möbius transf. with $\varphi(\mathbb{H}) = \mathbb{H}$
$$\varphi(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{R}, \quad ad-bc > 0.$$
Moreover: $\varphi(\infty) = \infty$; so
$\varphi(z) = az + b$, $a > 0$, $b \in \mathbb{R}$.
$\varphi(z) = z + o(1)$, so $a = 1$, $b = 0$,
and $\varphi = \mathrm{id}_{\hat{\mathbb{C}}}$.

Hence $f_1 = f_2$. $\square$

Thm. 7.6. (Herglotz representation for positive harmonic functions)
a) (disk version)
Let $h : \mathbb{D} \to (0, \infty)$ be a positive harmonic function. Then there ex. a unique pos. measure on $\partial\mathbb{D}$ with $0 < \mu(\partial\mathbb{D}) < \infty$ s.t.

(10)

$$h(z) = \int_{\partial\mathbb{D}} \mathrm{Re}\left[\frac{\xi+z}{\xi-z}\right] d\mu(\xi), \quad z\in\mathbb{D}$$

b) (half-plane version)
Let $h:\mathbb{H}\to(0,\infty)$ be a positive harmonic function. Then there ex. a unique const. $a\ge 0$ and a unique pos. measure ν on $\mathbb{R}$ s.t.

$$0 < a + \int_{\mathbb{R}} \frac{d\nu(t)}{1+t^2} < \infty,$$

and

$$h(z) = a\cdot \mathrm{Im}\, z + \int_{\mathbb{R}} \mathrm{Im}\left(\frac{1}{t-z}\right) d\nu(t),$$

$z\in\mathbb{H}$.

Proof: a)

$h \longleftrightarrow$ unique $f\in H(\mathbb{D})$

$\mathrm{Re} f = h > 0$,

$h(0)= f(0) > 0$

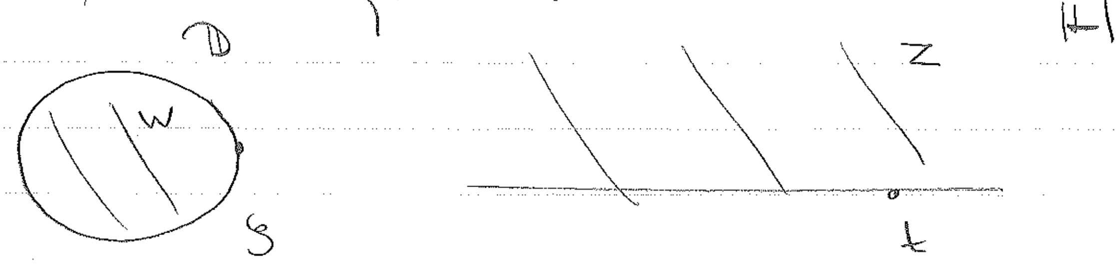
$\longleftrightarrow$ unique measure $\mu \ge 0$ on $\partial\mathbb{D}$

s.t. $$f(z) = \int_{\partial\mathbb{D}} \frac{\xi+z}{\xi-z}\, d\mu(\xi)$$

$$0 < \mu(\partial\mathbb{D}) = f(0) < \infty.$$

Existence and uniqueness follow.

b)



(11) $\mathbb{D}\cup\{1\} \longleftrightarrow \mathbb{H}\cup\{\infty\}$

$z = \varphi(w) = i\cdot\frac{1+w}{1-w}$, $w = \frac{z-i}{z+i} = \psi(z)$

$\varphi(1) = \infty$

$h : \mathbb{H} \to (0,\infty)$, $\Delta h = 0$

$g := h\circ\varphi$, $g > 0$ on $\mathbb{D}$, $\Delta g = 0$.

f unique holo. function on $\mathbb{D}$ s.t.

$\mathrm{Re}\, f = g$, $f(0) = g(0) > 0$.

By (a)

$$f(w) = a\cdot\frac{1+w}{1-w} + \int_{\partial\mathbb{D}\setminus\{1\}} \frac{\xi+w}{\xi-w}\, d\mu(\xi).$$

$a = \mu(\{1\}) \geq 0$

$\tau := \varphi_* \mu|_{\partial\mathbb{D}\setminus\{1\}}$ measure on $\mathbb{R}$

$\tau(A) = \mu(\varphi^{-1}(A))$ for $A \subseteq \mathbb{R}$

$\int_{\mathbb{R}} g\, d\tau = \int_{\partial\mathbb{D}\setminus\{1\}} (g\circ\varphi)\, d\mu$, $g \in L^1(\tau)$.

$0 < \mu(\partial\mathbb{D}) = a + \tau(\mathbb{R}) < \infty$.

(a,τ) unique

$\tilde{f}(z) = f(\psi(z))$, $h = \mathrm{Re}\, \tilde{f}$

$\frac{1+w}{1-w} = -iz$, $\mathrm{Re}\left(\frac{1+w}{1-w}\right) = \mathrm{Re}(-iz)$

$= \mathrm{Im}(z)$.

$\frac{\xi+w}{\xi-w} = -i\left(\frac{1+tz}{t-z}\right) = -i\left[\frac{1+t^2}{t-z} - t\right]$

$\xi = \frac{t-i}{t+i}$, $w = \frac{z-i}{z+i}$, $t\in\mathbb{R} \iff \xi \in \partial\mathbb{D}\setminus\{1\}$

(12) $$\operatorname{Re}\left(\frac{\xi+w}{\xi-w}\right) = \operatorname{Re}\left(-i\left[\frac{1+t^2}{z-t}\right]\right)$$
$$= \operatorname{Im}\left(\frac{1+t^2}{z-t}\right) = (1+t^2)\operatorname{Im}\left(\frac{1}{z-t}\right)$$

Define measure ν on $\mathbb{R}$ by
$$d\nu(t) = (1+t^2)\, d\sigma(t).$$
Then $\int_{\mathbb{R}} \frac{d\nu(t)}{1+t^2} = \int_{\mathbb{R}} d\sigma(t) < \infty.$

$$0 < a + \int_{\mathbb{R}} \frac{d\nu(t)}{1+t^2} = a + \sigma(\mathbb{R}) = \mu(\partial\mathbb{D}) < \infty.$$

$$f(z) = a(-iz) + \int_{\mathbb{R}} i\left[\frac{1+t^2}{t-z} + t\right] d\sigma(t)$$

$$h(z) = \operatorname{Re} f(z)$$
$$= a \operatorname{Im} z + \int_{\mathbb{R}} \operatorname{Im}\left(\frac{1}{t-z}\right) d\nu(t), \quad z \in \mathbb{D}$$

$z = x+iy$, $y>0$

$$+\operatorname{Im}\left(\frac{1}{t-z}\right) = \operatorname{Im}\left[\frac{1}{(t-x)-iy}\right] = \operatorname{Im}\left[\frac{(t-x)+iy}{(x-t)^2+y^2}\right]$$
$$= \frac{y}{(x-t)^2+y^2} \lesssim \frac{1}{1+t^2} \quad \text{for } x,y \text{ fixed, } |t| \text{ large}$$

Uniqueness of (a,ν) clear. □

Rem. 7.7 $g \in H(\mathbb{H})$, $\operatorname{Im} g \geq 0$

$f = -ig$, $\operatorname{Re} f \geq 0$ $\quad g = if$

(13) The proof shows that if $g \in H(\mathbb{H})$, $\operatorname{Im} g \geq 0$, then there ex. unique constants $a, b \in \mathbb{R}$, $a \geq 0$, and a unique finite measure $\sigma \geq 0$ on $\mathbb{R}$ s.t.

$$g(z) = az + b + \int_{\mathbb{R}} \left[\frac{1+t^2}{t-z} - t \right] d\sigma(t), \quad z \in \mathbb{H}.$$

<u>Thm. 7.8.</u> (Julia's Lemma)
Let $f: \mathbb{H} \longrightarrow \mathbb{H}$ be holomorphic, and

$$c := \inf_{z \in \mathbb{H}} \frac{\operatorname{Im} f(z)}{\operatorname{Im} z} \geq 0.$$

Then:
$$c = \lim_{y \to +\infty} \frac{\operatorname{Im} f(iy)}{y}. \qquad (1)$$

Suppose in addition that f is holomorphic near ∞, and has a Laurent exp. of the form

$$(*) \quad f(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \quad \text{near } \infty.$$

Then $c = 1$ and $a_1 \leq 0$ (so $a_1 \in \mathbb{R}$) with equality iff $f(z) = z$ for $z \in \mathbb{H}$.

<u>Note</u>: $\operatorname{Im} f(z) \geq \operatorname{Im} z$ for $z \in \mathbb{H}$, and so

$$f(\mathbb{H}_t) \subseteq \mathbb{H}_t, \quad t \geq 0 \qquad \mathbb{H}_t := \{ z \in \mathbb{C} : \operatorname{Im} z \geq t \}$$

(14) Proof: $h := \operatorname{Im} f$, $h \geq 0$, $\Delta h = 0$

Wlog $h > 0$ (otherwise $f \equiv a \in \mathbb{R}$; claim true)

By Thm. 7.6.

$$h(z) = a \cdot \operatorname{Im} z + \underbrace{\int_{\mathbb{R}} \underbrace{\operatorname{Im}\left(\frac{1}{t-z}\right)}_{\geq 0} d\nu(t)}_{\tilde{h}(z)}$$

$a \geq 0$, $\nu \geq 0$

$\hat{h} \geq 0$ for $z \in \mathbb{H}$.

$$\frac{h(z)}{\operatorname{Im} z} = a + \frac{\hat{h}(z)}{\operatorname{Im} z} \text{ ; so } c \geq a$$

For claim it suffices to show that

$$\lim_{y \to +\infty} \frac{\tilde{h}(iy)}{y} = 0 \quad \text{(then } c = a \text{ and (1) true)}$$

$$\operatorname{Im}\left(\frac{1}{t-iy}\right) = \frac{y}{t^2+y^2}$$

$$\frac{\hat{h}(iy)}{y} = \int_{\mathbb{R}} \frac{1}{t^2+y^2} d\nu(t) \longrightarrow 0 \text{ as } y \to +\infty$$

by Lebesgue Dom. Conv.

$$\frac{1}{t^2+y^2} \underset{y \geq 1}{\leq} \frac{1}{t^2+1} \in L^1(\nu)$$

$\downarrow$ pt.wise

0 as $y \to +\infty$

Suppose now in addition that f has exp. as in $(*)$.

Then

$$c = \lim_{y \to +\infty} \frac{\operatorname{Im} f(iy)}{y} = \lim_{y \to +\infty} \frac{y + o(1)}{y} = 1.$$

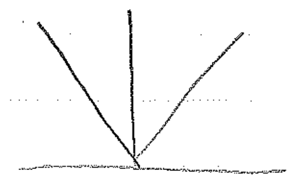
(15) So by def. of c:

$$\operatorname{Im} f(z) \geq \operatorname{Im} z \quad \text{for all } z \in \mathbb{H}.$$

$a_1 = \alpha + i\beta$; $z = x + iy \in \mathbb{H}$, $|z|$ large

$$\operatorname{Im}\left(\frac{a_1}{z}\right) = \operatorname{Im}\left(\frac{a_1 \cdot \bar{z}}{|z|^2}\right) = \frac{1}{|z|^2} \operatorname{Im}\left((\alpha + i\beta)(x - iy)\right)$$

$$= \frac{1}{|z|^2} (\beta x - \alpha y)$$



$$0 \leq |z| \left(\operatorname{Im} f(z) - \operatorname{Im} z\right)$$

$$= \frac{1}{|z|} (\beta x - \alpha y) + O\left(\frac{1}{|z|}\right)_{\text{small if}}$$

So

$$\beta x - \alpha y \geq 0 \quad \text{for } x + iy \in \mathbb{H} \quad \text{i.e. for } x \in \mathbb{R},\ y > 0.$$

$\Rightarrow \beta = 0$, $\alpha \leq 0$

$\Rightarrow a_1 \in \mathbb{R}$, $a_1 \leq 0$.

Case of equality: $a_1 = 0$.

Then inductively: $a_2 = a_3 = \dots 0$

$z = r \cdot e^{i\varphi}$, $\varphi \in (0, \pi)$, $r > 0$

Induction step:

$$f(z) = \frac{a_n}{z^n} + \dots$$

$$0 \leq |z|^n \left(\operatorname{Im} f(z) - \operatorname{Im} z\right)$$

$$= \operatorname{Im}\left(a_n \cdot e^{-in\varphi}\right) + O\left(\frac{1}{|z|}\right)$$

So $\operatorname{Im}\left(a_n e^{in\varphi}\right) \geq 0$, $\varphi \in (0, \pi)$ equiv.

$\operatorname{Im}\left(a_n \cdot e^{i\alpha}\right) \geq 0$ for all $\alpha \in [0, 2\pi]$

$\Rightarrow a_n = 0$. $\square$

(16) **Thm. 7.9.** (Integral representation)

$\Omega \subseteq \mathbb{H}$ simply connected s.t. $\mathbb{H} \setminus B(0,R) \subseteq \Omega$ for some $R > 0$, $f : \mathbb{H} \longleftrightarrow \Omega$ unique conf. map s.t.

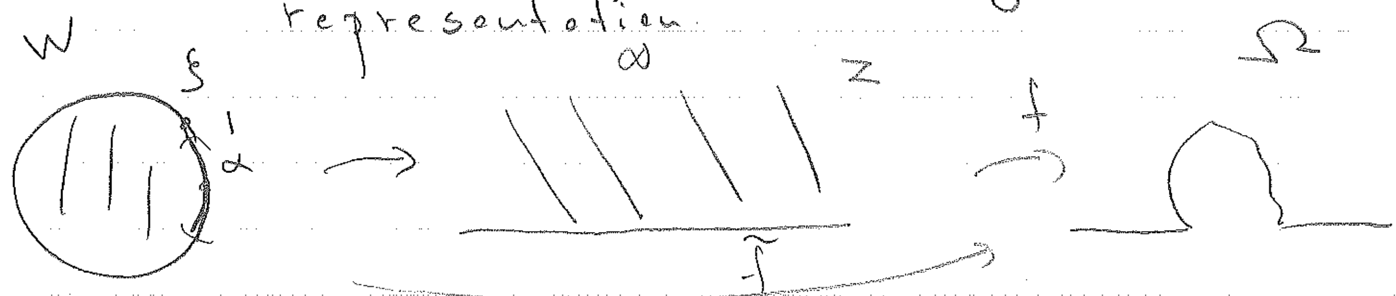
$$f(z) = z + \frac{a_1}{z} + \dots \quad \text{for } z \text{ near } \infty.$$

Then there ex. a unique finite Borel measure $\nu \geq 0$ on $\mathbb{R}$ with comp. support s.t.

$$f(z) = z + \int_{\mathbb{R}} \frac{1}{t-z}\, d\nu(t).$$

Proof: Uniqueness: follows from uniqueness of Herglotz represent. of $h = \operatorname{Im} f$.

Existence: Revisit proof of Herglotz representation



$$z = i\,\frac{1+w}{1-w} = \varphi(w) \longrightarrow \infty$$

By Schwarz reflection, $\tilde{f}$ has a holomorphic ext. across an open arc $\alpha \subseteq \partial\mathbb{D}$, $1 \in \alpha$

$\tilde{f}(\alpha) \subseteq \hat{\mathbb{R}}$; $\tilde{f}(\alpha \setminus \{1\}) \subseteq \mathbb{R}$

$\operatorname{Im} \tilde{f}(\xi) \equiv 0$ for $\xi \in \alpha \setminus \{1\}$ (cf. proof of Lem. 7.4)

$\operatorname{Im} \tilde{f} > 0$ on $\mathbb{H}$

$\tilde{g} = -i\tilde{f}$, $\operatorname{Re} \tilde{g} = \operatorname{Im} \tilde{f} \geq 0$. $\tilde{f} = i\tilde{g}$

$\operatorname{Re} \tilde{g}(\xi) \equiv 0$ for $\xi \in \alpha \setminus \{1\}$.

(17)

(1) $\operatorname{Re} \tilde{g}(r \cdot \xi) \longrightarrow 0$ loc. unif. as $r \longrightarrow 1^-$ for $\xi \in \alpha \setminus \{1\}$.

In Herglotz repr. for $\tilde{g}$ the measure μ on $\partial \mathbb{D}$ can be obtained as w^*-limits of measure μ_r on $\partial \mathbb{D}$ as $r \longrightarrow 1^-$, where

$$d\mu_r(\xi) = \operatorname{Re} \tilde{g}(r \cdot \xi) \frac{|d\xi|}{2\pi}.$$

Then (1) implies that

$$\operatorname{supp}(\mu) \subseteq \partial \mathbb{D} \setminus (\alpha \setminus \{1\}) = \partial \mathbb{D} \setminus \alpha \cup \{1\}$$

So

$$\tilde{f}(w) = b + i \int_{\partial \mathbb{D}} \frac{\xi + w}{\xi - w} \, d\mu(\xi), \quad b \in \mathbb{R}$$

going back to $\mathbb{H}$:

$$f(z) = az + b + \int_{\mathbb{R}} \left[\frac{1+t^2}{t-z} - t \right] d\sigma(t),$$

where $a = \mu(\{1\})$, $b \in \mathbb{R}$, and σ is finite measure with support in $\varphi(\partial \mathbb{D} \setminus \alpha) \subset\subset \mathbb{R}$.

$$d\nu(t) = (1+t^2)\, d\sigma(t)$$

$\nu \geq 0$ finite measure with cp supp.

$$\tilde{b} = b - \int_{\mathbb{R}} t \, d\sigma(t). \quad \text{Then}$$

$$f(z) = az + \tilde{b} + \underbrace{\int_{\mathbb{R}} \frac{1}{t-z} \, d\nu(t)}_{= O\left(\frac{1}{z}\right)};$$

on the other hand:

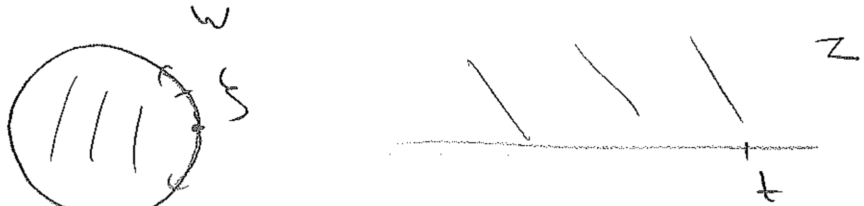
$$f(z) = z + o(1); \quad \text{so } a = 1, \ \tilde{b} = 0.$$

□

(17a) Rem.: If $f : \mathbb{H} \longleftrightarrow \Omega$ is as in Thm. 7.9. and $\operatorname{Im} f$ has a cont. extension to ~~$\mathbb{H}$~~ $\mathbb{H} \cup \mathbb{R}$, then

$$f(z) = z + \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{t-z} \operatorname{Im} f(t)\,dt ,$$

so $\boxed{d\nu(t) = \frac{1}{\pi}\operatorname{Im} f(t)\,dt}$



$z = i\,\frac{1+w}{1-w}$. $\qquad \tilde{g}(w) = -i f(z)$

$d\nu(t) = (1+t^2)\,d\tau(t) =$

$\tau = \varphi_*(\mu|\partial\mathbb{D}\setminus\{1\})$

$\mu_r \xrightarrow{w^*} \mu \qquad d\mu_r(\xi) = \operatorname{Re}\tilde{g}(r\xi)\frac{|d\xi|}{2\pi}$

If f has a cont. ext. to $\mathbb{R}$, then $\tilde{g}$ has cont. ext. to $\partial\mathbb{D}\setminus\{1\}$, and

$$d\mu(\xi) = \operatorname{Re}\tilde{g}(\xi)\,\frac{|d\xi|}{2\pi} \quad \text{on } \partial\mathbb{D}\setminus\{1\}$$

$w = \frac{z-i}{z+i} \qquad \xi = \frac{t-i}{t+i}$

$\frac{d\xi}{dt} = \frac{2i}{(t+i)^2} \qquad \left|\frac{d\xi}{dt}\right| = \frac{2}{(1+t^2)}$

$$d\tau(t) = \frac{1}{2\pi}\operatorname{Re}\tilde{g}(\xi) = \frac{1}{2\pi}\operatorname{Im} f(t)\left|\frac{d\xi}{dt}\right|dt = \frac{1}{\pi(1+t^2)}\operatorname{Im} f(t)\,dt$$

(18) Note: for $|z|$ large

$$\int_{\mathbb{R}} \frac{1}{t-z}\, d\nu(t) = -\frac{1}{z} \int_{\mathbb{R}} \frac{1}{\left(1-\frac{t}{z}\right)}\, d\nu(t)$$

$$= -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \int_{\mathbb{R}} t^n \, d\nu(t) \qquad (\text{unif. conv.})$$

On the other hand, if

$$f(z) = z + \sum_{n=1}^{\infty} \frac{a_n}{z^n}$$ is the Laurent exp. of f near ∞.

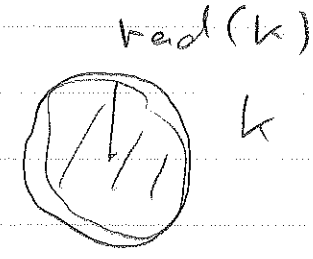
So $a_n = -\int t^{n-1}\, d\nu(t) \in \mathbb{R}$

$a_n \leq 0$ for $n \in \mathbb{N}$; if $a_1 = 0$, then $\nu(\mathbb{R}) = 0$, and $\nu = 0$. $f(z) \equiv z$.

The proof shows that $\operatorname{supp}(\nu) \subseteq I$, if I is an interval s.t. f has a holomorphic extension to $\mathbb{R} \setminus I$ with $f(\mathbb{R}\setminus I) \subseteq \mathbb{R}$.
In particular, if the Laurent expansion converges outside $\overline{B}(0,R)$, then $\operatorname{supp}(\nu) \subseteq [-R,R]$, and conversely the integral representation shows that if $\operatorname{supp}(\nu) \subseteq [-R,R]$, then Laurent exp. converges in $\mathbb{C} \setminus \overline{B}(0,R)$.

<u>Def. 7.10.</u> a) $K \subseteq \mathbb{C}$ bdd. Then

$\operatorname{rad}(K) = \sup\{|z| : z \in K\}$



b) A ~~set~~ bdd. set $A \subseteq \mathbb{H}$ is called an <u>$\mathbb{H}$-hull</u> if A is rel. closed in $\mathbb{H}$, i.e.,

(19) $A = \overline{A} \cap \mathbb{H}$, and if $\Omega_A = \mathbb{H} \setminus A$ is a simply connected region.
Then there ex. a unique cont. map $f_A : \mathbb{H} \longleftrightarrow \Omega_A$ with holomorphic ext. near ∞ of the form

$$f_A(z) = z + \frac{a_1}{z} + \dots$$

We call $\mathrm{hcap}(A) := -a_1 \geq 0$ the half-plane capacity of A.

c) $\mathcal{Q}$ = set of all $\mathbb{H}$-hulls.

Lem. 7.11. A $\mathbb{H}$-hull,

$$f_A(z) = z + \int_{\mathbb{R}} \frac{1}{t-z} \, d\nu_A(t)$$ integral repres. in 7.9.

Then: a) $\nu_A(\mathbb{R}) = \mathrm{hcap}(A)$ c) $\mathrm{hcap}(A) \lesssim \mathrm{rad}(A)^2$

b) $\mathrm{rad}(\mathrm{supp}(\nu_A)) \simeq \mathrm{rad}(A)$

(*)

Proof: a) $f_A(z) = z + \frac{a_1}{z} + \dots$ near ∞, where

$$a_1 = -\int_{\mathbb{R}} d\nu_A(t) = -\nu_A(\mathbb{R})$$
$$= -\mathrm{hcap}(A).$$

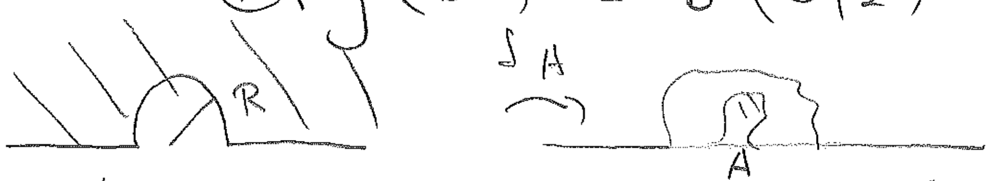
b) We know that $R := \mathrm{rad}(\mathrm{supp}(\nu_A))$ is smallest number s.t. f converges on $\mathbb{C} \setminus \overline{B}(0,R)$.
Then by Schwarz refl. f_A has a holo. ext. to a cont. map on $\hat{\mathbb{C}} \setminus \overline{B}(0,R)$

(20) into $\hat{\mathbb{C}}$.

Define

$$h(w) := \frac{1}{R} \cdot f_A(R \cdot w) = w + \frac{\tilde{a}_1}{w} + \dots \quad \text{for } w \in \mathbb{D}^+ = \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}.$$

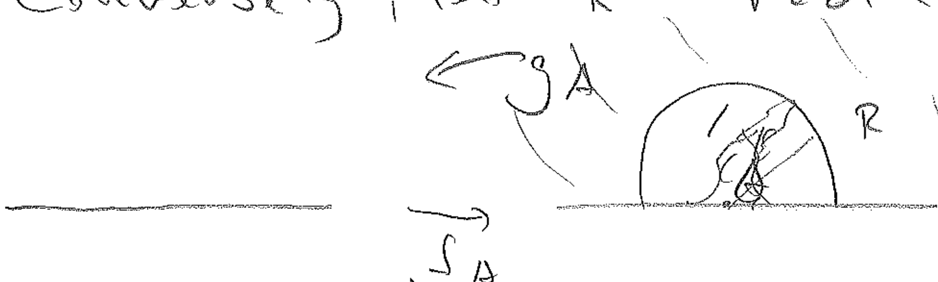
Then $h \in \Sigma$ (cf. Sect. 1), and so $\hat{\mathbb{C}} \setminus g(\mathbb{D}^+) \subseteq \overline{B}(0,2)$ (cf. Cor 1.3)



$\frac{1}{R} A \subseteq \overline{B}(0,2)$; and so $A \subseteq \overline{B}(0,2R)$, i.e.

rad$(A) \leq 2R$.

Conversely, let $\tilde{R} = \text{rad}(A)$



Then $g_A = f_A^{-1}$ has a conf. extension to

$\hat{\mathbb{C}} \setminus B(0, \tilde{R})$.

$\tilde{h}(w) := \frac{1}{\tilde{R}} g_A(\tilde{R} w) = w + \frac{b_1}{w} + \dots$ near ∞; so

$\tilde{h} \in \Sigma$, and

$\tilde{h}(\mathbb{D}^+) \supseteq \mathbb{C} \setminus \overline{B}(0,2)$, i.e.,

$g_A(\mathbb{C} \setminus \overline{B}(0,R)) \supseteq \mathbb{C} \setminus \overline{B}(0, 2\tilde{R})$.

So f_A is holomorphic on $\mathbb{C} \setminus \overline{B}(0, 2\tilde{R})$.

i.e., $R \leq 2\tilde{R} = 2\,\text{rad}(A)$; so $R \simeq \tilde{R}$.

(21) c) Notation as in b): $h \in \Sigma$

$$h(w) = \frac{1}{R} f_A(Rw) = w + \frac{a_1}{R^2 w} + \cdots$$

$$f_A(z) = z + \frac{a_1}{z} + \cdots$$

By the Area Thm. 1.2.:

$1 \geq \left|\frac{a_1}{R^2 w}\right|$ and so $\operatorname{hcap}(A) = -a_1$
$\leq R^2 \leq \tilde{R}^2 = \operatorname{rad}(A)^2$. $\square$

Rem. 7.12. $\mathcal{A}$ family of $\mathbb{H}$-hulls A,
$\mathcal{F} = \{f_A : A \in \mathcal{A}\}$ corr. family of
conf. maps $f_A : \mathbb{H} \longleftrightarrow \mathbb{H} \setminus A$
with usual normalization $f_A(z) = z + o(1)$ near ∞.
If $\operatorname{rad}(A)$ is unif. bdd.
for $A \in \mathcal{A}$ (i.e., if $\{\operatorname{rad}(A) : A \in \mathcal{A}\}$,
then one has good "a priori" control
for the maps in $\mathcal{F}$. For example:

i) $f_A(z) = z + \int_{\mathbb{R}} \frac{d\mu_A(t)}{t - z}$
The measure μ_A have unif. bdd. total
mass with supports contained in a
fixed interval I (follows from Lem. 7.11)
ii) $\mathcal{F}$ is loc. unif. bdd. and in particular
a normal family;
actually, $\mathcal{F}$ is unif. bdd. on bdd.
subsets of $\mathbb{H}$.

(22) Ex. $R > 0$ s.t. each $f_A \in \mathcal{F}$ has ext. to a conf. map on $\hat{\mathbb{C}} \setminus \bar{B}(0,R)$.

$h_A(w) := \frac{1}{R} f_A(Rw)$, $w \in \mathbb{D}^*$.

$h_A \in \Sigma$; so $h_A(\mathbb{D}^*) \supseteq \hat{\mathbb{C}} \setminus B(0,2)$

So $f_A(B(0,R) \cap \mathbb{H}) \subseteq B(0,2R)$.

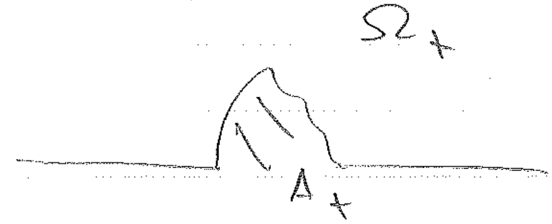
7.13.1 Chordal Loewner chains

(half-plane version of Loewner chains)

$I = [0,b]$, $b \in (0,\infty]$.

$\{\Omega_t\}_{t \in I}$ is a (geometric) chordal Loewner chain

if i) $\Omega_t \subseteq \mathbb{H}$ is a simply connected region of the form $\Omega_t = \mathbb{H} \setminus A_t$, where A_t is an $\mathbb{H}$-hull



ii) $\Omega_0 = \mathbb{H}$ $(A_0 = \emptyset)$

iii) $\Omega_s \subsetneq \Omega_t$ for $s > t$, $s,t \in I$

(equiv. $A_s \supsetneq A_t$)

iv) $\{\Omega_t\}_{t \in I}$ satisfies a continuity requirement (Lem. 7.14)

$f_t : \mathbb{H} \longleftrightarrow \Omega_t$ unique conf. map s.t. $f_t(z) = z + \frac{a_1(t)}{z} + \dots$ near ∞.

23 $\{f_t\}_{t\in I}$ is the corr. (analytic) Loewner chain

It is normalized if

$$f_t(z) = z - \frac{2t}{z} + \dots \quad \text{near } \infty \quad \text{for } t \in I,$$

i.e., $a_1(t) = -2t$, $t \in I$.

<u>Lem. 7.14.</u> $\{\Omega_t\}_{t\in I}$ chordal (pre-) Loewner chain corr. to analytic Loewnerchain $\{f_t\}_{t\in I}$. $\{t_n\}$ seq. in I with $t_n \longrightarrow t_\infty$ as $n \longrightarrow \infty$.

$\Omega_n = \Omega_{t_n}$, $f_n := f_{t_n}$ $\quad \Omega_n = \mathbb{H} \setminus A_n$

$$f_n(z) = z + \int_{\mathbb{R}} \frac{d\mu_n(u)}{u - z}, \quad z \in \mathbb{H}.$$

<u>TFAE:</u> i) $f_n \longrightarrow f_\infty$ loc. unif. on $\mathbb{H}$

ii) $\mu_n \xrightarrow{w^*} \mu_\infty$, i.e.

$$\int \varphi \, d\mu_n \longrightarrow \int \varphi \, d\mu_\infty \quad \text{for all } \varphi \in C_c(\mathbb{R})$$

(equiv. for all $\varphi \in C(\mathbb{R})$)

iii) $\operatorname{hcap}(A_n) \longrightarrow \operatorname{hcap}(A_\infty)$

iv) $\Omega_n \longrightarrow \Omega_\infty$ in the sense of Kernel convergence w.r.t. ∞

$\operatorname{Kern}_\infty(\{\Omega_n\})$

= set of all pts. $w \in \mathbb{C}$ for which there ex. an unbounded region U with $w \in U$ and $U \subseteq \Omega_n$ for all large n.

(24) <u>Proof:</u> $T = \sup\{t_n : n \in \mathbb{N} \cup \{\infty\}\} \in I$;
so $A_n \subseteq A_T$ and $\mathrm{rad}(A_n) \leq \mathrm{rad}(A_T) < \infty$ for $n \in \mathbb{N} \cup \{\infty\}$

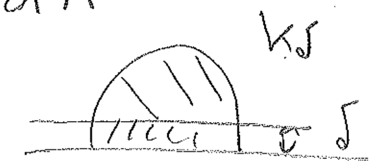
In particular, f_n, $n \in \mathbb{N} \cup \{\infty\}$, unif. bdd. on bdd. subsets of $\mathbb{H}$ and there ex. $C \geq 0$, $R \geq 0$ s.t.

$\mu_n(\mathbb{R}) \leq C_0$, $\mathrm{supp}(\mu_n) \subseteq [-R_0, R_0]$ for $n \in \mathbb{N} \cup \{\infty\}$.

i) $\to$ ii).

I) if $\varphi \in C_c(\mathbb{R}^2)$ is arb., then

$$\int_{\mathbb{H}} f_n \varphi \, dA \longrightarrow \int_{\mathbb{H}} f_\infty \varphi \, dA$$



$\mathrm{supp}(\varphi) \subseteq \bar{B}(0,R)$

$$\left| \int_{\mathbb{H}} (f_n - f_\infty) \varphi \, dA \right| \leq \int_{\mathbb{H} \cap \bar{B}(0,R)} |\varphi| \cdot |f_n - f_\infty| \, dA$$

$$\leq A(K_\delta) \|\varphi\|_\infty \cdot \sup_{z \in K_\delta} \{|f_n(z) - f_\infty(z)| : z \in K_\delta\}$$

$$4\delta R \|\varphi\|_\infty \cdot \underbrace{\sup\{|f_n(z)| : n \in \mathbb{N} \cup \{\infty\}, z \in B(0,R) \cap \mathbb{H}\}}_{< \infty}.$$

$$\leq C_1 \sup_{z \in K_\delta} \{|f_n(z) - f_\infty(z)| : z \in K_\delta\} + \varepsilon/2$$
$$\leq \varepsilon/2 + \varepsilon/2 \quad \text{if}$$

if $\delta > 0$ is suff. small, n is large.

II) Let P be an arb. polynomial (in z)

Then

$$\int P \, d\mu_n \longrightarrow \int P \, d\mu_\infty$$

(25) Pick $\chi \in C_c^\infty(\mathbb{C})$ s.t. $\chi | \bar{B}(0,R_0) \equiv 1$, and

$$h := \chi P. \quad h_{\bar z} = \chi_{\bar z} \cdot P \in C_c^\infty(\mathbb{C})$$

Hence

$$h(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{h_{\bar z}(w)}{w - z} \, dA(w), \quad z \in \mathbb{C}.$$

(cf. 246 B, Lec. 20.2).

So

$$\int_{\mathbb{R}} P \, d\mu_n = \int_{\mathbb{R}} \chi P \, d\mu_n = \int_{\mathbb{R}} h(u) \, d\mu_n(u)$$

$$= -\frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{C}} \frac{h_{\bar z}(w)}{w - u} \, dA(w) \, d\mu_n(u)$$

$$= +\frac{1}{\pi} \int_{\mathbb{C}} \Bigg(\underbrace{\int_{\mathbb{R}} \frac{d\mu_n(u)}{u - w}}_{= f_n(w) - w} \Bigg) h_{\bar z}(w) \, dA(w)$$

$$= \frac{1}{\pi} \int_{\mathbb{C}} (f_n(w) - w) \, h_{\bar z}(w) \, dA(w)$$

$$\longrightarrow \frac{1}{\pi} \int_{\mathbb{C}} (f_\infty(w) - w) \, h_{\bar z}(w) \, dA(w)$$

$$= \int_{\mathbb{R}} P \, d\mu.$$

III) Let $\varphi \in C(\mathbb{R})$ be arb. By the Weierstrass Approx. Thm. there ex. a polynomial s.t.

(26) $|P-\varphi| < \varepsilon$ on $[-R_0, R_0]$.

Then

$$\left| \int \varphi \, d\mu_n - \int \varphi \, d\mu_\infty \right|$$

$$\leq \varepsilon \cdot \mu_n(\mathbb{R}) + \varepsilon \mu_\infty(\mathbb{R}) + \left| \int P \, d\mu_n - \int P \, d\mu_\infty \right|$$

$$\leq 2 C_0 \varepsilon + \varepsilon \quad \text{for } n \text{ large.}$$

(ii) $\to$ (iii): Suppose $\mu_n \xrightarrow{w*} \mu_\infty$. Then

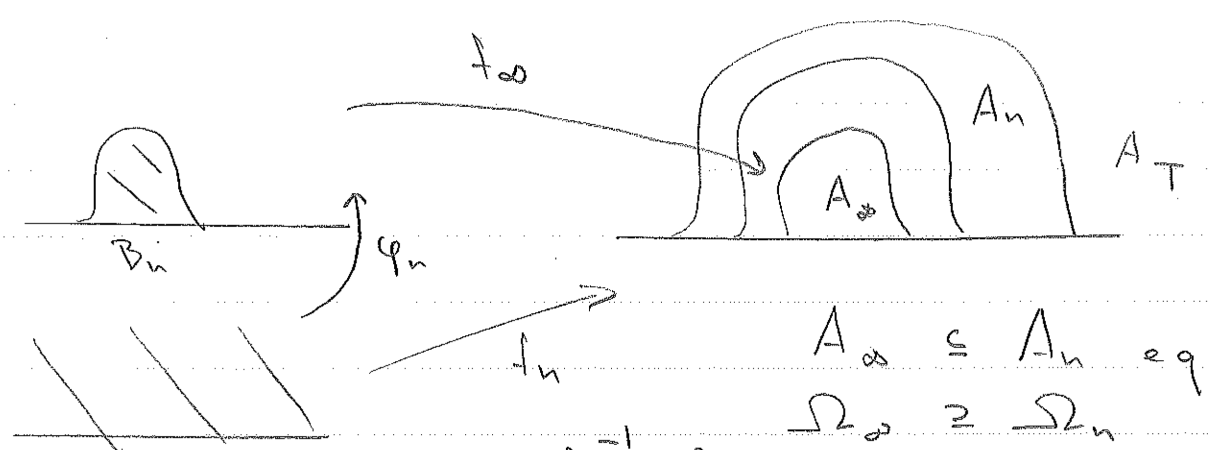
$$\operatorname{hcap}(A_n) = \mu_n(\mathbb{R}) = \int_{\mathbb{R}} 1 \cdot d\mu_n \longrightarrow \int_{\mathbb{R}} 1 \cdot d\mu_\infty = \mu_\infty(\mathbb{R}) = \operatorname{hcap}(A_\infty).$$

(iii) $\to$ (i): Suppose $\operatorname{hcap}(A_n) \longrightarrow \operatorname{hcap}(A_\infty)$.

WTS $f_n \longrightarrow f_\infty$ loc. unif. on $\mathbb{H}$

equiv. for all sequences $\{z_n\}$ in $\mathbb{H}$ with $z_n \longrightarrow z_\infty \in \mathbb{H}$, we have $f_n(z_n) \longrightarrow f_\infty(z_\infty)$.

Special case I: $t_\infty \leq t_n$ for all $n \in \mathbb{N}$.



$A_\infty \subseteq A_n$ equiv.

$\Omega_\infty \supseteq \Omega_n$

$\varphi_n := f_\infty^{-1} \circ f_n$, $n \in \mathbb{N}$. $f_n = f_\infty \circ \varphi_n$

$\varphi_n(\mathbb{H}) \subseteq \mathbb{H}$, cont. near ∞.

$\varphi_n(\mathbb{H}) = \mathbb{H} \setminus B_n$, where B_n $\mathbb{H}$-hull.

(27) $\psi(z) = z + \dfrac{a_1}{z} + \cdots$ near ∞. Then

$\psi^{-1}(z) = z - \dfrac{a_1}{z} + \cdots$ "

$\hat{\psi}(z) = z + \dfrac{b_1}{z} + \cdots$ "

$(\psi \circ \hat{\psi})(z) = z + \dfrac{a_1 + b_1}{z} + \cdots$ "

$$f_n(z) = z + \frac{a_n}{z} + \cdots, \quad a_n = \operatorname{hcap}(A_n)$$
$$f_\infty(z) = z + \frac{a_\infty}{z} + \cdots \quad a_\infty = \operatorname{hcap}(A_\infty)$$
$$\varphi_n(z) = z + \frac{a_n - a_\infty}{z} + \cdots$$

$$\operatorname{hcap}(B_n) = a_n - a_\infty$$
$$= \operatorname{hcap}(A_n) - \operatorname{hcap}(A_\infty) \longrightarrow 0$$

as $n \to \infty$.

$$\varphi_n(z) = z + \int_{\mathbb{R}} \frac{1}{u - z}\, d\nu_n(u)$$

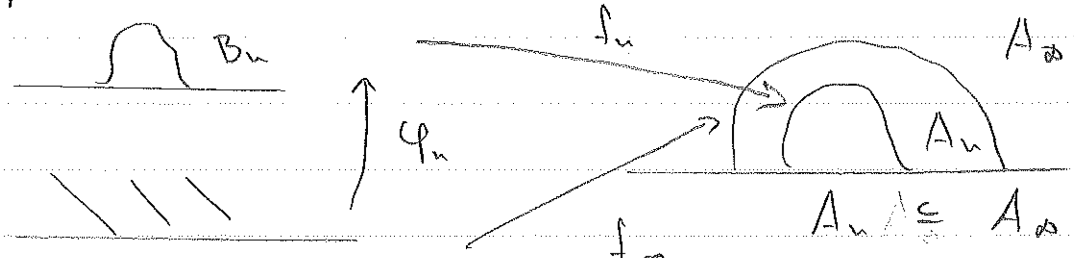
$\nu_n \geq 0$, $\operatorname{supp}(\nu_n) \subset\subset \mathbb{R}$, $\nu_n(\mathbb{R}) = \operatorname{hcap}(B_n) \longrightarrow 0$.

$$|\varphi_n(z) - z| \leq \frac{\nu_n(\mathbb{R})}{\operatorname{Im} z} \quad \text{for } z \in \mathbb{H}$$

$\Rightarrow$ $\varphi_n \longrightarrow \mathrm{id}_{\mathbb{H}}$ loc. unif. on $\mathbb{H}$.

If $z_n \in \mathbb{H} \longrightarrow z_\infty \in \mathbb{H}$, then $\varphi_n(z_n) \longrightarrow z_\infty$, and so $f_n(z_n) = f_\infty(\varphi_n(z_n)) \longrightarrow f_\infty(z_\infty)$.

<u>Special case II</u>: $t_n \leq t_\infty$ for all $n \in \mathbb{N}$.



$A_n \subsetneq A_\infty$ equiv.

$\Omega_n \supsetneq \Omega_\infty$

(28) $\varphi_n = f_n^{-1} \circ f_\infty$ equiv. $f_n \circ \varphi_n = f_\infty$

$\varphi_n(\mathbb{H}) \subseteq \mathbb{H}$, cont. near ∞

$\varphi_n(\mathbb{H}) = \mathbb{H} \setminus B_n$, B_n $\mathbb{H}$-hull

$$\mathrm{hcap}(B_n) = \mathrm{hcap}(A_\infty) - \mathrm{hcap}(A_n) \longrightarrow 0$$

$\Longrightarrow \varphi_n \longrightarrow \mathrm{id}_{\mathbb{H}}$ loc. unif. on $\mathbb{H}$.

$z_n \in \mathbb{H} \longrightarrow z_\infty \in \mathbb{H}$. $\varphi_n(z_n) \longrightarrow z_\infty$.

$\{f_n\}$ normal family (Rem. 7.12).

So $\{f_n\}$ is equicont. at z_∞.

$f_\infty(z_n) = f_\infty(z_\infty) + o(1)$

$f_\infty(z_n) = f_n(\varphi_n(z_n)) = f_n(z_\infty) + o(1)$

f_n equicont. at z_∞

$f_n(z_n) = f_n(z_\infty) + o(1)$

So $f_\infty(z_\infty) = f_n(z_n) + o(1)$.

Special cases I + II imply general case.

It remains to show that (i) $\Longleftrightarrow$ (iv).

(i) $\Rightarrow$ (iv): Assume $f_n \longrightarrow f_\infty$ loc. unif. on $\mathbb{H}$

<u>WTS</u> $\mathrm{Kern}_\infty(\{\Omega_n\}) = \Omega_\infty$

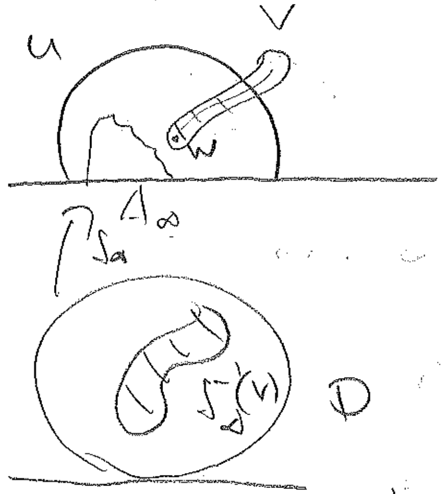
$\mathrm{Kern}_\infty :=$ (applied to all subsequences gives $\Omega_n \to \Omega_\infty$ w.r.t. ∞)

Note:

$\mathrm{rad}(A_n) \leq \tilde{R} < \infty$ for $n \in \mathbb{N} \cup \{\infty\}$; so $\mathbb{H} \setminus \bar{B}(0, \tilde{R}) =: U \subseteq \Omega_n$, $n \in \mathbb{N} \cup \{\infty\}$.

(29) I. $\Omega_\infty = f_\infty(\mathbb{H}) \subseteq \mathrm{Kern}_\infty$:

Let $w \in \Omega_\infty$ be arb. Then there ex.



$V \subset\subset \Omega_\infty$ open with $w \in V$ and $U \cap V \neq \emptyset$.

It is enough to show that $V \subset \Omega_n$ for large n ($\Rightarrow w \in \mathrm{Kern}_\infty$).

If not, there ex. $n_k \in \mathbb{N} \to \infty$, and $w_k \in V \setminus \Omega_{n_k}$.

Wlog. $w_k \longrightarrow w_\infty \in \bar{V} \subseteq \Omega_\infty$.

$f_{n_k} - w_k$ zero free on $\mathbb{H}$

$f_{n_k} - w_k \longrightarrow f_\infty - w_\infty$ loc. unif. on $\mathbb{H}$

$w_\infty \in \Omega_\infty$, so $f_\infty - w_\infty$ not zero-free, so $f_\infty - w_\infty \equiv 0$ by Hurwitz, and $f_\infty \equiv w_\infty$

Contradiction!

II. $\mathrm{Kern}_\infty \subseteq \Omega_\infty$:

Note: ex. $R_1, R_1' > 0$ and $C_1, C_1' \geq 0$ s.t.

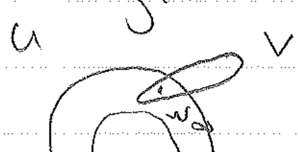
(1) $|f_n(z) - z| \leq C_1$ for $z \in \mathbb{H} \setminus \bar{B}(0, R_1)$,

(2) $|f_n^{-1}(w) - w| \leq C_1'$ $\quad w \in \mathbb{H} \setminus \bar{B}(0, R_1')$. $n \in \mathbb{N} \cup \{\infty\}$.

Let $w_\infty \in \mathrm{Kern}_\infty$ be arb.

<u>WTS</u> $w_\infty \in \Omega_\infty$, i.e., there ex. $z_\infty \in \mathbb{H}$ s.t. $f_\infty(z_\infty) = w_\infty$.

Since $w_\infty \in \mathrm{Kern}_\infty$, there ex. $V \subset\subset \mathbb{H}$ region with $V \cap U \neq \emptyset$, $w_\infty \in V$, and



$V \subset\subset \Omega_n$ for large n (unif. for all U_n).

$W = U \cup V \subseteq \Omega_n \subseteq \mathbb{H}$.

(30) $g_n := f_n^{-1}|W$.

<u>Claim</u> $\{g_n\}$ is loc. unif. bdd. and hence a normal family.

Proof by contradiction: Suppose not. Then there ex. $K \subseteq W$ comp. and a sequence $\{w_n\}$ in K s.t. $\{g_n(w_n)\}$ unbdd.

W.l.o.g. $w_n \longrightarrow w \in K$, $\underset{\in \mathbb{H}}{g_n(w_n)} \longrightarrow \infty$.

Then $w_n = f_n(g_n(w_n)) = g_n(w_n) + O(1)$ by (1) and $w_n \longrightarrow \infty$. Contradiction!

Using claim and passing to a subsequence, we may assume
$$g_n \longrightarrow g_\infty \in H(W) \text{ loc. unif. on } W.$$

$g_n(W) \subseteq \mathbb{H}$; so $g_\infty(W) \subseteq \mathbb{H} \cup \mathbb{R}$.

<u>Claim</u> $g_\infty(W) \subseteq \mathbb{H}$

Otherwise $g_\infty \equiv$ const. by open mapping thm.;

but by (2)
$$|g_\infty(w) - w| \leq C_1' \quad \text{for } w \in \mathbb{H} \text{ with } |w| \text{ large.}$$

Contradiction!

Define $z_\infty = g_\infty(w_\infty) \in \mathbb{H}$.

Then
$$f_\infty(z_\infty) = \lim_{n\to\infty} f_n(\overset{f_n^{-1}}{\overset{\shortparallel}{g_n}}(w_\infty)) = w_\infty.$$

$\uparrow$ loc. unif. conv. $f_n \longrightarrow f_\infty$

(31) (iv) $\longrightarrow$ (i) : Assume $\Omega_n \longrightarrow \Omega_\infty$

WTS $f_n \longrightarrow f_\infty$ loc. unif. on $\mathbb{H}$.

$\{f_n\}$ normal family; so it suffices to show every subsequence $\{\tilde{f}_n\}$ of $\{f_n\}$ has a subsequence that converges to f_∞ loc. unif. on $\mathbb{H}$.

$$\tilde{f}_n(z) = z + \int_{\mathbb{R}} \frac{d\tilde{\mu}_n(u)}{z-u}$$

$\text{supp}(\tilde{\mu}_n) \subseteq [-R_0, R_0]$, $\tilde{\mu}_n(\mathbb{R}) \leq C_0$

Passing to a subsequence, wlog

$\tilde{\mu}_n | [-R_0, R_0] \xrightarrow{w^*} \tilde{\mu}_\infty \geq 0$ measure on $[-R_0, R_0]$.

Then

$$\int_{\mathbb{R}} \varphi \, d\hat{\mu}_n \longrightarrow \int_{\mathbb{R}} \varphi \, d\tilde{\mu}_\infty \quad \text{for all } \varphi \in C(\mathbb{R}).$$

So

$$\hat{f}_n(z) = z + \int_{\mathbb{R}} \frac{d\hat{\mu}_n(u)}{u-z}$$

$$\longrightarrow \hat{f}_\infty(z) := z + \int_{\mathbb{R}} \frac{d\hat{\mu}_\infty(u)}{u-z}$$

pointwise for all $z \in \mathbb{H}$.

Since $\{\hat{f}_n\}$ is a normal family,

$\hat{f}_n \longrightarrow \tilde{f}_\infty$ loc. unif. on $\mathbb{H}$.

$\tilde{f}_\infty : \mathbb{H} \longrightarrow \tilde{\Omega}_\infty$ conf. map

$\hat{f}_\infty(z) = z + o(1)$ near ∞.

$\hat{f}_\infty : \mathbb{H} \longleftrightarrow \mathbb{H} \setminus \hat{A}_\infty = \tilde{\Omega}_\infty$, $\hat{A}_\infty$ $\mathbb{H}$-hull.

(32) By implication (i) $\longrightarrow$ (iv):

$$\tilde{\tilde{\Omega}}_\infty = \mathrm{Kern}_\infty(\{\tilde{\Omega}_n\}) = \Omega_\infty.$$

So $f_\infty, \tilde{f}_\infty : \mathbb{H} \longleftrightarrow \Omega_\infty = \tilde{\Omega}_\infty$ conf. maps s.t.

$$f_\infty(z) = z + o(1)$$
$$\tilde{f}_\infty(z) = z + o(1) \quad \text{near } \infty.$$

By uniqueness ((Cor. 7.5.)), $\tilde{f}_\infty = f_\infty$, and so $\tilde{f}_n \longrightarrow f_\infty$ loc. unif. on $\mathbb{H}$. $\square$

Lem. 7.15. A, B $\mathbb{H}$-hulls.

i) $\mathrm{hcap}(A) \geq 0$ with equality iff $A = \emptyset$.

ii) $\mathrm{hcap}(x + A) = \mathrm{hcap}(A)$, $x \in \mathbb{R}$.

iii) $\mathrm{hcap}(\lambda A) = \lambda^2\, \mathrm{hcap}(A)$, $\lambda > 0$,

iv) Suppose $A \subseteq B$. Then $\mathrm{hcap}(A) \leq \mathrm{hcap}(B)$ with equality iff $A = B$.

Proof: $f_A : \mathbb{H} \longleftrightarrow \mathbb{H} \setminus A$,

$$f_A(z) = z + \frac{a_1}{z} + \cdots \quad \text{near } \infty$$
$$= z + \int_{\mathbb{R}} \frac{d\mu_A(u)}{u - z}$$

$\mathrm{hcap}(A) = -a_1 = \mu_A(\mathbb{R})$.

i) So $\mathrm{hcap}(A) \geq 0$ with eq. iff $\mu_A \equiv 0$ iff $f_A(z) \equiv z$ iff $\mathbb{H} \setminus A = \mathbb{H}$ iff $A = \emptyset$.

(33) ii) $x \in \mathbb{R}$:

$$f_{x+A}(z) = x + f_A(z-x)$$

$$= z + \frac{a_1}{z-x} + \cdots$$

$$= z + \frac{a_1}{z} + \cdots$$

$$\frac{1}{z-x} = \frac{1}{z}\left(\frac{1}{1-\frac{x}{z}}\right) = \frac{1}{z} + \frac{x}{z^2} + \cdots$$

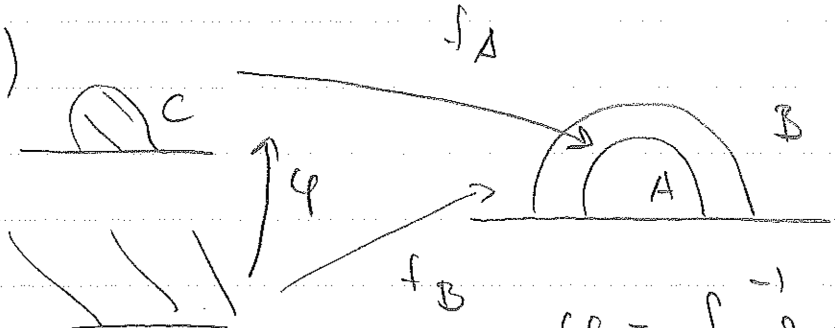
So $\mathrm{hcap}(x+A) = \mathrm{hcap}(A)$.

iii) $$f_{\lambda A}(z) = \lambda f_A(z/\lambda)$$

$$= z + \frac{a_1 \lambda}{z/\lambda} + \cdots = z + \frac{\lambda^2 a_1}{z^2} + \cdots$$

So $\mathrm{hcap}(\lambda A) = \lambda^2\, \mathrm{hcap}(A)$.

iv)



$$\varphi = f_A^{-1} \circ f_B$$

$$0 \le \mathrm{hcap}(C) = \mathrm{hcap}(B) - \mathrm{hcap}(A)$$

with equality iff $C = \emptyset$ iff $\varphi = \mathrm{id}_{\mathbb{H}}$ iff $f_A = f_B$ iff $A = B$. ∎

(34) <u>Rem. 7.16.</u> $\{\Omega_t\}_{t\in I}$ chordal Loewner chain,

$\Omega_t = \mathbb{H}\setminus A_t$, $A_t \subseteq \mathbb{H}$ $\mathbb{H}$-hull

$A_t \subsetneq A_s$, $t<s$, $A_0=\emptyset$.

$t \longmapsto \mathrm{hcap}(A_t)$ cont. (Lem. 7.14)

+ strictly increasing (Lem. 7.15).

So $t \longmapsto \mathrm{hcap}(A_t)$ homeo. of $I=[0,b]$ onto its image $J=[0,b']$

By reparametrizing t we may assume that $\mathrm{hcap}(A_t) = 2t$ for $t\in I$.

Then $f_t(z) = z - \frac{2t}{z} + \cdots$ near ∞, and $\{f_t\}$ is normalized.

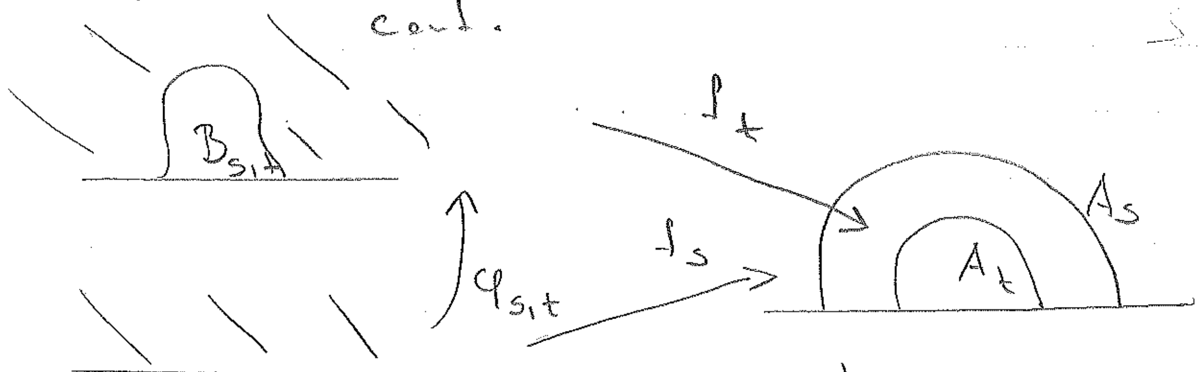
So wlog one can assume that a chordal Loewner chain is normalized.

<u>7.17. The associated semi-group.</u>

$\{f_t\}_{t\in I}$ chordal Loewner chain

$f_t : \mathbb{H} \xleftrightarrow[\text{cont.}]{} \Omega_t = \mathbb{H}\setminus A_t$ $\quad 0\le t\le s$

$\Omega_t = \mathbb{H}\setminus A_t$



$\varphi_{s,t} = f_t^{-1}\circ f_s$

(35) $f_s = f_t \circ \varphi_{s,t}$

$\varphi_{t,u} \circ \varphi_{s,t} = \varphi_{s,u}$, $0 \le u \le t \le s$
(semigroup property)

$\varphi_{t,t} = id_{\mathbb{H}}$.

Lem. I.18. $\{f_t\}_{t\in I}$ normalized chordal Loewner chain
with associated group $\varphi_{s,t}$.
Then for $t, s \in I$, $t \le s$,
$\varphi_{s,t}$ is a conformal map $\mathbb{H} \longleftrightarrow \mathbb{H} \setminus B_{s,t}$,
where $B_{s,t}$ is an $\mathbb{H}$-hull,
and $\varphi_{s,t}(z) = z + \frac{2(s-t)}{z} + \ldots$ near ∞;
ex. measure $\nu_{s,t} \ge 0$ supp $(\nu_{s,t}) \subset\subset \mathbb{R}$
s.t.
$$\varphi_{s,t}(z) = z + \int_{\mathbb{R}} \frac{d\nu_{s,t}(u)}{u-z}, \quad z \in \mathbb{H},$$
$\nu_{s,t}(\mathbb{R}) = 2(s-t)$.
Moreover, if $t \le s \le T$, then
rad $(B_{s,t}) \le T$ (and so supp $(\nu_{s,t})$
unif bdd.)

Proof: Clear that $\varphi_{s,t} = f_t^{-1} \circ f_s$ has
cont. ext. near ∞ that maps real axis
near ∞ into itself.
So $\varphi_{s,t}$ is conf. map of $\mathbb{H}$ onto
$\mathbb{H} \setminus$ compact set; i.e., $\varphi(\mathbb{H}) = \mathbb{H} \setminus B_{s,t}$,
where $B_{s,t}$ $\mathbb{H}$-hull.

36 $\quad \text{hcap}(B_{s,t}) = \text{hcap}(A_s) - \text{hcap}(A_t)$
$= 2(s-t)$,

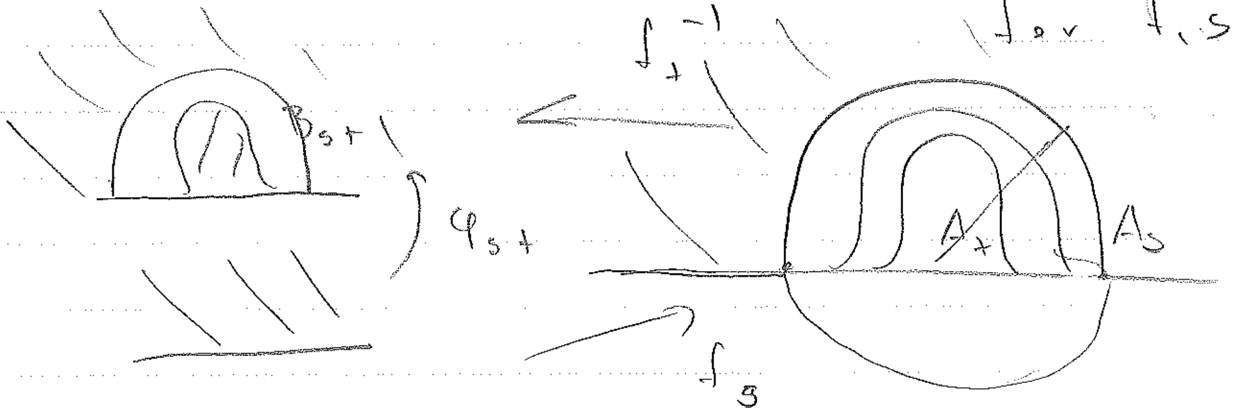
so, $\varphi_{s,t}(z) = z + \dfrac{2(s-t)}{z} + \cdots$ near ∞,

$$\varphi_{s,t}(z) = z + \int_{\mathbb{R}} \frac{d\nu_{s,t}(u)}{u-z}, \quad z \in \mathbb{H}$$

(Thm. 7.9).

We know $\nu_{s,t} \geq 0$,
$\text{supp}(\nu_{s,t}) \subset\subset \mathbb{R}$, $\nu_{s,t}(\mathbb{R}) = \text{hcap}(B_{s,t})$
$= 2(s-t)$.

Finally, $\text{rad}(B_{s,t}) \leq 2\,\text{rad}(A_s) \leq C_0$
for $t, s \leq T$



$\square$

<u>Lem. 7.19.</u> $\{f_t\}_{t \in I}$ normalized chordal Loewner chain,
$\varphi_{s,t} = f_t^{-1} \circ f_s$, $\; t \leq s$, $\; s, t \in I$.

Then for fixed $z \in \mathbb{H}$:

i) $\displaystyle |\varphi_{s,t}(z) - z| \leq \frac{2(s-t)}{\text{Im}\, z}$

ii) $\displaystyle |f_t(z) - f_s(z)| \leq \frac{2(s-t)}{(\text{Im}\, z)^3}\left(2t + (\text{Im}\, z)^2\right)$

iii) $\displaystyle |\varphi_{s,t}(z) - \varphi_{s,u}(z)| \leq \frac{2(t-u)}{\text{Im}\, z}$.

$u \leq t \leq s$, $\; u, t, s \in I$.

(37) iv) $|\varphi_{s,u}(z) - \varphi_{t,u}(z)|$
$$\leq \frac{2(s-t)}{(\operatorname{Im} z)^3}\left[2t + (\operatorname{Im} z)^2\right].$$

So the maps $(z,t) \longmapsto f_t(z)$,
$(z,t) \longmapsto \varphi_{s,t}(z)$,
$(z,t) \longmapsto \varphi_{s,u}(z)$
belong to $HL(\mathbb{H} \times I)$, $HL(\mathbb{H} \times [0,s])$, $HL(\mathbb{H} \times [u,b])$, resp.
$I = [0,b]$

Proof: $f_t(z) = z + \int_{\mathbb{R}} \frac{d\mu_t(u)}{u-z}$, $\mu_t(\mathbb{R}) = 2t$

$\varphi_{s,t}(z) = z + \int_{\mathbb{R}} \frac{d\nu_{s,t}(u)}{u-z}$, $\varphi_{s,t}(\mathbb{R}) = 2(s-t)$

$\operatorname{Im} \varphi_{s,t}(z) \geq \operatorname{Im} z$
(Julia's Lem. or integral repr.)

i) $$|\varphi_{s,t}(z) - z| \leq \int_{\mathbb{R}} \frac{d\nu_{s,t}(u)}{|u-z|} \leq \frac{\nu_{s,t}(\mathbb{R})}{\operatorname{Im} z} = \frac{2(s-t)}{\operatorname{Im} z}.$$

ii) $$f_t'(z) = 1 + \int_{\mathbb{R}} \frac{d\mu_t(z)}{(u-z)^2}$$

$$|f_t'(z)| \leq 1 + \frac{2t}{(\operatorname{Im} z)^2}$$

$$|f_t(z) - f_s(z)| = |f_t(z) - f_t(\varphi_{s,t}(z))|$$
$$\leq |z - \varphi_{s,t}(z)| \cdot \left(1 + \frac{2t}{(\operatorname{Im} z)^2}\right) \leq \frac{2(s-t)}{(\operatorname{Im} z)^3}(2t + (\operatorname{Im} z)^2)$$

(38) iii) $\varphi_{t,u} \circ \varphi_{s,t} = \varphi_{s,u}$

So $\left| \varphi_{s,t}(z) - \varphi_{s,u}(z) \right|$

$= \left| \varphi_{s,t}(z) - \varphi_{t,u}(\varphi_{s,t}(z)) \right|$

$\overset{(i)}{\leq} \dfrac{2(t-u)}{\operatorname{Im} \varphi_{s,t}(z)} \leq \dfrac{2(t-u)}{\operatorname{Im} z}$.

iv) $\left| \varphi'_{t,u}(z) \right| = \left| 1 - \int_{\mathbb{R}} \dfrac{d\nu_{t,u}(x)}{(x-z)^2} \right|$

$\leq 1 + \dfrac{2(t-u)}{(\operatorname{Im} z)^2} \leq 1 + \dfrac{2t}{(\operatorname{Im} z)^2}$.

So

$\left| \varphi_{s,u}(z) - \varphi_{t,u}(z) \right| = \left| \varphi_{t,u}(\varphi_{s,t}(z)) - \varphi_{t,u}(z) \right|$

$\leq \left| \varphi_{s,t}(z) - z \right| \cdot \left[1 + \dfrac{2t}{(\operatorname{Im} z)^2} \right]$

$\leq \dfrac{2(s-t)}{(\operatorname{Im} z)^3} \left[2t + (\operatorname{Im} z)^2 \right]$

Cor. 7.20. $\{f_t\}_{t\in I}$ normalized chordal Loewner chain, $\varphi_{s,t} = f_t^{-1} \circ f_s$, $t \leq s$, $s,t \in I$. $f(z,t) = f_t(z)$

Then there ex. a set $E \subseteq I$, $|E| = 0$

s.t.

i) f is diff. at each point $(z,t) \in \mathbb{H} \times I \setminus E$;

i.e. $f(z',t') = f(z,t) + \dfrac{\partial f}{\partial z}(z,t)(z'-z) + \dfrac{\partial f}{\partial t}(z,t)(t'-t) + o(|t'-t| + |z'-z|)$

near (z,t).

(39) In particular, $\frac{\partial f}{\partial t}(z,t)$ ex. for all $(z,t) \in \mathbb{H} \times I \setminus E$.

ii) $V(z,t) = \lim_{\varepsilon \to 0^+} \frac{\varphi_{t,t-\varepsilon}(z) - z}{\varepsilon}$

exist for all $(z,t) \in \mathbb{H} \times I \setminus E$,

and $\frac{\partial f}{\partial t}(z,t) = - V(z,t) \cdot \frac{\partial f}{\partial z}(z,t)$.

Proof: i) follows from Lem. 7.19 and Prop. 4.12.

ii) Let $(z,s) \in \mathbb{H} \times I \setminus E$, $t \leq s$, t near s.

$f_t \circ \varphi_{s,t} = f_s$; $z' = \varphi_{s,t}(z)$

$|z'-z| = |\varphi_{s,t}(z) - z| \underset{\text{Lem. 7.19.}}{\leq} C|s-t|$

$$0 = f_t(\varphi_{s,t}(z)) - f_s(z)$$

$$= f(z',t) - f(z,s)$$

$$= \underbrace{\frac{\partial f}{\partial z}(z,s)}_{\neq 0}(z'-z) + \frac{\partial f}{\partial s}(z,s)(t-s) + \underbrace{o(|t-s| + |z'-z|)}_{o(|t-s|)}.$$

So

$$V(z,s) = \lim_{t \to s^-} \frac{\varphi_{s,t}(z) - z}{(s-t)} = \lim_{t \to s^-} \frac{z'-z}{s-t}$$

$$= \lim_{t \to s^-} + \frac{\dot f(z,s)}{f'(z,s)} + o(1) = + \frac{\dot f(z,s)}{f'(z,s)}.$$

(40) **Thm. 7.21.** (Loewner-Kufarev equation; chordal case)

$\{f_t\}_{t \in I}$ normalized chordal Loewner chain, $f(z,t) = f_t(z)$

$\varphi_{s,t} = f_t^{-1} \circ f_s$, $t \leq s$, $s,t \in I$.

Then there ex. $E \subseteq I$, $|E| = 0$ s.t.

(a) $V(z,t) = \lim_{\varepsilon \to 0^+} \dfrac{\varphi_{t,t-\varepsilon}(z) - z}{\varepsilon}$

ex. for all $(z,t) \in \mathbb{H} \times I \setminus E$,

(b) $\dfrac{\partial f}{\partial t}(z,t)$ ex. for all $z \in \mathbb{H}$, $t \in I \setminus E$,

and

$$\boxed{\frac{\partial f}{\partial t}(z,t) = V(z,t) \frac{\partial f}{\partial z}(z,t)}$$

Loewner-Kufarev eq.

Moreover, $V(z,t)$ has the following properties:

i) $V(\cdot,t)$ is holomorphic on $\mathbb{H}$ for each $t \in I \setminus E$

ii) V is measurable on $\mathbb{H} \times I$

iii) for each $t \in I \setminus E$ there ex. a probability measure ν_t on $\mathbb{R}$, $\operatorname{supp}(\nu_t) \subset\subset \mathbb{R}$ s.t.

$$V(z,t) = 2 \int_{\mathbb{R}} \frac{d\nu_t(u)}{u - z}, \quad t \in I \setminus \mathbb{R}, \quad z \in \mathbb{H},$$

(41) Proof: We know that there ex. $E \subseteq I$, $|E| = 0$ s.t.

$$V(z,t) := \lim_{\varepsilon \to 0^+} \frac{\varphi_{t,t-\varepsilon}(z) - z}{\varepsilon}$$

ex. for all $z \in \mathbb{H}$, $t \in I \setminus E$,

$\frac{\partial f}{\partial t}(z,t)$ ex. for all $z \in \mathbb{H}$, $t \in I \setminus E$,

and

$$\frac{\partial f}{\partial t}(z,t) = V(z,t) \underbrace{\frac{\partial f}{\partial z}(z,t)}_{\neq 0}.$$

We know $\frac{\partial f}{\partial t}(\cdot,t) \in H(\mathbb{H})$ for $t \in I \setminus E$ (Prop. 4.12).

So

$$V(\cdot,t) = \frac{\dot{f}(\cdot,t)}{f'(\cdot,t)} \in H(\mathbb{H}) \text{ for } t \in I \setminus E,$$

and V measurable on $\mathbb{H} \times I$.

$$\frac{\varphi_{t,t-\varepsilon}(z) - z}{\varepsilon} = \frac{1}{\varepsilon} \int_{\mathbb{R}} \frac{d\nu_{t,t-\varepsilon}(u)}{u - z}.$$

Here $\nu_{t,t-\varepsilon}(\mathbb{R}) = 2\varepsilon$, $\nu_{t,t-\varepsilon}$ on $\mathbb{R}$.

Actually, the supports of $\nu_{t,t-\varepsilon}$ are unif. bdd. for $\varepsilon > 0$, t fixed. (Lem. 7.18).

Say $\mathrm{supp}(\nu_{t,t-\varepsilon}) \subseteq [-R_0, R_0]$.

Let

$$\sigma_\varepsilon := \frac{1}{2\varepsilon} \nu_{t,t-\varepsilon}.$$

Then σ_ε subconverges to a prob. measure ν_t on $[-R_0, R_0]$ as $\varepsilon \longrightarrow 0$ w.r.t. w^*-convergence.

So

$$V(z,t) := \lim_{\varepsilon \to 0^+} \frac{\varphi_{t,t-\varepsilon}(z) - z}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0^+} 2 \int_{[-R_0,R_0]} \frac{1}{u-z}\, d\sigma_\varepsilon(u)$$

$$= 2 \int_{[-R_0,R_0]} \frac{d\nu_t(u)}{u-z}, \quad z \in \mathbb{H},\ t \in I \setminus E.$$

<u>Rem. 7.22.</u> TFAE:

i) $$V(z) = \int_{\mathbb{R}} \frac{d\nu(u)}{u-z} \quad \text{for } z \in \mathbb{H},$$

where $\nu \geq 0$, $\nu(\mathbb{R}) = 1$, $\mathrm{supp}(\nu) \geq 0$.

ii) V is holomorphic on $\mathbb{H}$,

$\mathrm{Im}\, V(z) > 0$ for $z \in \mathbb{H}$,

V has holomorphic ext. near ∞ s.t.

$V(z) = -\frac{1}{z} + \dots$ near ∞,

and $\mathrm{Im}\, f(x) = 0$ for $x \in \mathbb{R}$, $|x|$ large.

<u>Proof:</u> (i) $\to$ (ii), $z = x+iy$

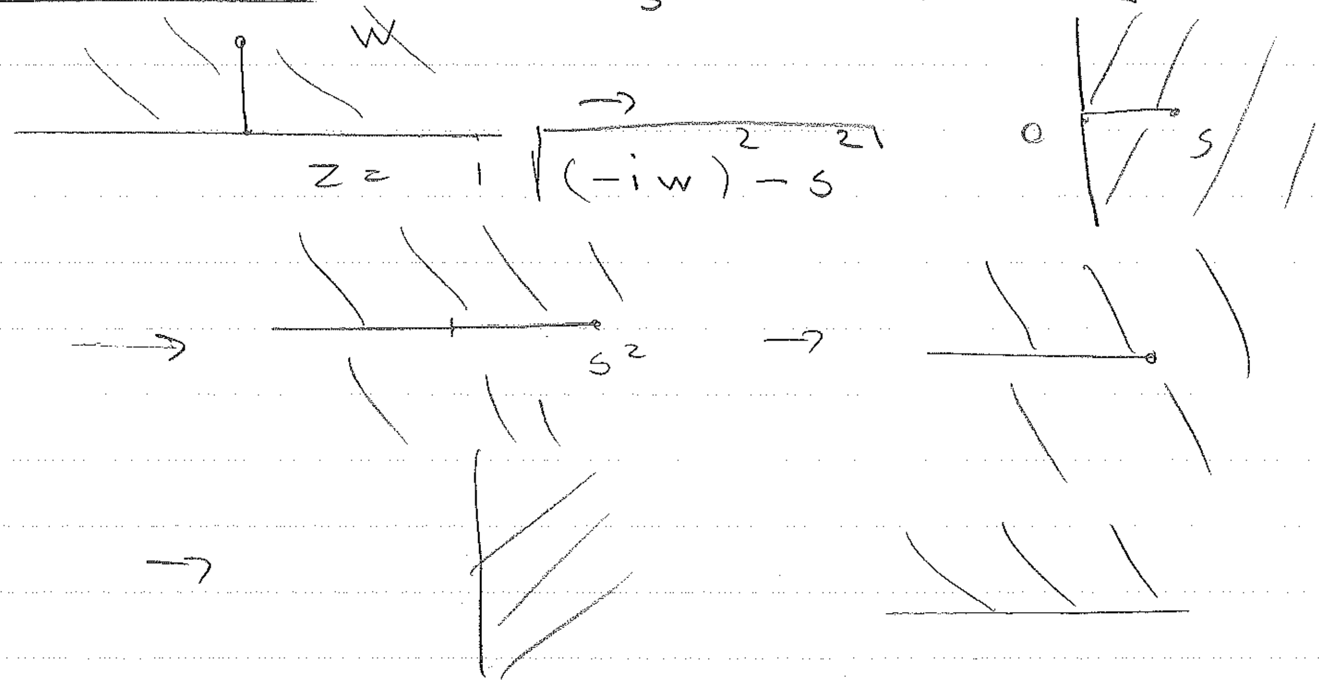
$$\mathrm{Im}\left(\frac{1}{u-z}\right) = \frac{y}{(u-x)^2+y^2} > 0 \quad \text{for } z \in \mathbb{H}.$$

(43) (ii) $\to$ (i): Follows as in the proof of Thm. 7.9. from Herglotz representation.

Note that if $\operatorname{Im} V$ has a cont. ext. to $\mathbb{R}$, then

$$d\nu(u) = \frac{1}{\pi} \operatorname{Im} V(u)\, du.$$

<u>Ex. 7.23.</u> $\Omega_s = \mathbb{H} \setminus [0, is]$.



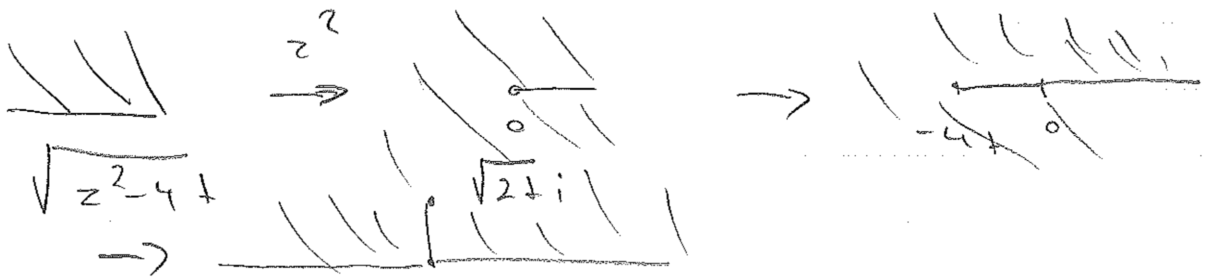
$$z = i\sqrt{-w^2 - s^2} = \sqrt{w^2 + s^2} = w\sqrt{1 + \frac{s^2}{w^2}}$$

$\sqrt{1+u} = 1 + \frac{1}{2}u \ldots$ near 0 $\qquad = w + \frac{s^2}{2w} + \ldots$ near ∞

$$2t = \frac{s^2}{2} \qquad s^2 = 4t$$

$$z = \sqrt{w^2 + 4t} \qquad z^2 = w^2 + 4t \qquad w = f_t(z) = \sqrt{z^2 - 4t}$$

(44) normalized Loewner chain.



$$\dot f_t(z) = \frac{1}{2}\,\frac{-4}{\sqrt{z^2-4t}} = -\frac{2}{\sqrt{z^2-4t}}$$

$$f_t'(z) = \frac{2z}{2\sqrt{z^2-4t}} = \frac{z}{\sqrt{z^2-4t}}$$

$$V(z,t) = \frac{\dot f_t(z)}{f_t'(z)} = -\frac{2}{z} = 2\int_{\mathbb{R}} \frac{d\delta_0(u)}{u-z}$$

So $\nu_t = \delta_0$ for all $t \geq 0$.

$$f_t(z) = z + \frac{1}{\pi}\int_{\mathbb{R}} \frac{\operatorname{Im} f_t(u)}{u-z}\, du$$

$$\operatorname{Im} f_t(u) = \begin{cases} \sqrt{4t-u^2} & \text{for } u \in [-2\sqrt{t}, 2\sqrt{t}] \\ 0 & \text{elsewhere} \end{cases}$$

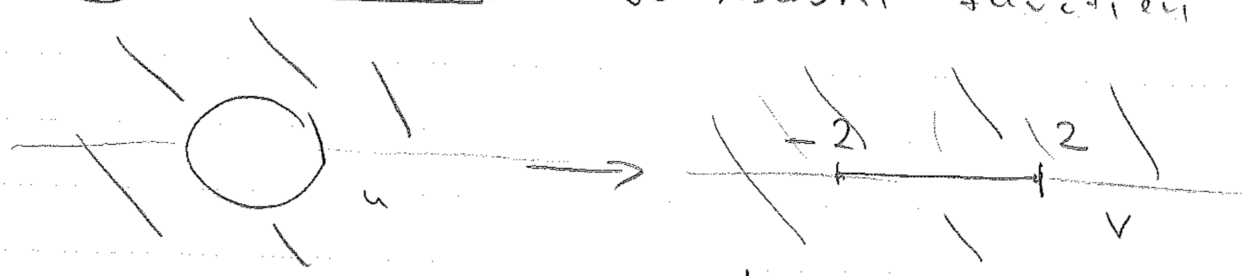
$$f_t(z) = z + \int \frac{d\mu_t(u)}{u-z}, \quad \text{where}$$

$$d\mu_t(u) = \frac{1}{\pi}\sqrt{4t-u^2}\,\chi_{[-2\sqrt{t},\,2\sqrt{t}]}(u)\, du, \quad t \geq 0$$

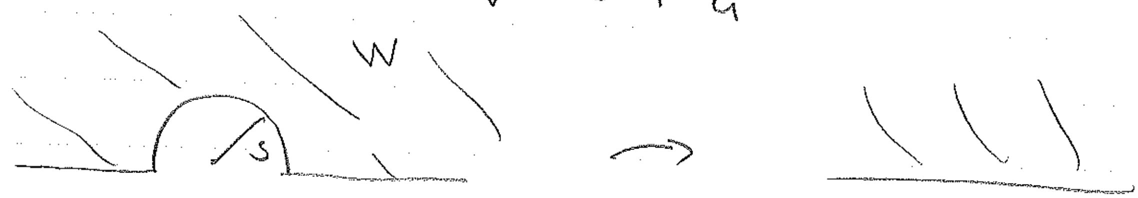
"semicircle law"

$$\mu_t(\mathbb{R}) = \frac{1}{\pi}\frac{\pi}{2}\cdot(2\sqrt{t})^2 = \underline{\underline{2t}}. \quad ✓$$

(45) Ex. 7.24. Joukowski-function



$$v = u + \frac{1}{u}$$



$$z = s\left(\frac{w}{s} + \frac{s}{w}\right) = w + \frac{s^2}{w} \quad , \; 2t = s^2$$

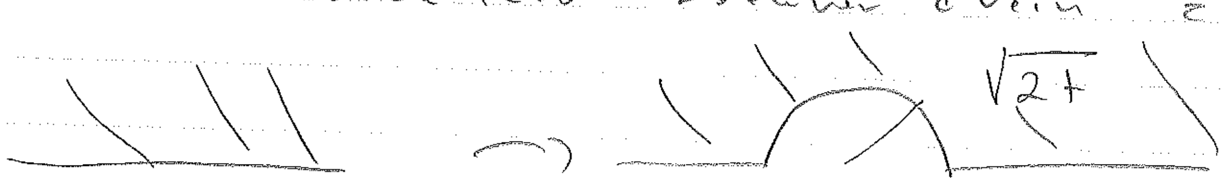
$$= w + \frac{2t}{w}$$

$$w^2 - zw + 2t = 0 \;, \quad w = \frac{z}{2} + \sqrt{\frac{z^2}{4} - 2t}$$

$$= \frac{1}{2}\left(z + \sqrt{z^2 - 8t}\right)$$

$$f_t(z) = \frac{1}{2}\left(z + \sqrt{z^2 - 8t}\right) \qquad \sqrt{z^2 - 8t} = z\sqrt{1 - \frac{8t}{z^2}}$$

normalized Loewner chain $\quad \approx z\left(1 - \frac{4t}{z^2}\right)$



$$\dot{f}_t = \frac{1}{4}\,\frac{-8}{\sqrt{z^2 - 8t}} = -\frac{2}{\sqrt{z^2 - 8t}}$$

$$f_t' = \frac{1}{2}\left(1 + \frac{2z}{2\sqrt{z^2 - 8t}}\right) = \frac{1}{2}\left(1 + \frac{z}{\sqrt{z^2 - 8t}}\right)$$

$$V(z,t) = \frac{\dot{f}_t}{f_t'} = -\frac{2}{\sqrt{\cdots}} \quad \frac{2}{\left(1 + \frac{z}{\sqrt{\quad}}\right)}$$

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$$= -\frac{4}{z+\sqrt{z^2-8t}}$$

$$= -4\,\frac{z-\sqrt{z^2-8t}}{z^2-(z^2-8t)}$$

$$= -\frac{1}{2t}\left(z-\sqrt{z^2-8t}\right) =$$

$$= -\frac{1}{2t}\left(z-z\sqrt{1-\frac{8t}{z^2}}\right) = -\frac{1}{2t}\left(\frac{4t}{z}+\cdots\right)$$

$$= -\frac{2}{z}+\cdots$$

$$\operatorname{Im} V(u,t) = \begin{cases} +\frac{1}{2t}\sqrt{8t-u^2} & \text{for } -\sqrt{8t}\le u\le\sqrt{8t} \\ 0 & \text{else} \end{cases}$$

$$V(u,t) = 2\int\frac{d\nu_t(u)}{u-z}, \text{ where}$$

$$d\nu_t(u) = \frac{1}{4\pi t}\sqrt{8t-u^2}\;\chi_{[-\sqrt{8t},\sqrt{8t}]}(u)\,du$$

"semicircle law"

$$\operatorname{Im} f_t(u) = \begin{cases} \frac{1}{2}\sqrt{8t-u^2} & -\sqrt{8t}\le u\le\sqrt{8t} \\ 0 & \text{else} \end{cases}$$

$$f_t(z) = z + \int_{\mathbb{R}}\frac{d\mu_t(u)}{u-z}, \text{ where}$$

$$d\mu_t(u) = \frac{1}{2\pi}\sqrt{8t-u^2}\;\chi_{[-\sqrt{8t},\sqrt{8t}]}(u)\,du$$

$$\mu_t(\mathbb{R}) = 2t \qquad \boxed{\frac{1}{2t}\mu_t = \nu_t}$$